A PROXIMAL POINT ALGORITHM FOR SOLVING THE GENERAL VARIATIONAL INCLUSIONS WITH $M(\cdot,\cdot)$-MONOTONE OPERATORS IN BANACH SPACES

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ABSTRACT. In this paper, a new monotonicity, $M(\cdot,\cdot)$-monotonicity, is introduced in Banach spaces, and the resolvent operator of an $M(\cdot,\cdot)$-monotone operator is proved to be single valued and Lipschitz continuous. By using the resolvent operator technique associated with $M(\cdot,\cdot)$-monotone operators, we construct a proximal point algorithm for solving a class of variational inclusions. And we prove the convergence of the sequences generated by the proximal point algorithms in Banach spaces. The results in this paper extend and improve some known results in the literature.

1. Introduction and preliminaries

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years, since they have wide applications in the optimization and control, economics and transportation equilibrium, engineering science. For these reasons, many existence results and iterative algorithms for various variational inclusions have been studied. For details, we refer the reader to [1-10]. Recently, Pennanen [11] introduced over-relaxed the Eckstein-Bertsekas proximal point algorithm [12] and using it has shown that the sequence converges linearly to a solution to the following variational inclusion problem: for finding $x \in H$, such that

$$0 \in T(x), \tag{1.1}$$

where H is Hilbert space, $T : H \to 2^H$ is a set-valued mapping on H. On the basis of this new version of the proximal point algorithm, Pennanen [11] studied a localized version of the maximal monotonicity, and has shown that it ensures the local convergence of the over-relaxed proximal point algorithm. Furthermore, the local convergence of multiplier methods for a general class of

problems is established. This, in a way, presents specializations as new convergence results for multiplier methods for nonmonotone variational inequalities and nonconvex nonlinear programming.

Verma [13] develop a hybrid version of the Eckstein-Bertsekas proximal point algorithm based on the notions of A-maximal monotonicity [4] and (A, η)-maximal monotonicity [14] for solving the variational inclusion problem (1.1). These notions generalize the general class of maximal monotone set-valued mappings, including the notion of H-maximal monotonicity introduced by Fang and Huang [15] in a Hilbert space.

Very recently, Juhe Sun, Liwei Zhang, Xiantao Xiao [17] introduced a new monotonicity, $M(\cdot, \cdot)$-monotonicity, and a proximal point algorithm is constructed to solve a class of variational inequalities in Hilbert space.

Motivated and inspired by the research work going on this field, in this paper, we consider the inclusion problem:

$$0 \in G(u) + T(u), \tag{1.2}$$

where $G : X \to X^*$ is a single-valued operator and $T : X \to 2^{X^*}$ is a set-valued mapping on X. And we develop the proximal point algorithm for solving this general variational inclusion in Banach space.

Let X be a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^*, and 2^{X^*} denote the family of all the nonempty subset of X^*. The generalized duality mapping $J_q(x) : X \to 2^X$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q = \|x\|^{q-2} J_2$, for all $x \in X$, and $J_q(x)$ is single-valued if X^* is strictly convex. In what follows, unless otherwise specified, we always suppose that X is a real Banach space such that $J_q(x)$ is single-valued. If X is a Hilbert space H, then J_2 becomes the identity mapping of H.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\| - 1) : \|x\| < t, \|y\| < t \right\}.$$

A Banach space X is called uniformly smooth of

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_X(t) \le ct^q, q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach space, Xu [18] proved the following lemma.

Lemma 1.1(Xu [18]) Let X be a real uniformly smooth Banach space. Then, X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 1.1 Let $A, B : X \to X$ be two single-valued mappings, $D : X \to X^*$ be another single-valued operator, and $M : X \times X \to X^*$ be a mapping,
(i)$M(A, \cdot)$ is said to be α-strong monotone with respect to A if there exists a positive constant α such that

$$\langle M(Ax, u) - M(Ay, u), x - y\rangle \geq \alpha\|x - y\|^2, \forall x, y, u \in X;$$

