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FUNCTIONS AND DIFFERENTIAL OPERATORS IN THE DUAL REDUCED QUATERNION FIELD

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Abstract. We research properties of ternary numbers and hyperholomorphic functions with values in $\mathbb{C}(2)$. We represent reduced quaternion numbers and obtain some propertries in dual reduced quaternion systems in view of Clifford analysis. Moreover, we obtain Cauchy theorems with respect to dual reduced quaternions.

1. Introduction

The skew field $\mathcal{T}$ of quaternions

$$p = \sum_{j=0}^{3} e_j x_j \ (x_j \in \mathbb{R}; j = 0, 1, 2, 3)$$

is a four dimensional non-commutative $\mathbb{R}$-field generated by four base elements

$$e_0 = id., \ e_1 = \sqrt{-1}, \ e_2, \ e_3$$

with the following non-commutative multiplication rule:

$$e_1e_2 = -e_2e_1 = e_3, \ e_2e_3 = -e_3e_2 = e_1, \ e_3e_1 = -e_1e_3 = e_2.$$

Let $\bar{e_1} = -e_1, \ \bar{e_2} = -e_2, \ \bar{e_3} = -e_3$. The absolute value $\|p\| = \sqrt{\sum_{j=0}^{3} |x_j|^2}$ coincides with the usual norm of $p \in \mathcal{T}$. Every quaternion p has an unique representation: $p = p_1 + p_2e_2$, where $p_1 = x_0 + e_1x_1$ and $p_2 = x_2 + e_1x_3$ are complex numbers. Then we can identify $\mathcal{T}$ with $\mathbb{C}^2$. A $\mathcal{T}$- valued function $f = \sum_{j=0}^{3} e_ju_j = f_1 + f_2e_2$ $(f_1 = u_0 + e_1u_1, \ f_2 = u_2 + e_1u_3)$ defined in a subset Ω of $\mathbb{C}^2$ and valued in $\mathcal{T}$:

$$f = f_1 + f_2e_2 : p = (p_1, p_2) \in \Omega \to f(p) = f_1(p_1, p_2) + f_2(p_1, p_2)e_2 \in \mathcal{T}.$$

Deavours [1] researched some properties of the quaternion calculus. Naser [10] obtained some theorems with respect to hyperholomorphic functions. We

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[6] regenerated quaternions and Clifford analysis. Nôno [11-13] obtained several results for hyperholomorphic functions and domains of hyperholomorphy. Gürsey and Tze [3] researched complex and quaternionic analyticity in Chiral and Gauge theories. Recently, Kula and Yayli [7] researched dual split quaternions and screw motion in Minkowski three space. Gürlebeck and Morais [2] calculated monogenic primitives. And Hengartner and Leutwiler [4] researched hyperholomorphic functions in the three dimensional real space. We [8, 9] obtained properties of hyperholomorphic functions in Clifford analysis and hyperholomorphic functions and hyper-conjugate harmonic functions of octonion variables. Also, we [5] researched some properties of hyperholomorphic functions on dual ternary numbers. In this paper, we investigate properties of ternary numbers and hyperholomorphic functions with values in $\mathbb{C}(2)$, and represent reduced quaternion numbers. We research some properties in dual reduced quaternions.

2. Notations on reduced quaternions and dual numbers

The skew field $T \cong \mathbb{R}^3$ of ternary numbers $a = \sum_{j=0}^{2} e_j x_j$ is a three dimensional non-commutative real field generated by bases $e_0 = id.$, $e_1 = \sqrt{-1}$, e_2, which satisfy the following identity:

$$e_1^2 = e_2^2 = -1.$$

Every element $a = (x_0, x_1, x_2) \in \mathbb{R}^3$ can be identified with the reduced quaternion number. The conjugate number a^* of $a \in T$ is $a^* = x_0 - e_1 x_1 - e_2 x_2$ and the corresponding norm is $|a| = \sqrt{aa^*} = \sqrt{\sum_{j=0}^{2} {x_j}^2}$. The algebra

