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SOME RESULTS ON KRONECKER PRODUCTS AND COMMUTATION MATRICES

Yong Sik Yun and Chul Kang*

Abstract. There are a lot of unsolved problems or issues regarding Kronecker products, vec-operator and Commutation matrices. We derived further properties and results of Kronecker products, vec-operator and Commutation Matrices.

1. Introduction

The central property of the Commutation matrix $I_{m,n}$ is that it transforms vec$A$ into vecA', where A is an arbitrary $m \times n$ matrix. We shall denote this matrix as $I_{m,n}$. It is known that $I_{m,n}$ can be used for reversing the order of Kronecker product, a property very uesful in the calculation of matrix derivatives and moment of matrix quadratic form.

In section 2, we review a number of known results on Kronecker products, vec-operator and Commutation matrices. In section 3, we shall derive further properties and results of Kronecker products, vec-operator and Commutation Matrices.

2. Basic Notations and Preliminary Results

In this section, we summarize some preliminary results on Kronecker products, vec-operator and Commutation matrices.

Definition 1. Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{kl})$ be a $p \times q$ matrix. The $mp \times nq$ matrix with $a_{ij}b_{kl}$ as the element in the (ik)th row and the (jl)th column is called Kronecker product of A and B and is denoted by

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*Corresponding author.

$A \otimes B$; that is,

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Definition 2. For an $m \times n$ matrix A, let vec$A$ denote the mn vector obtained by 'vectorizing' A; that is,

$$\text{vec}A = (a_1', a_2', \cdots, a_n')'$$

if $A = (a_1, a_2, \cdots, a_n)$, where a_i is an m vector.

Definition 3. The Commutation matrix $I_{m,n}$ is an $mn \times mn$ matrix containing mn blocks of order $m \times n$ such that the (ij)th block has a 1 in its (ji)th position and zeroes elsewhere. One has

$$I_{m,n} = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(H_{ij} \otimes H_{ij}'\right)$$

where H_{ij} is an $n \times m$ matrix with 1 in its (ij)th position and zeroes elsewhere, and can be written as $H_{ij} = e_i e_j'$ is the ith unit column vector of order n.

Some preliminary results on Kronecker products, vec operator and Commutation matrices are:

For $I_m : m \times m$ identity matrix,

$$I_m \otimes I_n = I_{mn}. \tag{1}$$

$$I_{m,n} I_{n,m} = I_{mn}. \tag{2}$$

Therefore,

$$(I_{m,n})^{-1} = (I_{m,n})' = I_{n,m}. \tag{3}$$

For $A : m \times n$ matrix and $B : p \times q$ matrix,

$$\text{vec}(A \otimes B) = (I_n \otimes I_{m,q} \otimes I_p)(\text{vec}A \otimes \text{vec}B). \tag{4}$$

For A, B, C, and D conformable matrices,

$$(AB) \otimes (CD) = (A \otimes C)(B \otimes D). \tag{5}$$

For details, see Magnus and Neudecker (1979), Neudecker and Wansbeek (1983) and Neudecker and Wansbeek (1987).

3. Some Results on Kronecker products and Commutation matrices

In this section, we will show a couple of relation between Kronecker products, vec-operator and Commutation matrices. Those relations are very useful in calculating moment of matrix quadratic form in multivariate analysis. They are also statistically meaningful in matrix algebra.

Theorem 3.1. $I_{mnq} \otimes I_{p,s} \otimes I_r = I_n \otimes I_{m,q} \otimes I_{prs}$.

Proof. Let $A : m \times n$, $B : p \times q$ and $C : r \times s$ be arbitrary matrices. Applying (4), (5) and (1), Tracy and Sultan (1993) have obtained the following equality.

$$\begin{aligned} & \text{vec}(A \otimes B \otimes C) \\ = \; & (I_n \otimes I_{m,qs} \otimes I_{pr})(I_{mnq} \otimes I_{p,s} \otimes I_r)(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C). \qquad (6) \end{aligned}$$