(ii)$M(\cdot, B)$ is said to be β-relaxed monotone with respect to B if there exists a positive constant β such that

$$\langle M(u, Bx) - M(u, By), x - y\rangle \geq -\beta\|x - y\|^2, \forall x, y, u \in X;$$

(iii)$M(\cdot, \cdot)$ is said to be $\alpha\beta$-symmetric monotone with respect to A and B if $M(A, \cdot)$ is α-strongly monotone with respect to A and $M(\cdot, B)$ is β-relaxed monotone with respect to B with $\alpha > \beta$ and $\alpha = \beta$ if and only if $x = y, \forall x, y, u \in X$;
(iv) $M(\cdot, \cdot)$ is said to be ξ-Lipschitz continuous with respect to the first argument if there exists a constant $\xi > 0$ satisfying

$$\|M(x, u) - M(y, u)\| \leq \xi\|x - y\|, \forall x, y, u \in X;$$

(v) A is said to be t-Lipschitz continuous if there exists a constant $t > 0$ satisfying

$$\|Ax - Ay\| \leq t\|x - y\|, \forall x, y \in X;$$

(vi) D is said to be r-strongly accretive with respect to $M(A, B)$ if there exists a constant $r > 0$ satisfying

$$\langle Dx - Dy, J_q^*(M(Ax, Bx) - M(Ay, By))\rangle \geq r\|x - y\|^q, \forall x, y \in X,$$

where $J_q^* : X^* \to X^{**}$ is the generalized duality mapping on X^*.

In a similar way to (iv), we can define the Lipschitz continuity of the mapping M with respect to the second argument. **Definition 1.2** Let $T : X \to 2^{X^*}$ be multivalued mapping. The map T is said to be:
(i) monotone if

$$\langle u^* - v^*, u - v\rangle \geq 0, \forall u, v \in X, u^* \in T(u), v^* \in T(v).$$

(ii) r-strong monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2, \forall u, v \in X, u^* \in T(u), v^* \in T(v).$$

(iii) maximal monotone if T is monotone and has no a proper monotone extension in X, i.e., for all $u, u_0 \in X, x \in Tu$,

$$\langle x - y_0, u - u_0 \rangle \geq 0$$

implies $y_0 \in Tu_0$.

When X is a reflexive Banach space, T is maximal monotone if and only if $(J + \lambda T)X = X^*$ for all $\lambda > 0$.

2. $M(\cdot, \cdot)$-monotone operator

In this section, we introduce $M(\cdot, \cdot)$-monotone operators and discuss its properties in Banach spaces.

Definition 2.1 Let $A, B : X \to X$ be two single-valued operators, $M : X \times X \to X^*$ be a mapping and $T : X \to 2^{X^*}$ be a multi-valued operator. T is said to be $M(\cdot, \cdot)$-monotone with respect to A and B, if T is monotone and $(M(A, B) + \lambda T)(X) = X^*$ holds for every $\lambda > 0$.

Remark 2.1. If $M(A, B) = H$, then the above definition reduces to H-monotonicity, which was studied in [3]. If $M(A, B) = J$, then the definition of J-monotonicity is just the maximal monotonicity.

Remark 2.2. Let T be a monotone operator and λ be a positive constant. If $T : X \to 2^{X^*}$ is an $M(\cdot, \cdot)$-monotone operator with respect to A and B, every element $z \in X^*$ can be written in exactly one way as $M(Ax, Bx) + \lambda u$, where $u \in T(x)$.

We use the same technique in [3] to finish the following proof.

Proposition 2.1. Let M be $\alpha\beta$-symmetric monotone with respect to A and B and $T : X \to 2^{X^*}$ be an $M(\cdot, \cdot)$-monotone operator with respect to A and B, then T is maximal monotone.

