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NORMAL FUZZY PROBABILITY FOR TRAPEZOIDAL FUZZY SETS

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Abstract. A fuzzy set A defined on a probability space $(\Omega, \mathfrak{F}, P)$ is called a fuzzy event. Zadeh defines the probability of the fuzzy event A using the probability P. We define the normal fuzzy probability on $\mathbb{R}$ using the normal distribution. We calculate the normal fuzzy probability for generalized trapezoidal fuzzy sets and give some examples.

1. Introduction

Four operations are based on the Zadeh's extension principle([6]). Zadeh defines the probability of fuzzy event as follows.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, where Ω denotes the sample space, $\mathfrak{F}$ the $\sigma-$algebra on Ω, and P a probability measure. A fuzzy set A on Ω is called a fuzzy event. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([7]) as

$$\widetilde{P}(A) = \int_\Omega \mu_A(\omega)\, dP(\omega), \qquad \mu_A(\omega) : \Omega \to [0,1].$$

We defined the normal fuzzy probability using the normal distribution and calculated the normal fuzzy probability for quadratic fuzzy number([1]). Then we had the explicit formula for the normal fuzzy probability for trigonometric fuzzy number. Furthermore we calculated the normal fuzzy probability for trigonometric fuzzy numbers driven by the above four operations([2]). We define the generalized trapezoidal fuzzy set and calculate four operations of two generalized trapezoidal fuzzy sets([3]).

In section 3, we correct the errors in the results for the normal fuzzy probability for generalized triangular fuzzy sets([4]). In section 4, we calculate the normal fuzzy probability for generalized trapezoidal fuzzy sets and give some examples.

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2. Preliminaries

Let X be a random variable defined on a probability space $(\Omega, \mathfrak{F}, P)$ and g be a real-valued Borel-measurable function on $\mathbb{R}$. Then $g(X)$ is also a random variable.

Definition 2.1. We say that the mathematical expectation of $g(X)$ exists if

$$E[g(X)] = \int_{\Omega} g(X(\omega))\, dP(\omega) = \int_{\Omega} g(X)\, dP$$

is finite.

We note that a random variable X defined on $(\Omega, \mathfrak{F}, P)$ induces a measure P_X on a Borel set $B \in \mathfrak{B}$ defined by the relation $P_X(B) = P\{X^{-1}(B)\}$. Then P_X becomes a probability measure on $\mathfrak{B}$ and is called the probability distribution of X. If $E[g(X)]$ exists, then g is also integrable over $\mathbb{R}$ with respect to P_X. Moreover, the relation

$$\int_{\Omega} g(X) dP = \int_{\mathbb{R}} g(t) dP_X(t)$$

holds. If g is continuous on $\mathbb{R}$ and $E[g(X)]$ exists, we have

$$\int_{\Omega} g(X) dP = \int_{\mathbb{R}} g dP_X = \int_{-\infty}^{\infty} g(x) dF(x),$$

where F is the distribution function corresponding to P_X, and the last integral is a Riemann-Stieltjes integral.

Let F be absolutely continuous on $\mathbb{R}$ with probability density function $f(x) = F'(x)$. Then $E[g(X)]$ exists if and only if the integral $\int_{-\infty}^{\infty} |g(x)| f(x) dx$ is finite and in that case we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) dF(x).$$

Example 2.2. *Let the random variable X have the normal distribution given by the probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

where $\sigma^2 > 0$ and $m \in \mathbb{R}$. Then $E[|X|^\gamma] < \infty$ for every $\gamma > 0$, and we have

$$E[X] = m \quad \text{and} \quad E[(X-m)^2] = \sigma^2.$$

The induced measure P_X is called the normal distribution.

A fuzzy set A on Ω is called a $fuzzy$ $event$. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([7]) as

$$\widetilde{P}(A) = \int_{\Omega} \mu_A(\omega)\, dP(\omega), \quad \mu_A(\omega) : \Omega \to [0,1].$$

Definition 2.3. The normal fuzzy probability $\widetilde{P}(A)$ of a fuzzy set A on $\mathbb{R}$ is defined by

$$\widetilde{P}(A) = \int_{\mathbb{R}} \mu_A(x)\, dP_X,$$

where P_X is the normal distribution.