$$\mathbb{C}(2) = \{a = \sum_{j=0}^{2} e_j x_j \mid x_j \in \mathbb{R}; j = 0, 1, 2\} \subset \mathcal{T}$$

is a non-commutative subalgebra of $\mathbb{R}^3$. We let

$$a = \sum_{j=0}^{2} e_j x_j, \; b = \sum_{j=0}^{2} e_j y_j.$$

A dual reduced quaternion number $z \in \mathbb{C}^3$ has the form $a + \varepsilon b$ and the dual identity ε is the dual symbol subjected to the rules

$$\varepsilon \neq 0, \; 0\varepsilon = 0, \; 1\varepsilon = \varepsilon 1 = \varepsilon, \; \varepsilon^2 = 0.$$

We use the altered Fueter variables:

$$z_1 := -\frac{1}{2} e_1 x_0 + x_1, \; z_2 := -\frac{1}{2} e_2 x_0 + x_2, w_1 := -\frac{1}{2} e_1 y_0 + y_1, \; w_2 := -\frac{1}{2} e_2 y_0 + y_2,$$

$$\bar{z_1} = \frac{1}{2} e_1 x_0 + x_1, \; \bar{z_2} = \frac{1}{2} e_2 x_0 + x_2, \bar{w_1} = \frac{1}{2} e_1 y_0 + y_1, \; \bar{w_2} = \frac{1}{2} e_2 y_0 + y_2.$$

Then, we have the following:

$$a=\sum_{j=1}^{2} e_j z_j,\ b=\sum_{j=1}^{2} e_j w_j.$$

Therefore, the dual reduced quaternion z and the conjugate number z^* are written as

$$\begin{aligned} z &= x_0+e_1x_1+e_2x_2+\varepsilon(y_o+e_1y_1+e_2y_2) \\ &= e_1z_1+e_2z_2+\varepsilon(e_1w_1+e_2w_2) \\ &= a+\varepsilon b, \\ z^* &= a^*+\varepsilon b^*, \end{aligned}$$

where $a=e_1z_1+e_2z_2,\ b=e_1w_1+e_2w_2,\ a^*=\bar{e_1}\bar{z_1}+\bar{e_2}\bar{z_2},\ b^*=\bar{e_1}\bar{w_1}+\bar{e_2}\bar{w_2}$. Let $\Omega_1,\Omega_2\subset\mathbb{R}^3$ be bounded domains. Consider functions

$$F:\Omega_1\times\Omega_2\to\mathbb{R}^3\times\mathbb{R}^3\cong T\times T$$

for $F(z)=F(a,b)=F(z_1,z_2,w_1,w_2)=F(x_0,x_1,x_2,y_0,y_1,y_2)=U+\varepsilon V$, where $U=\sum_{j=0}^{2} e_ju_j$ and $V=\sum_{j=0}^{2} e_jv_j$. We let

$$f_1=-\frac{1}{2}e_1u_0+u_1,\ f_2=-\frac{1}{2}e_2u_0+u_2,$$

$$g_1=-\frac{1}{2}e_1v_0+v_1,\ g_2=-\frac{1}{2}e_2v_0+v_2,$$

where u_j and v_j $(j=0,1,2)$ are real valued functions for $a=(x_0,x_1,x_2)$ and $b=(y_0,y_1,y_2)$ on $\mathbb{R}^3$. Then,

$$\begin{aligned} F(a,b) &= u_o(x_0,x_1,x_2)+e_1u_1(x_0,x_1,x_2)+e_2u_2(x_0,x_1,x_2) \\ &\quad +\varepsilon(v_0(y_0,y_1,y_2)+e_1v_1(y_0,y_1,y_2)+e_2v_2(y_0,y_1,y_2)) \\ &= e_1f_1(x_0,x_1,x_2)+e_2f_2(x_0,x_1,x_2)+\varepsilon(e_1g_1(y_0,y_1,y_2)+e_2g_2(y_0,y_1,y_2)) \\ &=: f(a)+\varepsilon g(b). \end{aligned}$$