Proof. Since T is monotone, it is sufficient to prove the following property: inequality $\langle x - y, u - v \rangle \geq 0$ for $(v, y) \in Graph(T)$ implies

$$x \in Tu. \tag{2.1}$$

Suppose, by contradiction, that there exists some $(u_0, x_0) \notin Graph(T)$ such that

$$\langle x_0 - y, u_0 - v \rangle \geq 0, \forall (v, y) \in Graph(T). \tag{2.2}$$

Since T is $M(\cdot, \cdot)$-monotone with respect to A and B, $(M(A, B) + \lambda T)(X) = X^*$ holds for every $\lambda > 0$, there exists $(u_1, x_1) \in Graph(T)$ such that

$$M(Au_1, Bu_1) + \lambda x_1 = M(Au_0, Bu_0) + \lambda x_0 \in X^*. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$
\begin{aligned}
0 &\leq \lambda\langle x_0 - x_1, u_0 - u_1\rangle \\
&= -\langle M(Au_0, Bu_0) - M(Au_1, Bu_1), u_0 - u_1\rangle \\
&= -\langle M(Au_0, Bu_0) - M(Au_1, Bu_0), u_0 - u_1\rangle \\
&\quad +\langle M(Au_1, Bu_0) - M(Au_1, Bu_1), u_0 - u_1\rangle \\
&\leq -(\alpha - \beta)\|u_0 - u_1\| \leq 0,
\end{aligned}
$$

which yields $u_1 = u_0$. By (2.3), we have that $x_1 = x_0$. Hence $(u_0, x_0) \in Graph(T)$, which is a contradiction. Therefore (2.1) holds and T is maximal monotone. This completes the proof. $\square$

The following example shows that a maximal monotone operator may not be $M(\cdot, \cdot)$-monotone for some A and B.

Example. Let $X = S^2$, where S^2 is the space of 2×2 symmetric matrices. The inner product is defined as $\langle S_1, S_2\rangle = tr(S_1 S_2)$. Let $T = I$ and $M(Ax, Bx) = E - x$ for all $x \in S^2$, where E is an identity matrix. Then it is easy to see that I is maximal monotone. For all $x \in S^2$, we have that

$$
\|(M(A, B) + I)(x)\|^2 = \|E - x + x\|^2 = \langle E, E\rangle = tr[E^2] = 2 > 0,
$$

which means that $0 \notin (M(A, B) + I)(S^2)$ and I is not $M(\cdot, \cdot)$-monotone with respect to A and B.

Theorem 2.1. Let $M(A, B)$ be $\alpha\beta$-symmetric monotone with respect to A and B and T be an $M(\cdot, \cdot)$-monotone operator with respect to A and B. Then the operator $(M(A, B) + \lambda T)^{-1}$ is single-valued.

Proof. For any given $u \in X$, let $x, y \in (M(A, B) + \lambda T)^{-1}(u)$. It follows that $\frac{1}{\lambda}(-M(Ax, Bx) + u) \in Tx$ and $\frac{1}{\lambda}(-M(Ay, By) + u) \in Ty$. From the mononicity of M and T, we have

$$
\begin{aligned}
0 &\leq \frac{1}{\lambda}\langle -M(Ax, Bx) + u - (-M(Ay, By) + u), x - y\rangle \\
&= -\frac{1}{\lambda}\langle M(Ax, Bx) - M(Ay, By), x - y\rangle \\
&= -\frac{1}{\lambda}\langle M(Ax, Bx) - M(Ay, Bx), x - y\rangle \\
&\quad -\langle M(Ay, Bx) - M(Ay, By), x - y\rangle \\
&\leq -\frac{1}{\lambda}(\alpha - \beta)\|x - y\|^2 \leq 0.
\end{aligned}
$$

Thus, we have $x = y$ and so $(M(A, B) + \lambda T)^{-1}$ is single-valued. This completes the proof. $\square$

Definition 2.2. Let $M(A, B)$ be α-strongly monotone with respect to A, β-relaxed monotone with respect to B, and $\alpha > \beta$. Let T be an $M(\cdot, \cdot)$-monotone

operator with respect to A and B. The resolvent operator $R_{T,\lambda}^M : X^* \to X$ is defined by $R_{T,\lambda}^M = (M(A, B) + \lambda T)^{-1}(u), \forall u \in X^*$.

The following theorem shows that the new resolvent operator has similarly useful properties as those discuss in [17].