Also, applying (4), (5) and (1), we get

$$\begin{aligned} & \text{vec}(A \otimes B \otimes C) \\ = \; & \text{vec}\big((A \otimes B) \otimes C\big) \\ = \; & (I_{nq} \otimes I_{mp,s} \otimes I_r)\big(\text{vec}(A \otimes B) \otimes \text{vec}C\big) \\ = \; & (I_{nq} \otimes I_{mp,s} \otimes I_r)\big((I_n \otimes I_{m,q} \otimes I_p)(\text{vec}A \otimes \text{vec}B) \otimes I_{rs}\text{vec}C\big) \\ = \; & (I_{nq} \otimes I_{mp,s} \otimes I_r)(I_n \otimes I_{m,q} \otimes I_p \otimes I_{rs})(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C) \\ = \; & (I_{nq} \otimes I_{mp,s} \otimes I_r)(I_n \otimes I_{m,q} \otimes I_{prs})(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C). \qquad (7) \end{aligned}$$

Since A, B and C be arbitrary,

$$I_{mnq} \otimes I_{p,s} \otimes I_r = I_n \otimes I_{m,q} \otimes I_{prs} \qquad (8)$$

using (1), (2) and (3). □

Theorem 3.2. *The following equalities hold.*

$$\begin{aligned} & (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnq} \otimes I_{p,sv} \otimes I_{ru})(I_{mnpqs} \otimes I_{r,v} \otimes I_u) \\ & = (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_n \otimes I_{m,q} \otimes I_{prusv})(I_{mnpqs} \otimes I_{r,v} \otimes I_u) \\ & = (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnqs} \otimes I_{pr,v} \otimes I_u)(I_{mnq} \otimes I_{p,s} \otimes I_{ruv}) \\ & = (I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_n \otimes I_{m,qs} \otimes I_{pruv})(I_{mnq} \otimes I_{p,s} \otimes I_{ruv}) \\ & = (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_{mnpqs} \otimes I_{r,v} \otimes I_u)(I_n \otimes I_{m,q} \otimes I_{prsuv}) \\ & = (I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_{nq} \otimes I_{mp,s} \otimes I_{ruv})(I_n \otimes I_{m,q} \otimes I_{prsuv}) \\ & = (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_n \otimes I_{m,q} \otimes I_{ps} \otimes I_{r,v} \otimes I_u) \end{aligned}$$

Proof. Let $A : m \times n$, $B : p \times q$, $C : r \times s$ and $D : u \times v$ be arbitrary matrices.

[1]

$$\begin{aligned}
&\text{vec}(A \otimes B \otimes C \otimes D) \\
=\;& \text{vec}\big(A \otimes B \otimes (C \otimes D)\big) \\
=\;& (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnq} \otimes I_{p,sv} \otimes I_{ru})\big(\text{vec}A \otimes \text{vec}B \otimes \text{vec}(C \otimes D)\big) \\
=\;& (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnq} \otimes I_{p,sv} \otimes I_{ru}) \\
&\quad \cdot\big((I_{mnpq}(\text{vec}A \otimes \text{vec}B)\big) \otimes \big((I_s \otimes I_{r,v} \otimes I_u)(\text{vec}C \otimes \text{vec}D)\big) \\
=\;& (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnq} \otimes I_{p,sv} \otimes I_{ru}) \\
&\quad \cdot(I_{mnpq} \otimes I_s \otimes I_{r,v} \otimes I_u)(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D) \\
=\;& (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnq} \otimes I_{p,sv} \otimes I_{ru})(I_{mnpqs} \otimes I_{r,v} \otimes I_u) \\
&\quad \cdot(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D),
\end{aligned}$$

using (6), (4), (5), (1) and (2).

[2]

$$\begin{aligned}
&\text{vec}(A \otimes B \otimes C \otimes D) \\
=\;& \text{vec}\big(A \otimes B \otimes (C \otimes D)\big) \\
=\;& (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_n \otimes I_{m,q} \otimes I_{prusv})\big(\text{vec}A \otimes \text{vec}B \otimes \text{vec}(C \otimes D)\big) \\
=\;& (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_n \otimes I_{m,q} \otimes I_{prusv}) \\
&\quad \cdot\big(I_{mnpq}(\text{vec}A \otimes \text{vec}B)\big) \otimes \big((I_s \otimes I_{r,v} \otimes I_u)(\text{vec}C \otimes \text{vec}D)\big) \\
=\;& (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_n \otimes I_{m,q} \otimes I_{prusv})(I_{mnpqs} \otimes I_{r,v} \otimes I_u) \\
&\quad \cdot(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D),
\end{aligned}$$

using (7), (4), (5), (1) and (2).