We consider that the numbers y_0,y_1 and y_2 are related numbers with respect to the numbers x_0,x_1 and x_2, respectively. And also, we consider the functions v_0,v_1 and v_2 are related functions with respect to the functions u_0,u_1 and u_2, respectively. We consider the following differential operators:

$$\frac{\partial}{\partial Z_1} := \frac{1}{2}e_1\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1},\ \frac{\partial}{\partial Z_2} := \frac{1}{2}e_2\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_2},$$
$$\frac{\partial}{\partial W_1} := \frac{1}{2}e_1\frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_1},\ \frac{\partial}{\partial W_2} := \frac{1}{2}e_2\frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_2},$$
$$\frac{\partial}{\partial \bar{Z}_1} = -\frac{1}{2}e_1\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1},\ \frac{\partial}{\partial \bar{Z}_2} = -\frac{1}{2}e_2\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_2},$$
$$\frac{\partial}{\partial \bar{W}_1} = -\frac{1}{2}e_1\frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_1},\ \frac{\partial}{\partial \bar{W}_2} = -\frac{1}{2}e_2\frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_2}.$$

Then, we have

$$\begin{aligned} D &:= \frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} + \varepsilon(\frac{\partial}{\partial y_0} - e_1\frac{\partial}{\partial y_1} - e_2\frac{\partial}{\partial y_2}), \\ &= \bar{e_1}\frac{\partial}{\partial Z_1} + \bar{e_2}\frac{\partial}{\partial Z_2} + \varepsilon(\bar{e_1}\frac{\partial}{\partial W_1} + \bar{e_2}\frac{\partial}{\partial W_2}) =: \frac{\partial}{\partial X} + \varepsilon\frac{\partial}{\partial Y}, \\ D^* &= e_1\frac{\partial}{\partial \bar{Z}_1} + e_2\frac{\partial}{\partial \bar{Z}_2} + \varepsilon(e_1\frac{\partial}{\partial \bar{W}_1} + e_2\frac{\partial}{\partial \bar{W}_2}) =: \frac{\partial}{\partial X^*} + \varepsilon\frac{\partial}{\partial Y^*}. \end{aligned}$$

For product algebras of reduced quaternions and the corresponding differential operators, we add a notation:

$$e_1e_2 = -e_2e_1$$

in the reduced quaternion field. The followings

$$\frac{\partial}{\partial X^*}F,\ \frac{\partial^2}{\partial X\partial X^*}F,\ D^*F,\ DD^*F$$

are differential operators for $F(a,b) = f(a) + \varepsilon g(b)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ with values in $\mathcal{T} \times \mathcal{T}$.

Definition 1. Let Ω_1 be an open subset of $\mathbf{R}^3$. A function $f(a) = \sum_{j=0}^{2} e_j u_j(a)$ is said to be hyperholomorphic in Ω_1 if
(a) $u_j(a)$ $(j = 0, 1, 2)$ are continuously differentiable in Ω_1,
(b) $\frac{\partial}{\partial X^*}f(a) = 0$ in Ω_1.

Equation (b) of Definition 1 operate on $f(a)$ with values in $\mathcal{T}$ as follows:

$$\begin{aligned} \frac{\partial}{\partial X^*}f &= (e_1\frac{\partial}{\partial \bar{Z}_1} + e_2\frac{\partial}{\partial \bar{Z}_2})(e_1f_1 + e_2f_2) \\ &= -(\frac{\partial f_1}{\partial \bar{Z}_1} + \frac{\partial f_2}{\partial \bar{Z}_2}) + e_1e_2(\frac{\partial f_2}{\partial Z_1} - \frac{\partial f_1}{\partial Z_2}). \end{aligned}$$

Thus, the above equation (b) of Definition 1 for $f(a)$ is equivalent to the following system of equations:

$$\frac{\partial f_1}{\partial \bar{Z}_1} = -\frac{\partial f_2}{\partial \bar{Z}_2},\ \frac{\partial f_2}{\partial Z_1} = \frac{\partial f_1}{\partial Z_2}. \tag{1}$$

Definition 2. The function $f = e_1 f_1 + e_2 f_2$ is harmonic on Ω_1 if each component f_1 and f_2 is harmonic on Ω_1.