Theorem 2.2. Let $M(A, B)$ be α-strongly monotone with respect to A, β-relaxed monotone with respect to B, and $\alpha > \beta$. Let T be an $M(\cdot, \cdot)$-monotone operator with respect to A and B. The resolvent operator $R_{T,\lambda}^M : X^* \to X$ is $\frac{1}{\alpha-\beta}$-Lipschitz continuous, that is :

$$\|R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\| \leq \frac{1}{\alpha - \beta}\|u - v\|, \forall u, v \in X^*.$$

Proof. Let u, v be any given points in X^*. It follows from definition 2.2 that $R_{T,\lambda}^M(u) = (M(A, B) + \lambda T)^{-1}(u)$ and $R_{T,\lambda}^M(v) = (M(A, B) + \lambda T)^{-1}(v)$. This implies that

$$\frac{1}{\lambda}(u - M(A(R_{T,\lambda}^M(u)), B(R_{T,\lambda}^M(u))) \in T(R_{T,\lambda}^M(u))$$

and

$$\frac{1}{\lambda}(v - M(A(R_{T,\lambda}^M(v)), B(R_{T,\lambda}^M(v))) \in T(R_{T,\lambda}^M(v)).$$

Since T is $M(\cdot, \cdot)$-monotone,

$$
\begin{aligned}
&\frac{1}{\lambda} && \langle u - M(A(R_{T,\lambda}^M(u)), B(R_{T,\lambda}^M(u)) - (v - M(A(R_{T,\lambda}^M(v)), B(R_{T,\lambda}^M(v)), \\
& && R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\rangle \\
&= \frac{1}{\lambda} && \langle u - v - (M(A(R_{T,\lambda}^M(u)), B(R_{T,\lambda}^M(u))) - M(A(R_{T,\lambda}^M(v)), B(R_{T,\lambda}^M(v)))), \\
& && R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\rangle \\
&= && \langle T(R_{T,\lambda}^M(u)) - T(R_{T,\lambda}^M(v)), R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\rangle \\
&\geq && 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|u - v\| \quad &\|R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\| \geq \langle u - v, R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\rangle \\
\geq \quad & \langle M(A(R_{T,\lambda}^M(u)), B(R_{T,\lambda}^M(u))) - M(A(R_{T,\lambda}^M(v)), B(R_{T,\lambda}^M(v))), \\
& R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\rangle \\
\geq \quad & (\alpha - \beta)\|R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\|^2
\end{aligned}
$$

and so

$$\|R_{T,\lambda}^M(u) - R_{T,\lambda}^M(v)\| \leq \frac{1}{\alpha - \beta}\|u - v\|.$$

3. An algorithm for variational inclusions

In this section, we suggest a proximal point algorithm to solve the variational inclusion (1.2) and prove the global convergence of the algorithm.

Lemma 3.1 Let $M(A, B)$ be $\alpha\beta$-symmetric monotone with respect to A and B and T be an $M(\cdot, \cdot)$-monotone operator with respect to A and B. Then $u \in X$ is a solution of the variational inclusion (1.2) if and only if u satisfies

$$u = R_{T,\lambda}^M[M(A(u), B(u)) - \lambda G(u)]. \tag{3.1}$$

proof. The conclusion can be drawn directly from the definition of the resolvent operator $R_{T,\lambda}^M$. □

Based on (3.1), we can construct the following algorithm.

Algorithm 3.1 For any given $x_0 \in X$, compute $\{x_n\} \subset X$ as follows:

$$x_{n+1} = (1 - \rho_n)x_n + \rho_n y_n,$$

and y_n satisfies

$$\|y_n - R_{T,\lambda}^M[M(A(x_n), B(x_n)) - \lambda_n G(x_n)]\| \le \varepsilon_n.$$