Definition 2.4. The addition, subtraction, multiplication, and division of two fuzzy numbers are defined as

1. Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

2. Subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

3. Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

4. Division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

Definition 2.5. A fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \le x \\ \frac{c(x-a_1)}{a_2-a_1}, & a_1 \le x < a_2 \\ c, & a_2 \le x < a_3 \\ \frac{c(a_4-x)}{a_4-a_3}, & a_3 \le x < a_4 \end{cases}$$

where $a_i \in \mathbb{R}, i = 1,2,3,4$ and $0 < c < 1$, is called *a generalized trapezoidal fuzzy set* and will be denoted by $A = (a_1, a_2, c, a_3, a_4)$.

Theorem 2.6. *([3]) Let $A = (a_1, a_2, m_1, a_3, a_4)$ and $B = (b_1, b_2, m_2, b_3, b_4)$, where $a_i, b_i \in \mathbb{R}, i = 1,2,3,4, 0 < m_1 \le m_2 < 1$ and $\mu_B(x) \ge m_1$ in $[p, r]$. Then we have the following.*

1. *Addition : The membership function $\mu_{A(+)B}(z)$ is*

$$\begin{cases} 0, & z < a_1 + b_1, a_4 + b_4 \le z \\ \frac{m_1 m_2 (z - a_1 - b_1)}{m_2(a_2 - a_1) + m_1(b_2 - b_1)}, & a_1 + b_1 \le z < a_2 + b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2} \end{cases}$$

$$\begin{cases} m_1, & a_2+b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\le z \\ & \quad < a_3+b_4-(b_4-b_3)\cdot\frac{m_1}{m_2} \\ \frac{m_1m_2(a_4+b_4-z)}{m_2(a_4-a_3)+m_1(b_4-b_3)}, & a_3+b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\le z< a_4+b_4 \end{cases}$$

i.e. $A(+)B$ *is a generalized trapezoidal fuzzy set.*

2. *Subtraction : The membership function* $\mu_{A(-)B}(z)$ *is*

$$\begin{cases} 0, & z<a_1-b_4, a_4-b_1\le z \\ \frac{m_1m_2(z+b_4-a_1)}{m_2(a_2-a_1)+m_1(b_4-b_3)}, & a_1-b_4\le z<a_2-\big(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\big) \\ m_1, & a_2-\big(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\big)\le z \\ & \quad < a_3-\big(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\big) \\ \frac{m_1m_2(a_4-b_1-z)}{m_2(a_4-a_3)+m_1(b_2-b_1)}, & a_3-\big(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\big)\le z<a_4-b_1 \end{cases}$$

i.e. $A(-)B$ *is a generalized trapezoidal fuzzy set.*

3. *Multiplication : The membership function* $\mu_{A(\cdot)B}(z)$ *is*

$$\begin{cases} 0, & z<a_1b_1, a_4b_4\le z \\ \frac{-D_1+\sqrt{D^2+4m_1m_2(b_2-b_1)(a_2-a_1)z}}{2(b_2-b_1)(a_2-a_1)}, & a_1b_1\le z<a_2\big(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\big) \\ m_1, & a_2\big(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\big)\le z \\ & \quad < a_3\big(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\big) \\ \frac{\widetilde{D}_1-\sqrt{\widetilde{D}^2+4m_1m_2(b_4-b_3)(a_4-a_3)z}}{2m_1(b_4-b_3)}, & a_3\big(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\big)\le z<a_4b_4 \end{cases}$$

where

$$\begin{aligned} D &= b_1m_2(a_2-a_1)-a_1m_1(b_2-b_1) \\ D_1 &= b_1m_2(a_2-a_1)+a_1m_1(b_2-b_1) \\ \widetilde{D} &= a_4m_1(b_4-b_3)-b_4m_2(a_4-a_3) \\ \widetilde{D}_1 &= a_4m_1(b_4-b_3)+b_4m_2(a_4-a_3) \end{aligned}$$

i.e. $A(\cdot)B$ *is a fuzzy set on* $\big(a_1b_1, a_4b_4\big)$, *but need not to be a generalized trapezoidal fuzzy set.*