Definition 3. The function f is harmonic on Ω_1 with respect to Z_1 and Z_2 if the Laplacian $\Delta_{Z_1 Z_2} = \frac{\partial^2 f}{\partial Z_1 \partial \bar{Z}_1} + \frac{\partial^2 f}{\partial Z_2 \partial \bar{Z}_2} = 0$ on Ω_1.

Theorem 2.1. *If the function f is hyperholomorphic on Ω_1, then f is harmonic on Ω_1.*

Proof. Under the assumption $\frac{\partial f}{\partial X^*} = 0$, we have

$$\begin{aligned} \frac{\partial^2 f}{\partial X \partial X^*} &= (\bar{e_1}\frac{\partial f}{\partial Z_1} + \bar{e_2}\frac{\partial f}{\partial Z_2})(e_1\frac{\partial f}{\partial \bar{Z}_1} + e_2\frac{\partial f}{\partial \bar{Z}_2}) \\ &= \frac{\partial^2 f}{\partial Z_1 \partial \bar{Z}_1} + \frac{\partial^2 f}{\partial Z_2 \partial \bar{Z}_2} = \Delta_{Z_1 Z_2} f = 0. \end{aligned}$$

Therefore, f is harmonic with respect to Z_1 and Z_2 on Ω_1. □

Theorem 2.2. *Let $f(a)$ be a hyperholomorphic function in a domain Ω_1 of $\mathbf{R}^3$ and let*

$$\kappa_1 = (e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2)e_2 - (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1).$$

Then for any domain $G \subset \Omega_1$ with smooth boundary bG, we have

$$\int_{bG} \kappa_1 f(a) = 0,$$

where $\kappa_1 f(a)$ is the product of ternary numbers of the form κ_1 and the function $f(a)$ with values in $\mathcal{T}$.

Proof. By the rule of the multiplication of ternary numbers, we have

$$\begin{aligned} \kappa_1 f &= ((e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2)e_2 - (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1))(e_1 f_1 + e_2 f_2) \\ &= (e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2)e_2 e_1 f_1 + (e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2)e_2 e_2 f_2 \\ &\quad -(dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1)e_1 f_1 - (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1)e_2 f_2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(\kappa_1 f) &= (\frac{\partial}{dZ_1}dZ_1 + \frac{\partial}{dZ_2}dZ_2 + \frac{\partial}{d\bar{Z}_1}d\bar{Z}_1 + \frac{\partial}{d\bar{Z}_2}d\bar{Z}_2)(\kappa_1 f) \\ &= e_2\frac{\partial f_1}{d\bar{Z}_1}dZ_1\wedge dZ_2\wedge d\bar{Z}_1\wedge d\bar{Z}_2 - e_1\frac{\partial f_2}{dZ_1}dZ_1\wedge dZ_2\wedge d\bar{Z}_1\wedge d\bar{Z}_2 \\ &\quad + e_1\frac{\partial f_1}{dZ_2}dZ_1\wedge dZ_2\wedge d\bar{Z}_1\wedge d\bar{Z}_2 + e_2\frac{\partial f_2}{d\bar{Z}_2}dZ_1\wedge dZ_2\wedge d\bar{Z}_1\wedge d\bar{Z}_2. \end{aligned}$$

By Equation (1), we have $d(\kappa_1 f(a)) = 0$. By Stoke's theorem, we have

$$\int_{bG}\kappa_1 f(a) = \int_G d(\kappa_1 f(a)) = 0.$$

□

Definition 4. Let Ω be an open subset of $\mathbf{C}^3$. A function $F(z) = f(a)+\varepsilon g(b) = \sum_{j=0}^2 e_j u_j(a) + \varepsilon(\sum_{j=0}^2 e_j v_j(b))$ is said to be hyperholomorphic in Ω if
(a) $u_j(a)$ and $v_j(b)(j = 0, 1, 2)$ are continuously differentiable in Ω,
(b) $D^*F(z) = 0$ in Ω.