Theorem 3.1 Let X^* be a q-uniformly smooth Banach space, $A : X \to X$ be τ-Lipschitz continuous and B be t-Lipschitz continuous, $M(A, \cdot) : X \to X^*$ be α-strongly monotone with respect to A and $M(\cdot, B) : X \to X^*$ is β-relaxed monotone with respect to B with $\alpha > \beta$. Let $M(\cdot, \cdot)$ is ξ-Lipschitz continuous with respect to the first argument and ζ-Lipschitz continuous with respect to the second argument, T be an $M(\cdot, \cdot)$-monotone and $G : X \to X^*$ be γ-Lipschitz continuous and r-strongly accretive with respect $M(\cdot, \cdot)$. Let x_0 be given, $\{\varepsilon_n\} \subset [0, +\infty)$ satisfy $E = \sum_{n=1}^{\infty} \varepsilon_n < \infty$, $\{\lambda_n\} \subset (\lambda_0, \infty)$, where $\lambda_0 > 0$ and

$$(\alpha - \beta)^q > (\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n,$$

which implies that

$$\tilde{L} = \left[1 - \frac{1}{\alpha - \beta} \sqrt[q]{(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n}\right]$$

$$\times \left\{1 + q\frac{1}{\alpha - \beta} \sqrt[q]{(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n}\right.$$

$$\left.+ c_q \frac{1}{(\alpha - \beta)^q}[(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n]\right\}^{-1}$$

$$> 0.$$

If $\{\rho_n\} \subseteq [R_m, R_M]$, where $0 < R_m < R_M < \sqrt[q-1]{\frac{q\tilde{L}}{c_q}}$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to a solution of the variational inclusion (1.2).

Proof. We introduce a new map

$$Q_n \equiv I - R_{T,\lambda_n}^M (M(A,B) - \lambda_n G).$$

Clearly, any zero of $G + T$ is also a zero of Q_n. For $x, y \in X$, we know that

$$
\begin{aligned}
&\langle Q_n(x) - Q_n(y), J_q(x-y) \rangle \\
=\; &\langle x - y - \\
&R_{T,\lambda_n}^M (M(A(x), B(x)) - \lambda_n G(x)) + R_{T,\lambda_n}^M (M(A(y), B(y)) - \lambda_n G(y)), \\
&J_q(x-y) \rangle \\
=\; &\|x - y\|^q \\
&- \langle R_{T,\lambda_n}^M (M(A(x), B(x)) - \lambda_n G(x)) - R_{T,\lambda_n}^M (M(A(y), B(y)) - \lambda_n G(y)), \\
&J_q(x-y) \rangle \\
\geq\; &\|x - y\|^q - \frac{1}{\alpha - \beta} \|M(A(x), B(x)) - M(A(y), B(y)) - \lambda_n (G(x) - G(y))\| \\
&\times \|J_q(x-y)\| \\
=\; &\|x - y\|^q - \frac{1}{\alpha - \beta} \|M(A(x), B(x)) - M(A(y), B(y)) - \lambda_n (G(x) - G(y))\| \\
&\times \|x - y\|^{q-1},
\end{aligned}
\tag{3.2}
$$

by the assumptions and Lemma 1.1, we have

$$
\begin{aligned}
&\|M(A(x), B(x)) - M(A(y), B(y)) - \lambda_n (G(x) - G(y))\|^q \\
\leq\; &\|M(A(x), B(x)) - M(A(y), B(y))\|^q + \lambda_n^q c_q \|G(x) - G(y)\|^q \\
&- q\lambda_n \langle G(x) - G(y), J_q^* (M(A(x), B(x)) - M(A(y), B(y))) \rangle \\
\leq\; &(\|M(A(x), B(x)) - M(A(y), B(x))\| + \|M(A(y), B(x)) - M(A(y), B(y))\|)^q \\
&+ \lambda_n^q c_q \gamma^q \|x - y\|^q - rq\lambda_n \|x - y\|^q \\
=\; &[(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n] \|x - y\|^q,
\end{aligned}
\tag{3.3}
$$

by (3.3), we have

$$
\begin{aligned}
&\|M(A(x), B(x)) - M(A(y), B(y)) - \lambda_n (G(x) - G(y))\| \\
\leq\; &\sqrt[q]{(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n} \|x - y\|.
\end{aligned}
\tag{3.4}
$$

