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CONVERGENCE THEOREMS FOR GENERALIZED EQUILIBRIUM PROBLEMS AND ASYMPTOTICALLY $k-$STRICT PSEUDO-CONTRACTIONS IN HILBERT SPACES

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Abstract. In this paper, we introduce an iterative scheme for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of a finite family of asymptotically k-strict pseudo-contractions in Hilbert spaces. Weak and strong convergence theorems are established for the iterative scheme.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(A1) $F(x, x) = 0, \forall x \in C$;

(A2)F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;

(A3)$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y), \forall x, y, z \in C$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let $A : C \to H$ be a nonlinear mapping. Then, we consider the following generalized equilibrium problem(GEP) which is to find $z \in C$ such that

$$\text{GEP:} F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.1}$$

In the case of $A \equiv 0$, this problem (1.1) reduces to the equilibrium problem(EP), which is to find $z \in C$ such that

$$\text{EP: } F(z, y) \geq 0, \forall y \in C. \tag{1.2}$$

In the case of $F \equiv 0$, this problem (1.1) reduces to the variational inequality problem(VIP), which is to find $z \in C$ such that

$$\text{VIP: } \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.3}$$

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Denote the set of solutions of GEP by Ω, the set of solutions of EP by $EP(F)$ and the set of solutions of VIP by $VI(C,A)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, for instance, [1]. Recall that a mapping $T: C \to C$ is said to be asymptotically k-strictly pseudo-contractive (The class of asymptotically k-strictly pseudo-contractive mappings was first introduced in Hilbert spaces by [7].) if there exists a sequence $\{k_n\} \subset [1,+\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that there exists $k \in [0,1)$ such that

$$\|T^n x - T^n y\|^2 \le k_n^2 \|x-y\|^2 + k\|(I-T^n)x - (I-T^n)y\|^2, \qquad (1.4)$$

for all $x, y \in C$ and $n \in Z^+$.

Note that the class of asymptotically k-strict pseudo-contractions strictly includes the class of asymptotically nonexpansive mappings [4] which are mappings T on C such that

$$\|T^n x - T^n y\| \le k_n \|x-y\|, \quad \forall x, y \in C,$$

where the sequence $\{k_n\}$ in $[1,+\infty)$ satisfies $\lim_{n\to\infty} k_n = 1$. That is, T is asymptotically nonexpansive if and only if T is asymptotically 0-strictly pseudo-contractive.

Recall that a mapping $T: C \to C$ is said to be a $k-$strict pseudo-contraction if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \qquad (1.5)$$

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \le \|x-y\|, \quad \forall x, y \in C.$$

That is, T is nonexpansive if and only if T is 0-strict pseudo-contractive.

The set of fixed points of T is denoted by $F(T)$. Many iterative methods for finding a common element of the set of solutions of the equilibrium problem(EP) or the variational inequality problem(VIP) and the set of fixed points of a nonexpansive mapping have been extensively investigated by many authors(see, e.g.,[2],[5],[9],[11],[12]). However iterative methods for finding a common element of the set of solutions of the generalized equilibrium problem(GEP) and the set of common fixed points of a finite family of asymptotically k-strict pseudo-contractions are rarely studied.

Recently, Takahashi and Takahashi [10] introduced an iterative method for finding a common element of the set of solutions of the generalized equilibrium problem(GEP) and the set of fixed points of a nonexpansive mapping. More precisely, they proved the following theorem.

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F: C \times C \to R$ be a bifunction satisfying (A1)-(A4). Let A be an*

α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S)\bigcap\Omega \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)z_n], & \forall n \in N, \end{cases}$$

where $\{\alpha_n\} \subset [0,1], \{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy

$$0 < c \leq \beta_n \leq d < 1, \quad 0 < a \leq \lambda_n \leq b < 2\alpha,$$

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $z = P_{F(S)\bigcap\Omega}u$, where $P_{F(S)\bigcap\Omega}$ is the metric projection from C onto $F(S)\bigcap\Omega$.

Very recently, X.L.Qin et al.[8] introduced the following algorithm for asymptotically $k-$strict pseudo-contractions.