4. *Division : The membership function* $\mu_{A(/)B}(z)$ *is*

$$\mu_{A(/)B}(z)=\begin{cases} 0, & z<\frac{a_1}{b_4}, \frac{a_4}{b_1}\le z \\ \frac{m_1m_2(b_4z-a_1)}{m_1(b_4-b_3)z+m_2(a_2-a_1)}, & \frac{a_1}{b_4}\le z<\frac{a_2}{b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}} \\ m_1, & \frac{a_2}{b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}}\le z<\frac{a_3}{b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}} \\ \frac{m_1m_2(a_4-b_1z)}{m_1(b_2-b_1)z+m_2(a_4-a_3)}, & \frac{a_3}{b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}}\le z<\frac{a_4}{b_1} \end{cases}$$

i.e. $A(/)B$ *is a fuzzy set on* $\left(\frac{a_1}{b_4}, \frac{a_4}{b_1}\right)$, *but need not to be a generalized trapezoidal fuzzy set.*

3. Normal fuzzy probability for generalized triangular fuzzy sets

In [4], we proved the following theorem.

Theorem 3.1. *([4], Theorem 4.1) Let* $X \sim N(m, \sigma^2)$ *and* $A = ((a_1, c, a_2))$ *be generalized triangular fuzzy set. Then the normal fuzzy probability of a generalized triangular fuzzy set* A *is*

$$\begin{aligned}\tilde{P}(A) = &\frac{2mc}{\sqrt{2\pi}(a_2 - a_1)}\Big(2N(\frac{a_1 + a_2 - 2m}{2\sigma}) - N(\frac{a_1 - m}{\sigma}) - N(\frac{a_2 - m}{\sigma})\Big)\\ &- \frac{2c(a_1 + \sigma)}{\sqrt{2\pi}(a_2 - a_1)}\Big(N(\frac{a_1 + a_2 - 2m}{2\sigma}) - N(\frac{a_1 - m}{\sigma})\Big)\\ &+ \frac{2c(a_2 + \sigma)}{\sqrt{2\pi}(a_2 - a_1)}\Big(N(\frac{a_2 - m}{\sigma}) - N(\frac{a_1 + a_2 - 2m}{2\sigma})\Big),\end{aligned}$$

where $N(\alpha)$ *is the standard normal distribution, that is,*

$$N(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp(-\frac{t^2}{2})dt.$$

However, we made a mistake and the correct result is

$$\begin{aligned}\tilde{P}(A) = &\frac{2mc}{a_2 - a_1}\Big(2N(\frac{a_1 + a_2 - 2m}{2\sigma}) - N(\frac{a_1 - m}{\sigma}) - N(\frac{a_2 - m}{\sigma})\Big)\\ &- \frac{2ca_1}{a_2 - a_1}\Big(N(\frac{a_1 + a_2 - 2m}{2\sigma}) - N(\frac{a_1 - m}{\sigma})\Big)\\ &+ \frac{2ca_2}{a_2 - a_1}\Big(N(\frac{a_2 - m}{\sigma}) - N(\frac{a_1 + a_2 - 2m}{2\sigma})\Big)\\ &+ \frac{2c\sigma}{\sqrt{2\pi}(a_2 - a_1)}\Big(\exp(-\frac{(a_2 - m)^2}{2\sigma^2}) - \exp(-\frac{(a_1 - m)^2}{2\sigma^2})\\ &\qquad - 2\exp(-\frac{(a_1 + a_2 - 2m)^2}{8\sigma^2})\Big)\end{aligned}$$