[3]

$$\begin{aligned}
&\text{vec}(A \otimes B \otimes C \otimes D) \\
=\;& \text{vec}\big(A \otimes (B \otimes C) \otimes D\big) \\
=\;& (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnqs} \otimes I_{pr,v} \otimes I_u)\big(\text{vec}A \otimes \text{vec}(B \otimes C) \otimes \text{vec}D\big) \\
=\;& (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnqs} \otimes I_{pr,v} \otimes I_u) \\
&\quad \cdot\big((I_{mn}\text{vec}A) \otimes \big((I_q \otimes I_{p,s} \otimes I_r)(\text{vec}B \otimes \text{vec}C)\big) \otimes (I_{uv}\text{vec}D)\big) \\
=\;& (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnqs} \otimes I_{pr,v} \otimes I_u) \\
&\quad \cdot(I_{mn} \otimes I_q \otimes I_{p,s} \otimes I_r \otimes I_{uv})(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D) \\
=\;& (I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnqs} \otimes I_{pr,v} \otimes I_u)(I_{mnq} \otimes I_{p,s} \otimes I_{ruv}) \\
&\quad \cdot(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D),
\end{aligned}$$

using (6), (5), (4), (1) and (2).

[4]

$$\begin{aligned}
&\text{vec}(A \otimes B \otimes C \otimes D) \\
=\ &\text{vec}\big(A \otimes (B \otimes C) \otimes D\big) \\
=\ &(I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_n \otimes I_{m,qs} \otimes I_{pruv})\big(\text{vec}A \otimes \text{vec}(B \otimes C) \otimes \text{vec}D\big) \\
=\ &(I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_n \otimes I_{m,qs} \otimes I_{pruv}) \\
&\qquad \cdot\big((I_{mn}\text{vec}A) \otimes \big((I_q \otimes I_{p,s} \otimes I_r)(\text{vec}B \otimes \text{vec}C)\big) \otimes (I_{uv}\text{vec}D)\big) \\
=\ &(I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_n \otimes I_{m,qs} \otimes I_{pruv}) \\
&\qquad \cdot(I_{mn} \otimes I_q \otimes I_{p,s} \otimes I_r \otimes I_{uv})(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D) \\
=\ &(I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_n \otimes I_{m,qs} \otimes I_{pruv})(I_{mnq} \otimes I_{p,s} \otimes I_{ruv}) \\
&\qquad\qquad \cdot(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D),
\end{aligned}$$

using (7), (5), (4), (1) and (2).

[5]

$$\begin{aligned}
&\text{vec}(A \otimes B \otimes C \otimes D) \\
=\ &\text{vec}\big((A \otimes B) \otimes C \otimes D\big) \\
=\ &(I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_{mnpqs} \otimes I_{r,v} \otimes I_u)\big(\text{vec}(A \otimes B) \otimes \text{vec}C \otimes \text{vec}D\big) \\
=\ &(I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_{mnpqs} \otimes I_{r,v} \otimes I_u) \\
&\qquad \cdot\big(((I_n \otimes I_{m,q} \otimes I_p)(\text{vec}A \otimes \text{vec}B)) \otimes (I_{rs}\text{vec}C) \otimes (I_{uv}\text{vec}D)\big) \\
=\ &(I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_{mnpqs} \otimes I_{r,v} \otimes I_u) \\
&\qquad \cdot(I_n \otimes I_{m,q} \otimes I_p \otimes I_{rs} \otimes I_{uv})(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D) \\
=\ &(I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_{mnpqs} \otimes I_{r,v} \otimes I_u)(I_n \otimes I_{m,q} \otimes I_{prsuv}) \\
&\qquad\qquad \cdot(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D),
\end{aligned}$$

using (6), (5), (8), (1) and (2).