Let $x_0 \in C$ and $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(0,1)$. The sequence $\{x_n\}_{n=1}^{\infty}$ is generated by the following way:

$$\begin{cases} x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_1 x_0, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_2 x_1, \\ \dots & \\ x_N &= \alpha_{N-1}x_{N-1} + (1-\alpha_{N-1})T_N x_{N-1}, \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_1^2 x_N, \\ \dots & \\ x_{2N} &= \alpha_{2N-1} + (1-\alpha_{2N-1})T_N^2 x_{2N-1}, \\ x_{2N+1} &= \alpha_{2N} + (1-\alpha_{2N})T_1^3 x_{2N}, \\ \dots & \end{cases}$$

Since, for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, ...N\}, h = h(n) \geq 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. Hence the above table can be rewritten in the following compact form:

$$x_n = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)} x_{n-1}, \quad \forall n \geq 0. \tag{1.6}$$

They proved a weak convergence theorem for a finite family of asymptotically $k-$strict pseudo-contractions by algorithm (1.6) in the framework of Hilbert spaces.

Motivated and inspired by these facts, we introduce an iteration scheme for finding a common element of the set of solutions of the generalized equilibrium problem(GEP) and the set of common fixed points of a finite family of asymptotically k-strict pseudo-contractions in Hilbert spaces. We obtain weak and strong convergence theorems.

2. Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by Z^+ and R the sets of positive integers and real numbers, respectively. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \qquad \forall y \in C.$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and satisfies the following property:

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C. \tag{2.1}$$

Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C x \Leftrightarrow \langle x - u, u - y\rangle \geq 0, \forall y \in C. \tag{2.2}$$

A mapping $A : C \to H$ is called inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \forall x, y \in C.$$

Such a mapping A is also called α-inverse-strongly monotone. If A is an α-inverse-strongly monotone mapping of C to H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|(x-y) - \lambda(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda(\lambda - 2\alpha)\|Ax-Ay\|^2. \end{aligned} \tag{2.3}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.
A mapping $T : C \to C$ is said to be semi-compact, if for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$. A mapping $T : C \to C$ is said to be uniformly $L-$Lipschitzian, if there exists some $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in C \text{ and } \forall n \in Z^+.$$

Lemma 2.1. *(*[1], [3]*) Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ into R satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Further, if $T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, then the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e.,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \forall x, y \in H;$$

(3) $F(T_r) = EP(F)$; (4) $EP(F)$ is closed and convex.

Lemma 2.2. *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.3. *([13])Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n \quad \text{for all } n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.4. *([6]) Let H be a real Hilbert space, C a nonempty subset of H and $T : C \to C$ be a k-strictly asymptotically pseudo-contractive mapping. Then T is uniformly L-Lipschitzian.*

Lemma 2.5. *([8])Let H be a real Hilbert space, C a nonempty closed convex subset of H and $T : C \to C$ be a k-strictly asymptotically pseudo-contractive mapping. Then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.6. *([8])Let $N \ge 1$ be an integer. Let, for each $1 \le i \le N$, $T_i : C \to C$ be a s_i-strictly asymptotically pseudocontractive mapping for some $0 \le s_i < 1$ with a sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$, then there exist a constant $s = \max\{s_i : 1 \le i \le N\}$ and a sequence $\{k_n\} = \max\{k_{n,i} : 1 \le i \le N\}$ such that*

$$\|T_i^n x - T_i^n y\|^2 \le k_n^2\|x-y\|^2 + s\|(I - T_i^n)x - (I - T_i^n)y\|^2$$

for all $1 \le i \le N$, where $\lim_{n\to\infty} k_n = 1$.

Lemma 2.7. *([6])Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H and $T : C \to C$ be a k-strictly asymptotically pseudo-contractive mapping for some $0 \le k < 1$ with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and the fixed points set of T is nonempty. Then $(I - T)$ is demiclosed at zero.*

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let $N \ge 1$ be an integer. Let, for each $1 \le i \le N, T_i : C \to C$ be an asymptotically s_i-strictly pseudo-contractive mapping for some $0 \le s_i < 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $s = \max\{s_i : 1 \le i \le N\}$ and*

$\{k_n\} = \max\{k_{n,i} : 1 \le i \le N\}$. *Assume that* $F = \bigcap_{i=1}^{N} F(T_i) \bigcap \Omega \neq \emptyset$. *For any* $x_0 \in C$, *define the following sequence* $\{x_n\}$:

$$\begin{cases} y_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ x_n \in C \text{ such that} \\ F(x_n, y) + \langle Ay_{n-1}, y - x_n\rangle + \frac{1}{\lambda_{n-1}}\langle y - x_n, x_n - y_{n-1}\rangle \ge 0, \forall y \in C, \quad n \in Z^+, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfy*