Proof. Since

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \ \ a_2 \le x, \\ \frac{2c(x-a_1)}{a_2-a_1}, & a_1 \le x < \frac{a_1+a_2}{2}, \\ \frac{-2c(x-a_2)}{a_2-a_1}, & \frac{a_1+a_2}{2} \le x < a_2, \end{cases}$$

we have

$$\begin{aligned}\widetilde{P}(A) &= \int_{\mathbb{R}} \mu_A(x)\, dP_X \\ &= \int_{a_1}^{\frac{a_1+a_2}{2}} \frac{2c(x-a_1)}{a_2-a_1} f(x)\, dx + \int_{\frac{a_1+a_2}{2}}^{a_2} \frac{-2c(x-a_2)}{a_2-a_1} f(x)\, dx,\end{aligned}$$

where $f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\dfrac{(x-m)^2}{2\sigma^2}\Big)$. Putting $\frac{x-m}{\sigma} = t$, then

$$\begin{aligned}\widetilde{P}(A) &= \int_{a_1}^{\frac{a_1+a_2}{2}} \frac{2c(x-a_1)}{a_2-a_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx \\ &\quad + \int_{\frac{a_1+a_2}{2}}^{a_2} \frac{-2c(x-a_2)}{a_2-a_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx \\ &= \frac{2c}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_1+a_2-2m}{2\sigma}} (m+\sigma t) e^{-\frac{t^2}{2}}\, dt \\ &\quad - \frac{2ca_1}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_1+a_2-2m}{2\sigma}} e^{-\frac{t^2}{2}}\, dt \\ &\quad - \frac{2c}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1+a_2-2m}{2\sigma}}^{\frac{a_2-m}{\sigma}} (m+\sigma t) e^{-\frac{t^2}{2}}\, dt \\ &\quad + \frac{2ca_2}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1+a_2-2m}{2\sigma}}^{\frac{a_2-m}{\sigma}} e^{-\frac{t^2}{2}}\, dt \\ &= \frac{2mc}{a_2-a_1} \Big(N(\frac{a_1+a_2-2m}{2\sigma}) - N(\frac{a_1-m}{\sigma}) \Big) \\ &\quad - \frac{2mc}{a_2-a_1} \Big(N(\frac{a_2-m}{\sigma}) - N(\frac{a_1+a_2-2m}{2\sigma}) \Big) \\ &\quad - \frac{2ca_1}{a_2-a_1} \Big(N(\frac{a_1+a_2-2m}{2\sigma}) - N(\frac{a_1-m}{\sigma}) \Big) \\ &\quad + \frac{2ca_2}{a_2-a_1} \Big(N(\frac{a_2-m}{\sigma}) - N(\frac{a_1+a_2-2m}{2\sigma}) \Big) \\ &\quad - \frac{2c\sigma}{\sqrt{2\pi}(a_2-a_1)} \Big(\exp(-\frac{(a_1+a_2-2m)^2}{8\sigma^2}) - \exp(-\frac{(a_1-m)^2}{2\sigma^2}) \Big) \\ &\quad + \frac{2c\sigma}{\sqrt{2\pi}(a_2-a_1)} \Big(\exp(-\frac{(a_2-m)^2}{2\sigma^2}) - \exp(-\frac{(a_1+a_2-2m)^2}{8\sigma^2}) \Big)\end{aligned}$$

$$= \frac{2mc}{a_2 - a_1}\Big(2N(\frac{a_1 + a_2 - 2m}{2\sigma}) - N(\frac{a_1 - m}{\sigma}) - N(\frac{a_2 - m}{\sigma})\Big)$$
$$- \frac{2ca_1}{a_2 - a_1}\Big(N(\frac{a_1 + a_2 - 2m}{2\sigma}) - N(\frac{a_1 - m}{\sigma})\Big)$$
$$+ \frac{2ca_2}{a_2 - a_1}\Big(N(\frac{a_2 - m}{\sigma}) - N(\frac{a_1 + a_2 - 2m}{2\sigma})\Big)$$
$$+ \frac{2c\sigma}{\sqrt{2\pi}(a_2 - a_1)}\Big(\exp(-\frac{(a_2 - m)^2}{2\sigma^2}) - \exp(-\frac{(a_1 - m)^2}{2\sigma^2})$$
$$- 2\exp(-\frac{(a_1 + a_2 - 2m)^2}{8\sigma^2})\Big)$$

Thus the proof is complete. □

Although we made a mistake with Theorem 3.1, the following example is correct.