So, we have

$$\langle Q_n(x) - Q_n(y), J_q(x-y) \rangle \geq \left[1 - \frac{1}{\alpha - \beta} \sqrt[q]{(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n} \right] \|x - y\|^q. \tag{3.5}$$

By Lemma 1.1

$$\|Q_n(x) - Q_n(y)\|^q$$
$$= \|x - y -$$
$$\left(R^M_{T,\lambda_n}(M(A(x), B(x)) - \lambda_n G(x)) - R^M_{T,\lambda_n}(M(A(y), B(y)) - \lambda_n G(y))\right)\|^q$$
$$\leq \|x - y\|^q -$$
$$q\langle R^M_{T,\lambda_n}(M(A(x), B(x)) - \lambda_n G(x)) - R^M_{T,\lambda_n}(M(A(y), B(y)) - \lambda_n G(y)), J_q(x - y)\rangle$$
$$+ c_q \|R^M_{T,\lambda_n}(M(A(x), B(x)) - \lambda_n G(x)) - R^M_{T,\lambda_n}(M(A(y), B(y)) - \lambda_n G(y))\|^q$$
$$\leq \|x - y\|^q +$$
$$q\|R^M_{T,\lambda_n}(M(A(x), B(x)) - \lambda_n G(x)) - R^M_{T,\lambda_n}(M(A(y), B(y)) - \lambda_n G(y))\|\|x - y\|^{q-1}$$
$$+ c_q \frac{1}{(\alpha - \beta)^q} \|M(A(x), B(x)) - \lambda_n G(x) - (M(A(y), B(y)) - \lambda_n G(y))\|$$
$$\leq \|x - y\|^q + q\frac{1}{\alpha - \beta} \|M(A(x), B(x)) - \lambda_n G(x) - (M(A(y), B(y)) - \lambda_n G(y))\|$$
$$\times \|x - y\|^{q-1} + c_q \frac{1}{(\alpha - \beta)^q}[(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n]\|x - y\|^q$$
$$\leq \|x - y\|^q + q\frac{1}{\alpha - \beta} \sqrt[q]{(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n}\|x - y\|^q$$
$$+ c_q \frac{1}{(\alpha - \beta)^q}[(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n]\|x - y\|^q$$
$$= \left\{1 + q\frac{1}{\alpha - \beta}\sqrt[q]{(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n} + \right.$$
$$\left. c_q \frac{1}{(\alpha - \beta)^q}[(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n]\right\}\|x - y\|^q \qquad (3.6)$$

From (3.2) to (3.6), we have

$$\langle Q_n(x) - Q_n(y), J_q(x - y)\rangle$$
$$\geq \left[1 - \frac{1}{\alpha - \beta}\sqrt[q]{(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n}\right]$$
$$\times \left\{1 + q\frac{1}{\alpha - \beta}\sqrt[q]{(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n} + \right.$$
$$\left. c_q \frac{1}{(\alpha - \beta)^q}[(\xi\tau + \zeta t)^q + \lambda_n^q c_q \gamma^q - rq\lambda_n]\right\}^{-1}\|Q_n(x) - Q_n(y)\|^q$$
$$= \tilde{L}\|Q_n(x) - Q_n(y)\|^q.$$

For all n we denote by

$$z_{n+1} = (1 - \rho_n)x_n + \rho_n R^M_{T,\lambda_n}(M(A(x_n), B(x_n)) - \lambda_n G(x_n)).$$

For every zero x^* of $T + G$, we can write:

$$\|z_{n+1} - x^*\|^q = \|x_n - \rho_n Q_n(x_n) - x^*\|^q$$
$$\leq \|x_n - x^*\|^q - q\rho_n \langle Q_n(x_n) - Q_n(x^*), J_q(x_n - x^*) \rangle + c_q \rho_n^q \|Q(x_n)\|^q$$
$$\leq \|x_n - x^*\|^q - q\rho_n \tilde{L} \|Q_n(x_n)\|^q + c_q \rho_n^q \|Q(x_n)\|^q$$
$$\leq \|x_n - x^*\|^q - (q\rho_n \tilde{L} - c_q \rho_n^q)\|Q_n(x_n)\|^q$$
$$\leq \|x_n - x^*\|^q - \rho_n(q\tilde{L} - c_q \rho_n^{q-1})\|Q_n(x_n)\|^q$$
$$\leq \|x_n - x^*\|^q - R_m(q\tilde{L} - c_q R_M^{q-1})\|Q_n(x_n)\|^q$$
$$\leq \|x_n - x^*\|^q.$$