$B1 : k + \epsilon \le \alpha_n \le 1 - \epsilon$, *for all* $n \ge 0$ *and some* $\epsilon \in (0, 1)$;

$B2 : \lambda_n \in [a, b]$ *for some* $0 < a < b < 2\alpha$. *Then* $\{x_n\}$ *converges weakly to* $z \in F$, *where* $z = \lim_{n\to\infty} P_F x_n$. *Further, if one of* $T_1, T_2, ...T_N$ *is completely continuous, then* $\{x_n\}$ *converges strongly to* $z \in F$. *Again, if one of* $T_1, T_2, ...T_N$ *is semi-compact, then* $\{x_n\}$ *also converges strongly to* $z \in F$.

Proof. Note that x_n can be rewritten as $x_n = T_{\lambda_{n-1}}(y_{n-1} - \lambda_{n-1}Ay_{n-1})$ for each $n \in Z^+$. Let $p \in F$. Since $p = T_{\lambda_{n-1}}(p - \lambda_{n-1}Ap)$, then, by Lemma 2.1 and (2.3), we have $\|x_n - p\| \le \|y_{n-1} - p\|$. Using (3.1) and lemma 2.2, we have

$$\begin{aligned} \|y_{n-1} - p\|^2 &= \|\alpha_{n-1}(x_{n-1} - p) + (1-\alpha_{n-1})(T_{i(n)}^{h(n)}x_{n-1} - p)\|^2 \\ &= \alpha_{n-1}\|x_{n-1} - p\|^2 + (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}x_{n-1} - p\|^2 \\ &- \alpha_{n-1}(1-\alpha_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &\le \alpha_{n-1}\|x_{n-1} - p\|^2 - \alpha_{n-1}(1-\alpha_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &+ (1-\alpha_{n-1})(k_{h(n)}^2\|x_{n-1} - p\|^2 + s\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2) \\ &\le k_{h(n)}^2\|x_{n-1} - p\|^2 - (1-\alpha_{n-1})(\alpha_{n-1} - s)\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &\le \big(1 + (k_{h(n)}^2 - 1)\big)\|x_{n-1} - p\|^2, \end{aligned} \tag{3.2}$$

and so,

$$\begin{aligned} &\|x_n - p\|^2 \le \|y_{n-1} - p\|^2 \le \big(1 + (k_{h(n)}^2 - 1)\big)\|x_{n-1} - p\|^2 \\ &\le \prod_{i=1}^{n}(1 + (k_{h(i)}^2 - 1))\|x_0 - p\|^2 \\ &\le e^{\sum_{i=1}^{n}(k_{h(i)}^2 - 1)}\|x_0 - p\|^2. \end{aligned} \tag{3.3}$$

Since $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$, we have $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ and hence $\sum_{n=0}^{\infty}(k_{h(n)}^2 - 1) < \infty$, then $\{x_n\}$ is bounded. It implies that there exists a constant $M > 0$ such that $\|x_n - p\|^2 \le M$ for all $n \in Z^+$. So,

$$\|x_n - p\|^2 \le \|x_{n-1} - p\|^2 + (k_{h(n)}^2 - 1)M.$$

It follows from lemma 2.3 that $\lim_{n\to\infty} \|x_n - p\|$ exists. By (2.3) and (3.2), we have

$$\begin{aligned}\|x_n - p\|^2 &\leq \|y_{n-1} - p\|^2 + \lambda_{n-1}(\lambda_{n-1} - 2\alpha)\|Ay_{n-1} - Ap\|^2 \\ &\leq k_{h(n)}^2\|x_{n-1} - p\|^2 - (1-\alpha_{n-1})(\alpha_{n-1} - s)\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &+ \lambda_{n-1}(\lambda_{n-1} - 2\alpha)\|Ay_{n-1} - Ap\|^2.\end{aligned}$$

Hence,

$$(1-\alpha_{n-1})(\alpha_{n-1} - s)\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \leq k_{h(n)}^2\|x_{n-1} - p\|^2 - \|x_n - p\|^2$$

and

$$-\lambda_{n-1}(\lambda_{n-1} - 2\alpha)\|Ay_{n-1} - Ap\|^2 \leq k_{h(n)}^2\|x_{n-1} - p\|^2 - \|x_n - p\|^2.$$