Equation (b) of Definition 2 operate on $F(z)$ with values in $\mathcal{T}\times\mathcal{T}$ as follows:

$$\begin{aligned} D^*F &= -(\frac{\partial f_1}{\partial\bar{Z}_1} + \frac{\partial f_2}{\partial\bar{Z}_2}) + e_1e_2(\frac{\partial f_2}{\partial Z_1} - \frac{\partial f_1}{\partial Z_2}) \\ &\quad +\varepsilon(-(\frac{\partial g_1}{\partial\bar{Z}_1} + \frac{\partial g_2}{\partial\bar{Z}_2} + \frac{\partial f_1}{\partial\bar{W}_1} + \frac{\partial f_2}{\partial\bar{W}_2}) + e_1e_2(\frac{\partial g_2}{\partial Z_1} + \frac{\partial f_2}{\partial W_1} - \frac{\partial g_1}{\partial Z_2} - \frac{\partial f_1}{\partial W_2})). \end{aligned}$$

The above equation (b) of Definition 2 for $F(z)$ is equivalent to the following system of equations:

$$\begin{gathered} \frac{\partial f_1}{\partial\bar{Z}_1} = -\frac{\partial f_2}{\partial\bar{Z}_2},\ \frac{\partial f_2}{\partial Z_1} = \frac{\partial f_1}{\partial Z_2}, \\ \frac{\partial g_1}{\partial\bar{Z}_1} + \frac{\partial g_2}{\partial\bar{Z}_2} + \frac{\partial f_1}{\partial\bar{W}_1} + \frac{\partial f_2}{\partial\bar{W}_2} = 0,\ \frac{\partial g_2}{\partial Z_1} + \frac{\partial f_2}{\partial W_1} = \frac{\partial g_1}{\partial Z_2} + \frac{\partial f_1}{\partial W_2}. \end{gathered} \tag{2}$$

We consider the following condition of integrability:

$$\frac{\partial f_1}{\partial\bar{W}_1} + \frac{\partial f_2}{\partial\bar{W}_2} = 0,\ \frac{\partial f_2}{\partial W_1} = \frac{\partial f_1}{\partial W_2}. \tag{3}$$

Definition 5. The function $F(z) = e_1f_1(a) + e_2f_2(a) + \varepsilon(e_1g_1(b) + e_2g_2(b))$ is harmonic on $\mathcal{T}\times\mathcal{T}$ if each component f_1, f_2, g_1 and g_2 is harmonic on $\mathcal{T}\times\mathcal{T}$.

Definition 6. The function $F(z)$ is harmonic on $\Omega\subset\mathbf{C}^3\approx\mathbf{R}^3\times\mathbf{R}^3$ with respect to Z_1, Z_2, W_1 and W_2 if $\Delta_S = \sum_{j=1}^2\frac{\partial^2}{\partial S_j\partial S_j^*} = 0$ on Ω, where $\frac{\partial}{\partial S_j} := \frac{\partial}{\partial Z_j} + \varepsilon\frac{\partial}{\partial W_j}$ and $\frac{\partial}{\partial S_j^*} = \frac{\partial}{\partial\bar{Z}_j} + \varepsilon\frac{\partial}{\partial\bar{W}_j}$.

Theorem 2.3. *If the function $F(z)$ is hyperholomorphic on $\Omega \subset \mathbf{C}^3$, then $F(z)$ is harmonic on Ω.*

Proof. Under the assumption $D^*F(z) = 0$, we find

$$\begin{aligned} DD^*f &= (\bar{e_1}\frac{\partial f}{\partial Z_1} + \bar{e_2}\frac{\partial f}{\partial Z_2} + \varepsilon(\bar{e_1}\frac{\partial f}{\partial W_1} + \bar{e_2}\frac{\partial f}{\partial W_2}))(e_1\frac{\partial f}{\partial \bar{Z_1}} + e_2\frac{\partial f}{\partial \bar{Z_2}} + \varepsilon(e_1\frac{\partial f}{\partial \bar{W_1}} + e_2\frac{\partial f}{\partial \bar{W_2}})) \\ &= \frac{\partial^2 f}{\partial Z_1 \partial \bar{Z_1}} + \frac{\partial^2 f}{\partial Z_2 \partial \bar{Z_2}} + \varepsilon(\frac{\partial^2 f}{\partial Z_1 \partial \bar{W_1}} + \frac{\partial^2 f}{\partial Z_2 \partial \bar{W_2}} + \frac{\partial^2 f}{\partial W_1 \partial \bar{Z_1}} + \frac{\partial^2 f}{\partial W_2 \partial \bar{Z_2}}) \\ &= \sum_{j=1}^{2} \frac{\partial^2 f}{\partial S_j \partial S_j^*} = \Delta_S f = 0. \end{aligned}$$