Since $\|x_{n+1} - z_{n+1}\| \leq \rho_n \varepsilon_n$, we get that

$$\|x_{n+1} - x^*\| \leq \|z_{n+1} - x^*\| + \|x_{n+1} - z_{n+1}\|$$
$$\leq \|x_n - x^*\| + \rho_n \varepsilon_n$$
$$\leq \|x_0 - x^*\| + \sum_{j=1}^{n} \rho_j \varepsilon_j$$
$$\leq \|x_0 - x^*\| + \sqrt[q-1]{\frac{q}{c_q}} E,$$

so that the sequence $\{x_n\}$ is bounded. We can also know that

$$\|x_{n+1} - x^*\|^q = \|z_{n+1} - x^* + x_{n+1} - z_{n+1}\|^q$$
$$\leq \|z_{n+1} - x^*\|^q + q\langle x_{n+1} - z_{n+1}, J_q(z_{n+1} - x^*)\rangle + c_q \|x_{n+1} - z_{n+1}\|^q$$
$$\leq \|z_{n+1} - x^*\|^q + q\|x_{n+1} - z_{n+1}\|\|z_{n+1} - x^*\|^{q-1} + c_q \|x_{n+1} - z_{n+1}\|^q$$
$$\leq \|x_n - x^*\|^q + q\rho_n \varepsilon_n (\|x_0 - x^*\| + \sqrt[q-1]{\frac{q}{c_q}} E)^{q-1} + \rho_n^q \varepsilon_n^q$$
$$- R_m(q\tilde{L} - c_q R_M^{q-1})\|Q_n(x_n)\|^q.$$

We have for every n

$$\|x_{n+1} - x^*\|^q \leq \|x_0 - x^*\|^q + q(\|x_0 - x^*\| + 2E)^{q-1} \sum_{n=}^{\infty} \rho_n \varepsilon_n$$
$$+ \sum_{n=1}^{\infty} \rho_n^q \varepsilon_n^q - R_m(q\tilde{L} - c_q R_M^{q-1}) \sum_{n=1}^{\infty} \|Q_n(x_n)\|^q.$$

Passing $n \to \infty$, we deduce that $\sum_{n=1}^{\infty} \|Q_n(x_n)\|^q < \infty$. It follows that
$$\lim_{n\to\infty} Q_n(x_n) = 0.$$

According to the remark 2.2, for every n there exist a unique pair $(u_n, v_n) \in Graph(T)$, such that

$$w_n = M(A(x_n), B(x_n)) - \lambda_n G(x_n) = M(A(u_n), B(u_n)) + \lambda_n v_n.$$

Then $R_{T,\lambda_n}^M(M(A(x_n), B(x_n)) - \lambda_n G(x_n) = u_n$, so that $Q_n(x_n) \to 0$ implies that $(x_n - u_n) \to 0$. Since x_n is bounded, it has at lest a limit point. Let x^* be such a limit point and assume that the sequence $\{x_{n_k}\}$ converges to x^*. It follows that $\{u_{n_k}\}$ also converges to x^* and $\{v_{n_k}\}$ converges to $-G(x^*)$. For every $(u, v) \in Graph(T)$, by the monotonicity of T, we have $\langle u - u_n, v - v_n \rangle \geq 0$. Let $k \to \infty$, we get $\langle u - x^*, v + G(x^*) \rangle \geq 0$. We see that T is $M(\cdot, \cdot)$-monotone due to Proposition 2.1, this implies $(x^*, -G(x^*)) \in Graph(T)$, that is $-G(x^*) \in T(x^*)$. This completes the proof.
$\square$

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