It follows from $B1$ and $B2$ that

$$\epsilon^2\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \leq k_{h(n)}^2\|x_{n-1} - p\|^2 - \|x_n - p\|^2 \tag{3.4}$$

and

$$a(2\alpha - b)\|Ay_{n-1} - Ap\|^2 \leq k_{h(n)}^2\|x_{n-1} - p\|^2 - \|x_n - p\|^2. \tag{3.5}$$

Taking the limit as $n \to \infty$ yields that

$$\lim_{n\to\infty} \|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\| = 0 \text{ and } \lim_{n\to\infty} \|Ay_{n-1} - Ap\| = 0. \tag{3.6}$$

Using (3.1), we have

$$\|y_{n-1} - x_{n-1}\| = (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}x_{n-1} - x_{n-1}\| \to 0, \text{ as } n \to \infty. \tag{3.7}$$

Using lemma 2.1 and (3.1), we have

$$\begin{aligned}\|x_n - p\|^2 &= \|T_{\lambda_{n-1}}(y_{n-1} - \lambda_{n-1}Ay_{n-1}) - T_{\lambda_{n-1}}(p - \lambda_{n-1}Ap)\|^2 \\ &\leq \langle (y_{n-1} - \lambda_{n-1}Ay_{n-1}) - (p - \lambda_{n-1}Ap), x_n - p\rangle \\ &= \tfrac{1}{2}(\|(y_{n-1} - \lambda_{n-1}Ay_{n-1}) - (p - \lambda_{n-1}Ap)\|^2 + \|x_n - p\|^2 \\ &- \|(y_{n-1} - \lambda_{n-1}Ay_{n-1}) - (p - \lambda_{n-1}Ap) - (x_n - p)\|^2) \\ &\leq \tfrac{1}{2}(\|y_{n-1} - p\|^2 + \|x_n - p\|^2 - \|(y_{n-1} - x_n) - \lambda_{n-1}(Ay_{n-1} - Ap)\|^2) \\ &= \tfrac{1}{2}(\|y_{n-1} - p\|^2 + \|x_n - p\|^2 - \|y_{n-1} - x_n\|^2 \\ &- \lambda_{n-1}^2\|Ay_{n-1} - Ap\|^2 + 2\lambda_{n-1}\langle y_{n-1} - x_n, Ay_{n-1} - Ap\rangle),\end{aligned}$$

so, we have

$$\|x_n - p\|^2 \leq \|y_{n-1} - p\|^2 - \|y_{n-1} - x_n\|^2 - \lambda_{n-1}^2\|Ay_{n-1} - Ap\|^2 + 2\lambda_{n-1}\langle y_{n-1} - x_n, Ay_{n-1} - Ap\rangle. \tag{3.8}$$

Then, from (3.2) and (3.8), we have

$$\|x_n - p\|^2 \leq k_{h(n)}^2\|x_{n-1} - p\|^2 - \|y_{n-1} - x_n\|^2 - \lambda_{n-1}^2\|Ay_{n-1} - Ap\|^2 + 2\lambda_{n-1}\langle y_{n-1} - x_n, Ay_{n-1} - Ap\rangle.$$

So, we have

$$\|y_{n-1} - x_n\|^2 \leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + (k_{h(n)}^2 - 1)\|x_{n-1} - p\|^2 + 2\lambda_{n-1}\langle y_{n-1} - x_n, Ay_{n-1} - Ap\rangle.$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} \|Ay_{n-1} - Ap\| = 0$, we have

$$\lim_{n\to\infty} \|y_{n-1} - x_n\| = 0. \tag{3.9}$$

It follows from (3.7) and (3.9) that

$$\|x_n - x_{n-1}\| \le \|x_n - y_{n-1}\| + \|y_{n-1} - x_{n-1}\| \to 0, \text{ as } n \to \infty. \tag{3.10}$$

Observe that

$$\begin{aligned} \|x_{n-1} - T_{i(n)}^{h(n)} x_n\| &\le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + \|T_{i(n)}^{h(n)} x_{n-1} - T_{i(n)}^{h(n)} x_n\| \\ &\le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + L\|x_{n-1} - x_n\|. \end{aligned}$$

Thus, combining (3.6) with (3.10) gives

$$\lim_{n\to\infty} \|x_{n-1} - T_{i(n)}^{h(n)} x_n\| = 0. \tag{3.11}$$

On the other hand, it follows from (3.10) that

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0, \quad \forall j = 1, 2, ...N. \tag{3.12}$$