[6]

$$\begin{aligned}
&\text{vec}(A \otimes B \otimes C \otimes D) \\
=\ &\text{vec}\big((A \otimes B) \otimes C \otimes D\big) \\
=\ &(I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_{nq} \otimes I_{mp,s} \otimes I_{ruv})\big(\text{vec}(A \otimes B) \otimes \text{vec}C \otimes \text{vec}D\big) \\
=\ &(I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_{nq} \otimes I_{mp,s} \otimes I_{ruv}) \\
&\qquad \cdot\big(((I_n \otimes I_{m,q} \otimes I_p)(\text{vec}A \otimes \text{vec}B)) \otimes (I_{rs}\text{vec}C) \otimes (I_{uv}\text{vec}D)\big) \\
=\ &(I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_{nq} \otimes I_{mp,s} \otimes I_{ruv})(I_n \otimes I_{m,q} \otimes I_{prsuv}) \\
&\qquad\qquad \cdot(\text{vec}A \otimes \text{vec}B \otimes \text{vec}C \otimes \text{vec}D),
\end{aligned}$$

using (7), (5), (4), (1) and (2).

[7]

$$
\begin{aligned}
& \operatorname{vec}(A \otimes B \otimes C \otimes D) \\
= \; & \operatorname{vec}\big((A \otimes B) \otimes (C \otimes D)\big) \\
= \; & (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})\big(\operatorname{vec}(A \otimes B) \otimes \operatorname{vec}(C \otimes D)\big) \\
= \; & (I_{nq} \otimes I_{mp,sv} \otimes I_{ru}) \\
& \cdot\big((I_n \otimes I_{m,q} \otimes I_p)(\operatorname{vec}A \otimes \operatorname{vec}B) \otimes (I_s \otimes I_{r,v} \otimes I_u)(\operatorname{vec}C \otimes \operatorname{vec}D)\big) \\
= \; & (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_n \otimes I_{m,q} \otimes I_{ps} \otimes I_{r,v} \otimes I_u) \\
& \qquad\qquad \cdot(\operatorname{vec}A \otimes \operatorname{vec}B \otimes \operatorname{vec}C \otimes \operatorname{vec}D),
\end{aligned}
$$

using (4), (5), (1) and (2).

□

Theorem 3.3. *The following equalities hold.*

$$(I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnq} \otimes I_{p,sv} \otimes I_{ru}) = (I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_n \otimes I_{m,q} \otimes I_{prusv})$$

$$(I_n \otimes I_{m,qsv} \otimes I_{pru})(I_{mnqs} \otimes I_{pr,v} \otimes I_u) = (I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_n \otimes I_{m,qs} \otimes I_{pruv})$$

$$(I_{nq} \otimes I_{mp,sv} \otimes I_{ru})(I_{mnpqs} \otimes I_{r,v} \otimes I_u) = (I_{nqs} \otimes I_{mpr,v} \otimes I_u)(I_{nq} \otimes I_{mp,s} \otimes I_{ruv})$$

$$(I_{mnpqs} \otimes I_{r,v} \otimes I_u)(I_n \otimes I_{m,q} \otimes I_{prsuv}) = I_n \otimes I_{m,q} \otimes I_{ps} \otimes I_{r,v} \otimes I_u$$

Proof. Using results on Theorem 3.2, we can show easily that the above equalities hold.

□

Corollary 3.4. *The following equalities hold.*

$$
\begin{aligned}
& (I_n \otimes I_{m,n^3} \otimes I_{m^3})(I_{mn^2} \otimes I_{m,n^2} \otimes I_{m^2})(I_{m^2n^3} \otimes I_{m,n} \otimes I_m) \\
& = (I_{n^2} \otimes I_{m^2,n^2} \otimes I_{m^2})(I_n \otimes I_{m,n} \otimes I_{m^3n^2})(I_{m^2n^3} \otimes I_{m,n} \otimes I_m) \\
& = (I_n \otimes I_{m,n^3} \otimes I_{m^3})(I_{mn^3} \otimes I_{m^2,n} \otimes I_m)(I_{mn^2} \otimes I_{m,n} \otimes I_{m^2n}) \\
& = (I_{n^3} \otimes I_{m^3,n} \otimes I_m)(I_n \otimes I_{m,n^2} \otimes I_{m^3n})(I_{mn^2} \otimes I_{m,n} \otimes I_{m^2n}) \\
& = (I_{n^2} \otimes I_{m^2,n^2} \otimes I_{m^2})(I_{m^2n^3} \otimes I_{m,n} \otimes I_m)(I_n \otimes I_{m,n} \otimes I_{m^3n^2}) \\
& = (I_{n^3} \otimes I_{m^3,n} \otimes I_m)(I_{n^2} \otimes I_{m^2,n} \otimes I_{m^2n})(I_n \otimes I_{m,n} \otimes I_{m^3n^2}) \\
& = (I_{n^2} \otimes I_{m^2,n^2} \otimes I_{m^2})(I_n \otimes I_{m,n} \otimes I_{mn} \otimes I_{m,n} \otimes I_m)
\end{aligned}
$$

Proof. Let A, B, C and D be arbitrary $m \times n$ matrices. Substituting $p = r = u = m$, $q = s = v = n$ and applying Theorem 3.2, we get the following relations.