Example 3.2. *([4], Example 4.2) 1. Let $A = (2, \frac{1}{2}, 8)$ be a generalized triangular fuzzy number. Then the normal fuzzy probability of A with respect to $X \sim N(3, 2^2)$ is 0.1777.*

2. Let $B = (1, \frac{3}{4}, 5)$ be a generalized triangular fuzzy number. Then the normal fuzzy probability of A with respect to $X \sim N(3, 3^2)$ is 0.1924.

4. Normal fuzzy probability for generalized trapezoidal fuzzy sets

In this section, we derive the explicit formula for the normal fuzzy probability for generalized trapezoidal fuzzy sets and give some examples.

Theorem 4.1. *Let $X \sim N(m, \sigma^2)$ and $A = (a_1, a_2, c, a_3, a_4)$ be generalized trapezoidal fuzzy set. Then the normal fuzzy probability of a generalized trapezoidal fuzzy set A is*

$$\tilde{P}(A) = \frac{c(m - a_1)}{a_2 - a_1}\Big(N(\frac{a_2 - m}{\sigma}) - N(\frac{a_1 - m}{\sigma})\Big)$$
$$- \frac{c\sigma}{\sqrt{2\pi}(a_2 - a_1)}\Big(\exp(-\frac{(a_2 - m)^2}{2\sigma^2}) - \exp(-\frac{(a_1 - m)^2}{2\sigma^2})\Big)$$
$$+ c\Big(N(\frac{a_3 - m}{\sigma}) - N(\frac{a_2 - m}{\sigma})\Big)$$
$$+ \frac{c(a_4 - m)}{a_4 - a_3}\Big(N(\frac{a_4 - m}{\sigma}) - N(\frac{a_3 - m}{\sigma})\Big)$$
$$+ \frac{c\sigma}{\sqrt{2\pi}(a_4 - a_3)}\Big(\exp(-\frac{(a_4 - m)^2}{2\sigma^2}) - \exp(-\frac{(a_3 - m)^2}{2\sigma^2})\Big)$$

where $N(\alpha)$ is the standard normal distribution, that is,

$$N(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp(-\frac{t^2}{2})dt.$$

Proof. Since

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \le x \\ \frac{c(x-a_1)}{a_2-a_1}, & a_1 \le x < a_2 \\ c, & a_2 \le x < a_3 \\ \frac{c(a_4-x)}{a_4-a_3}, & a_3 \le x < a_4 \end{cases}$$

we have

$$\begin{aligned} \widetilde{P}(A) &= \int_{\mathbb{R}} \mu_A(x)\, dP_X \\ &= \int_{a_1}^{a_2} \frac{c(x-a_1)}{a_2-a_1} f(x)\, dx + \int_{a_2}^{a_3} cf(x)\, dx + \int_{a_3}^{a_4} \frac{c(a_4-x)}{a_4-a_3} f(x)\, dx, \end{aligned}$$

where $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(x-m)^2}{2\sigma^2}\Big)$. Putting $\frac{x-m}{\sigma} = t$, then