Since, for any positive integer $n > N$, it can be written as $n = (h(n)-1)N + i(n)$, where $i(n) \in \{1, 2, ...N\}$, observe that

$$\begin{aligned} \|x_{n-1} - T_n x_{n-1}\| &\le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + \|T_{i(n)}^{h(n)} x_{n-1} - T_n x_{n-1}\| \\ &\le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + \|T_{i(n)}^{h(n)} x_{n-1} - T_{i(n)} x_{n-1}\| \\ &\le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + L\|T_{i(n)}^{h(n)-1} x_{n-1} - x_{n-1}\| \\ &\le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + L(\|T_{i(n)}^{h(n)-1} x_{n-1} - T_{i(n-N)}^{h(n)-1} x_{n-N}\| \\ &+ \|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_{n-1}\|). \end{aligned} \tag{3.13}$$

Since, for each $n > N$, $n = (h(n) - 1)N + i(n)$, we have $n - N = (h(n) - 1 - 1)N + i(n) = (h(n - N) - 1)N + i(n - N)$, that is $h(n - N) = h(n) - 1$, $i(n - N) = i(n)$. Observe that

$$\|T_{i(n)}^{h(n)-1} x_{n-1} - T_{i(n-N)}^{h(n)-1} x_{n-N}\| = \|T_{i(n)}^{h(n)-1} x_{n-1} - T_{i(n)}^{h(n)-1} x_{n-N}\| \le L\|x_{n-1} - x_{n-N}\| \tag{3.14}$$

and

$$\begin{aligned} \|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\| &\le \|T_{i(n-N)}^{h(n)-1} x_{n-N} - T_{i(n-N)}^{h(n-N)} x_{(n-N)-1}\| \\ &+ \|T_{i(n-N)}^{h(n-N)} x_{(n-N)-1} - x_{(n-N)-1}\| \\ &\le L\|x_{n-N} - x_{(n-N)-1}\| + \|T_{i(n-N)}^{h(n-N)} x_{(n-N)-1} - x_{(n-N)-1}\|. \end{aligned} \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), we can obtain

$$\begin{aligned} \|x_{n-1} - T_n x_{n-1}\| &\le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + L(L\|x_{n-1} - x_{n-N}\| + L\|x_{n-N} - x_{(n-N)-1}\| \\ &+ \|T_{i(n-N)}^{h(n-N)} x_{(n-N)-1} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_{n-1}\|). \end{aligned}$$

It follows from (3.6) and (3.12) that

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_{n-1}\| = 0. \tag{3.16}$$

Notice that

$$\begin{aligned}\|x_n - T_n x_n\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_{n-1}\| + \|T_n x_{n-1} - T_n x_n\| \\ &\leq (1+L)\|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_{n-1}\|.\end{aligned}$$

From (3.10) and (3.16), we can easily see that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.17}$$

On the other hand, from (3.12) and (3.17), we obtain that

$$\begin{aligned}\|x_n - T_{n+j} x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| + \|T_{n+j} x_{n+j} - T_{n+j} x_n\| \\ &\leq (1+L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \to 0, \text{ as } n \to \infty,\end{aligned}$$

for any $j \in \{1, 2, ..., N\}$. This gives that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in \{1, 2, ..., N\}. \tag{3.18}$$

Noticing that $\{x_n\}$ is bounded, we obtain that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$. By Lemma 2.7, we have $w \in \bigcap_{l=1}^{N} F(T_l)$. Let us show $w \in \Omega$. Since $x_n = T_{\lambda_{n-1}}(y_{n-1} - \lambda_{n-1} A y_{n-1})$, for any $y \in C$ we have

$$F(x_n, y) + \langle y - x_n, Ay_{n-1}\rangle + \frac{1}{\lambda_{n-1}}\langle y - x_n, x_n - y_{n-1}\rangle \geq 0.$$

From (A2), we also have

$$\langle y - x_n, Ay_{n-1}\rangle + \frac{1}{\lambda_{n-1}}\langle y - x_n, x_n - y_{n-1}\rangle \geq F(y, x_n). \tag{3.19}$$