It follows that F is harmonic with respect to Z_1, Z_2, W_1 and W_2 on Ω. □

Theorem 2.4. *Under the condition of integrability* (3), *let $F(z)$ be a hyperholomorphic function in a domain Ω of $\mathbf{C}^3$ and*

$$\begin{aligned} &\kappa_2 = (e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 \\ &+dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\ &+\varepsilon((dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge e_1 dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 \\ &+dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1). \end{aligned}$$

Then for any domain $G \subset \Omega$ with smooth boundary bG,

$$\int_{bG} \kappa_2 F(z) = 0,$$

where $\kappa_2 F(z)$ is the product of ternary numbers of the form κ_2 and the function $F(z)$ with values in $\mathcal{T} \times \mathcal{T}$.

Proof. By the rule of the multiplication of ternary numbers, we have

$$
\begin{aligned}
& \kappa_2 F \\
= \;& ((e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 \\
& + dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\
& + \varepsilon((dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge e_1 dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 \\
& + dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1)(e_1 f_1 + e_2 f_2 + \varepsilon(e_1 g_1 + e_2 g_2)) \\
= \;& (e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 e_1 f_1 \\
& + (e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 e_2 f_2 \\
& + (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_1 f_1 \\
& + (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 f_2 \\
& + \varepsilon((e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 e_1 g_1 \\
& + (e_1 dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 e_2 g_2 \\
& + (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_1 g_1 \\
& + (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 g_2 \\
& + (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge e_1 dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 e_1 f_1 \\
& + (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge e_1 dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2)e_2 e_2 f_2 \\
& + (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1)e_1 f_1 \\
& + (dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1)e_2 f_2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& d(\kappa_2 F) \\
= \ & (\frac{\partial}{\partial Z_1} dZ_1 + \frac{\partial}{\partial Z_2} dZ_2 + \frac{\partial}{\partial \bar{Z}_1} d\bar{Z}_1 + \frac{\partial}{\partial \bar{Z}_2} d\bar{Z}_2 \\
& + \varepsilon(\frac{\partial}{\partial W_1} dW_1 + \frac{\partial}{\partial W_2} dW_2 + \frac{\partial}{\partial \bar{W}_1} d\bar{W}_1 + \frac{\partial}{\partial \bar{W}_2} d\bar{W}_2))(\kappa_2 F) \\
= \ & e_2 \frac{\partial f_1}{\partial \bar{Z}_1} dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\
& - e_1 \frac{\partial f_2}{\partial Z_1} dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\
& + e_1 \frac{\partial f_1}{\partial Z_2} dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\
& + e_2 \frac{\partial f_2}{\partial \bar{Z}_2} dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\
& + \varepsilon(e_2 \frac{\partial g_1}{\partial \bar{Z}_1} dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\
& - e_1 \frac{\partial g_2}{\partial Z_1} dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\
& + e_1 \frac{\partial g_1}{\partial Z_2} dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2 \\
& + e_2 \frac{\partial g_2}{\partial \bar{Z}_2} dZ_1 \wedge dZ_2 \wedge d\bar{Z}_1 \wedge d\bar{Z}_2 \wedge dW_1 \wedge dW_2 \wedge d\bar{W}_1 \wedge d\bar{W}_2).
\end{aligned}
$$

By Equations (2) and (3), we have $d(\kappa_2 F(z)) = 0$. By Stoke's theorem, we have

$$\int_{bG} \kappa_2 F(z) = \int_G d(\kappa_2 F(z)) = 0.$$

□

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