$$\begin{aligned} \widetilde{P}(A) &= \frac{c}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} (m+\sigma t)\exp(-\frac{t^2}{2})dt \\ &\quad - \frac{ca_1}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} \exp(-\frac{t^2}{2})dt + \frac{c}{\sqrt{2\pi}} \int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} \exp(-\frac{t^2}{2})dt \\ &\quad - \frac{c}{\sqrt{2\pi}(a_4-a_3)} \int_{\frac{a_3-m}{\sigma}}^{\frac{a_4-m}{\sigma}} (m+\sigma t)\exp(-\frac{t^2}{2})dt \\ &\quad + \frac{ca_4}{\sqrt{2\pi}(a_4-a_3)} \int_{\frac{a_3-m}{\sigma}}^{\frac{a_4-m}{\sigma}} \exp(-\frac{t^2}{2})dt \\ &= \frac{cm}{a_2-a_1}\Big(N(\frac{a_2-m}{\sigma}) - N(\frac{a_1-m}{\sigma})\Big) \\ &\quad - \frac{c\sigma}{\sqrt{2\pi}(a_2-a_1)}\Big(\exp(-\frac{(a_2-m)^2}{2\sigma^2}) - \exp(-\frac{(a_1-m)^2}{2\sigma^2})\Big) \\ &\quad - \frac{ca_1}{a_2-a_1}\Big(N(\frac{a_2-m}{\sigma}) - N(\frac{a_1-m}{\sigma})\Big) \\ &\quad + c\Big(N(\frac{a_3-m}{\sigma}) - N(\frac{a_2-m}{\sigma})\Big) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{ca_4}{a_4-a_3}\Big(N(\frac{a_4-m}{\sigma}) - N(\frac{a_3-m}{\sigma})\Big)\\
&- \frac{cm}{a_4-a_3}\Big(N(\frac{a_4-m}{\sigma}) - N(\frac{a_3-m}{\sigma})\Big)\\
&+ \frac{c\sigma}{\sqrt{2\pi}(a_4-a_3)}\Big(\exp(-\frac{(a_4-m)^2}{2\sigma^2}) - \exp(-\frac{(a_3-m)^2}{2\sigma^2})\Big)\\
=& \frac{c(m-a_1)}{a_2-a_1}\Big(N(\frac{a_2-m}{\sigma}) - N(\frac{a_1-m}{\sigma})\Big)\\
&- \frac{c\sigma}{\sqrt{2\pi}(a_2-a_1)}\Big(\exp(-\frac{(a_2-m)^2}{2\sigma^2}) - \exp(-\frac{(a_1-m)^2}{2\sigma^2})\Big)\\
&+ c\Big(N(\frac{a_3-m}{\sigma}) - N(\frac{a_2-m}{\sigma})\Big)\\
&+ \frac{c(a_4-m)}{a_4-a_3}\Big(N(\frac{a_4-m}{\sigma}) - N(\frac{a_3-m}{\sigma})\Big)\\
&+ \frac{c\sigma}{\sqrt{2\pi}(a_4-a_3)}\Big(\exp(-\frac{(a_4-m)^2}{2\sigma^2}) - \exp(-\frac{(a_3-m)^2}{2\sigma^2})\Big)
\end{aligned}
$$

Thus the proof is complete. □

Example 4.2. *Let* $A = (2, 4, \frac{2}{3}, 6, 7)$ *be a generalized trapezoidal fuzzy set. Then the normal fuzzy probability of* A *with respect to* $X \sim N(5, 2^2)$ *is 0.4017. In fact, putting* $\frac{x-5}{2} = t$*, we have*

$$
\begin{aligned}
\widetilde{P}(A) =& \int_2^4 \frac{x-2}{3}\cdot\frac{1}{2\sqrt{2\pi}}e^{-\frac{(x-5)^2}{8}}dx + \int_4^6 \frac{2}{3}\cdot\frac{1}{2\sqrt{2\pi}}e^{-\frac{(x-5)^2}{8}dx}\\
&+ \int_6^7 \frac{2(7-x)}{3}\cdot\frac{1}{2\sqrt{2\pi}}e^{-\frac{(x-5)^2}{8}}dx\\
=& \frac{1}{3\sqrt{2\pi}}\int_{-3/2}^{-1/2}(5+2t)e^{-\frac{t^2}{2}}dt - \frac{2}{3\sqrt{2\pi}}\int_{-3/2}^{-1/2}e^{-\frac{t^2}{2}}dt\\
&+ \frac{2}{3\sqrt{2\pi}}\int_{-1/2}^{1/2}e^{-\frac{t^2}{2}}dt\\
&- \frac{2}{3\sqrt{2\pi}}\int_{1/2}^{1}(5+2t)e^{-\frac{t^2}{2}}dt + \frac{14}{3\sqrt{2\pi}}\int_{1/2}^{1}e^{-\frac{t^2}{2}}dt\\
=& N(-1/2) - N(-3/2) + \frac{4}{3}\Big(N(1) - N(1/2)\Big)\\
&- \frac{2}{3\sqrt{2\pi}}\Big(3e^{-\frac{1}{8}} - 2e^{-\frac{1}{2}} - e^{-\frac{9}{8}}\Big) + \frac{2}{3}\Big(N(1/2) - N(-1/2)\Big)\\
=& 0.4017.
\end{aligned}
$$

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