Put $z_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.19) we have

$$\begin{aligned}\langle z_t - x_n, Az_t\rangle &\geq \langle z_t - x_n, Az_t\rangle - \langle z_t - x_n, Ay_{n-1}\rangle \\ &- \langle z_t - x_n, \frac{x_n - y_{n-1}}{\lambda_{n-1}}\rangle + F(z_t, x_n) \\ &= \langle z_t - x_n, Az_t - Ax_n\rangle + \langle z_t - x_n, Ax_n - Ay_{n-1}\rangle \\ &- \langle z_t - x_n, \frac{x_n - y_{n-1}}{\lambda_{n-1}}\rangle + F(z_t, x_n).\end{aligned}$$

Since $\|x_n - y_{n-1}\| \to 0$, we have $\|Ax_n - Ay_{n-1}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - x_n, Az_t - Ax_n\rangle \geq 0$. So, replacing n by n_k, from (A4) we have

$$\langle z_t - w, Az_t\rangle \geq F(z_t, w), \text{ as } k \to \infty. \tag{3.20}$$

From (A1),(A4) and (3.20), we also have

$$\begin{aligned}0 &= F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, w) \\ &\leq tF(z_t, y) + (1-t)\langle z_t - w, Az_t\rangle \\ &= tF(z_t, y) + (1-t)t\langle y - w, Az_t\rangle,\end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle y - w, Az_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \le F(w, y) + \langle y - w, Aw \rangle.$$

This implies $w \in \Omega$. Therefore, $w \in F$. Define $u_n = P_F x_n$ for all $n \in Z^+$. Since $w \in F$, we have $\|u_n - x_n\| \le \|w - x_n\|$, then, $\{u_n\}$ is bounded. From (3.3), we have

$$\|x_n - u_{n-1}\|^2 \le \|x_{n-1} - u_{n-1}\|^2 + (k^2_{h(n)} - 1)\|x_{n-1} - u_{n-1}\|^2. \tag{3.21}$$

By $u_n = P_F x_n$ and $u_{n-1} = P_F x_{n-1} \in F$, we have

$$\|u_n - x_n\|^2 \le \|u_{n-1} - x_n\|^2 \le \|u_{n-1} - x_{n-1}\|^2 + (k^2_{h(n)} - 1)M^*,$$

where $M^* = \sup\{\|x_n - u_n\|^2, n \in Z^+\}$. Since $\sum_{n=1}^{\infty}(k^2_{h(n)} - 1) < \infty$, it follows from Lemma 2.3 that $\lim_{n\to\infty} \|u_n - x_n\|$ exists. Again, using (3.21), for all $m \in Z^+$, we have

$$\|x_{n+m} - u_{n-1}\|^2 \le \prod_{i=0}^{m} k^2_{h(n+i)} \|x_{n-1} - u_{n-1}\|^2.$$

From $u_{n+m} = P_F x_{n+m}$ and $u_{n-1} = P_F x_{n-1} \in F$, we have

$$\begin{aligned}
\|u_{n-1} - u_{n+m}\|^2 &\le \|u_{n-1} - x_{n+m}\|^2 - \|u_{n+m} - x_{n+m}\|^2 \\
&\le \prod_{i=0}^{m} k^2_{h(n+i)} \|x_{n-1} - u_{n-1}\|^2 - \|u_{n+m} - x_{n+m}\|^2 \\
&\le e^{\sum_{i=0}^{m}(k^2_{h(n+i)} - 1)} \|x_{n-1} - u_{n-1}\|^2 - \|u_{n+m} - x_{n+m}\|^2.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k^2_{h(n+i)} - 1) < \infty$ and $\lim_{n\to\infty} \|u_n - x_n\|$ exists, we obtain that $\{u_n\}$ is a Cauchy sequence. Since F is closed, we have that $\{u_n\}$ converges strongly to $z \in F$. On the other hand, noticing that $w \in F$ and $u_n = P_F x_n$, we have

$$\langle x_{n_k} - u_{n_k}, u_{n_k} - w \rangle \ge 0.$$

Letting $k \to \infty$, we have

$$\langle w - z, z - w \rangle \ge 0.$$

Hence, $w = z$. Therefore, $\{x_n\}$ converges weakly to $z \in F$, where $z = \lim_{n\to\infty} P_F x_n$.

If one of $T_1, T_2, ..., T_N$ is completely continuous, without loss of generality, we may assume that $T_l x_n \to z, l \in \{1, 2, ..., N\}$ as $n \to \infty$. By (3.18), we have $x_n \to z$.