[1]

$$\begin{aligned}&\mathrm{vec}(A\otimes B\otimes C\otimes D)\\ =\;&(I_n\otimes I_{m,n^3}\otimes I_{m^3})(I_{mn^2}\otimes I_{m,n^2}\otimes I_{m^2})(I_{m^2n^3}\otimes I_{m,n}\otimes I_m)\\ &\cdot(\mathrm{vec}A\otimes\mathrm{vec}B\otimes\mathrm{vec}C\otimes\mathrm{vec}D).\end{aligned}$$

[2]

$$\begin{aligned}&\mathrm{vec}(A\otimes B\otimes C\otimes D)\\ =\;&(I_{n^2}\otimes I_{m^2,n^2}\otimes I_{m^2})(I_n\otimes I_{m,n}\otimes I_{m^3n^2})(I_{m^2n^3}\otimes I_{m,n}\otimes I_m)\\ &\cdot(\mathrm{vec}A\otimes\mathrm{vec}B\otimes\mathrm{vec}C\otimes\mathrm{vec}D).\end{aligned}$$

[3]

$$\begin{aligned}&\mathrm{vec}(A\otimes B\otimes C\otimes D)\\ =\;&(I_n\otimes I_{m,n^3}\otimes I_{m^3})(I_{mn^3}\otimes I_{m^2,n}\otimes I_m)(I_{mn^2}\otimes I_{m,n}\otimes I_{m^2n})\\ &\cdot(\mathrm{vec}A\otimes\mathrm{vec}B\otimes\mathrm{vec}C\otimes\mathrm{vec}D).\end{aligned}$$

[4]

$$\begin{aligned}&\mathrm{vec}(A\otimes B\otimes C\otimes D)\\ =\;&(I_{n^3}\otimes I_{m^3,n}\otimes I_m)(I_n\otimes I_{m,n^2}\otimes I_{m^3n})(I_{mn^2}\otimes I_{m,n}\otimes I_{m^2n})\\ &\cdot(\mathrm{vec}A\otimes\mathrm{vec}B\otimes\mathrm{vec}C\otimes\mathrm{vec}D).\end{aligned}$$

[5]

$$\begin{aligned}&\mathrm{vec}(A\otimes B\otimes C\otimes D)\\ =\;&(I_{n^3}\otimes I_{m^3,n}\otimes I_m)(I_{n^2}\otimes I_{m^2,n}\otimes I_{m^2n})(I_n\otimes I_{m,n}\otimes I_{m^3n^2})\\ &\cdot(\mathrm{vec}A\otimes\mathrm{vec}B\otimes\mathrm{vec}C\otimes\mathrm{vec}D).\end{aligned}$$

[6]

$$\begin{aligned}&\mathrm{vec}(A\otimes B\otimes C\otimes D)\\ =\;&(I_{n^3}\otimes I_{m^3,n}\otimes I_m)(I_{n^2}\otimes I_{m^2,n}\otimes I_{m^2n})(I_n\otimes I_{m,n}\otimes I_{m^3n^2})\\ &\cdot(\mathrm{vec}A\otimes\mathrm{vec}B\otimes\mathrm{vec}C\otimes\mathrm{vec}D).\end{aligned}$$

[7]

$$\begin{aligned}&\mathrm{vec}(A\otimes B\otimes C\otimes D)\\ =\;&(I_{n^2}\otimes I_{m^2,n^2}\otimes I_{m^2})(I_n\otimes I_{m,n}\otimes I_{mn}\otimes I_{m,n}\otimes I_m)\\ &\cdot(\mathrm{vec}A\otimes\mathrm{vec}B\otimes\mathrm{vec}C\otimes\mathrm{vec}D).\end{aligned}$$