If one of $T_1, T_2, ..., T_N$ is semi-compact, then, by (3.18), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $q \in C$. By using the same argument as the proof of $w \in F$, we can obtain $q \in F$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, then $\{x_n\}$ converges strongly to q. Since $\{x_n\}$ converges weakly to $z \in F$, then we have $q = z$, where $z = \lim_{n\to\infty} P_F x_n$. □

Remark 1. Taking $s_i = 0$ in theorem 3.1 for each $i \in \{1, 2, ..., N\}$, we can obtain weak and strong convergence theorems for the common element of the set of solutions of the generalized equilibrium problem(GEP) and the set of common fixed points of a finite family of asymptotically nonexpansive mappings in Hilbert spaces.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N, T_i : C \to C$ be a s_i-strictly pseudo-contractive mapping for some $0 \leq s_i < 1$. Let $s = \max\{s_i : 1 \leq i \leq N\}$. Assume that $F = \bigcap_{i=1}^{N} F(T_i) \bigcap \Omega \neq \emptyset$. For any $x_0 \in C$, define the following sequence $\{x_n\}$:*

$$\begin{cases} y_{n-1} = \alpha_{n-1}x_{n-1} + (1 - \alpha_{n-1})T_{i(n)}x_{n-1}, \\ x_n \in C \text{ such that} \\ F(x_n, y) + \langle Ay_{n-1}, y - x_n\rangle + \frac{1}{\lambda_{n-1}}\langle y - x_n, x_n - y_{n-1}\rangle \geq 0, \forall y \in C, \quad n \in Z^+, \end{cases}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy

$B1 : k + \epsilon \leq \alpha_n \leq 1 - \epsilon$, for all $n \geq 0$ and some $\epsilon \in (0, 1)$;

$B2 : \lambda_n \in [a, b]$ for some $0 < a < b < 2\alpha$. Then $\{x_n\}$ converges weakly to $z \in F$, where $z = \lim_{n\to\infty} P_F x_n$. Further, if one of $T_1, T_2, ...T_N$ is completely continuous, then $\{x_n\}$ converges strongly to $z \in F$. Again, if one of $T_1, T_2, ...T_N$ is semi-compact, then $\{x_n\}$ also converges strongly to $z \in F$.

Proof. Taking $k_{n,i} \equiv 1$ for each $n \geq 0$ and $i \in \{1, 2, ..., N\}$ in theorem 3.1, we can easily obtain the desired result. □

Remark 2. Taking $s_i = 0$ in Corollary 3.3 for each $i \in \{1, 2, ..., N\}$, we can obtain weak and strong convergence theorems for the common element of the set of solutions of the generalized equilibrium problem(GEP) and the set of common fixed points of a finite family of nonexpansive mappings in Hilbert spaces. Corollary 3.3 generalize the result of Takahashi and Takahashi [10] from a nonexpansive mapping to a finite family of s_i-strict pseudo-contractions.

References

[1] E. Blum, and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud. **63** (1994),123-145.

[2] J. Chen, L. Zhang, and T. Fan, *Viscosity approximation methods for nonexpansive mappings and monotone mappings*, J. Math. Anal. Appl. **334** (2007), 1450-1461.

[3] P.L. Combettes,and A. Hirstoaga, *Equilibrium programming in Hilbert spaces*,J. Nonlinear Convex Anal. **6** (2005), 117-136.

[4] K. Goebel, and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171-174.

[5] H. Iiduka, and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal. **61** (2005), 341-350.

[6] T.H. Kim, and H.K. Xu, *Convergence of the modified Manns iteration method for asymptotically strict pseudo-contractions*, Nonlinear Anal.**68** (2008),2828-2836.

[7] L. Qihou, *Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings*, Nonlinear Anal. **26** (1996), 1835-1842.

[8] X. Qin, Y. Cho, S.Kang, and M. Shang, *A hybrid iterative scheme for asymptotically k-strict pseudo-contractions in Hilbert spaces*, Nonlinear Anal.**70**(2009),1902-1911.

[9] A. Tada, and W. Takahashi, *Strong convergence theorem for an equilibrium problem and a nonexpansive mapping*, J. Optim. Theory Appl. **133** (2007),359-370.

[10] S. Takahashi, and W. Takahashi, *Strong convergence theorems for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*,Nonlinear Anal. **69** (2008), 1025-1033.

[11] S. Takahashi, and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (2007), 506-515.

[12] W. Takahashi, and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417-428.

[13] K.K. Tan, and H.K. Xu, *Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301-308.

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