□

Corollary 3.5. *The following equalities hold.*

$$(I_n \otimes I_{m,n^3} \otimes I_{m^3})(I_{mn^2} \otimes I_{m,n^2} \otimes I_{m^2}) = (I_{n^2} \otimes I_{m^2,n^2} \otimes I_{m^2})(I_n \otimes I_{m,n} \otimes I_{m^3n^2})$$

$$(I_n \otimes I_{m,n^3} \otimes I_{m^3})(I_{mn^3} \otimes I_{m^2,n} \otimes I_m) = (I_{n^3} \otimes I_{m^3,n} \otimes I_m)(I_n \otimes I_{m,n^2} \otimes I_{m^3n})$$

$$(I_{n^2} \otimes I_{m^2,n^2} \otimes I_{m^2})(I_{m^3n^2} \otimes I_{m,n} \otimes I_m) = (I_{n^3} \otimes I_{m^3,n} \otimes I_m)(I_{n^2} \otimes I_{m^2,n} \otimes I_{m^2n})$$

$$(I_{m^2n^3} \otimes I_{m,n} \otimes I_m)(I_n \otimes I_{m,n} \otimes I_{m^3n^2}) = I_n \otimes I_{m,n} \otimes I_{mn} \otimes I_{m,n} \otimes I_m$$

Proof. Using results on Corollary 3.4, we can show easily that the above equalities is true. □

Corollary 3.6. *The following equalities hold.*

$$\begin{aligned}
&(I_n \otimes I_{n,n^3} \otimes I_{n^3})(I_{n^3} \otimes I_{n,n^2} \otimes I_{n^2})(I_{n^5} \otimes I_{n,n} \otimes I_n)\\
&= (I_{n^2} \otimes I_{n^2,n^2} \otimes I_{n^2})(I_n \otimes I_{n,n} \otimes I_{n^5})(I_{n^5} \otimes I_{n,n} \otimes I_n)\\
&= (I_n \otimes I_{n,n^3} \otimes I_{n^3})(I_{n^4} \otimes I_{n^2,n} \otimes I_n)(I_{n^3} \otimes I_{n,n} \otimes I_{n^3})\\
&= (I_{n^3} \otimes I_{n^3,n} \otimes I_n)(I_n \otimes I_{n,n^2} \otimes I_{n^4})(I_{n^3} \otimes I_{n,n} \otimes I_{n^3})\\
&= (I_{n^2} \otimes I_{n^2,n^2} \otimes I_{n^2})(I_{n^5} \otimes I_{n,n} \otimes I_n)(I_n \otimes I_{n,n} \otimes I_{n^5})\\
&= (I_{n^3} \otimes I_{n^3,n} \otimes I_n)(I_{n^2} \otimes I_{n^2,n} \otimes I_{n^3})(I_n \otimes I_{n,n} \otimes I_{n^5})\\
&= (I_{n^2} \otimes I_{n^2,n^2} \otimes I_{n^2})(I_n \otimes I_{n,n} \otimes I_{n^2} \otimes I_{n,n} \otimes I_n)
\end{aligned}$$

Proof. Let A, B, C and D be arbitrary $n \times n$ matrices. Substituting $m = p = q = r = s = u = v = n$ and applying Theorem 3.2, Corollary 3.6 is obtained.

[1]

$$\begin{aligned}
&\mathrm{vec}(A \otimes B \otimes C \otimes D)\\
= \ &(I_n \otimes I_{n,n^3} \otimes I_{n^3})(I_{n^3} \otimes I_{n,n^2} \otimes I_{n^2})(I_{n^5} \otimes I_{n,n} \otimes I_n)\\
&\cdot(\mathrm{vec}A \otimes \mathrm{vec}B \otimes \mathrm{vec}C \otimes \mathrm{vec}D).
\end{aligned}$$

[2]

$$\begin{aligned}
&\mathrm{vec}(A \otimes B \otimes C \otimes D)\\
= \ &(I_{n^2} \otimes I_{n^2,n^2} \otimes I_{n^2})(I_n \otimes I_{n,n} \otimes I_{n^5})(I_{n^5} \otimes I_{n,n} \otimes I_n)\\
&\cdot(\mathrm{vec}A \otimes \mathrm{vec}B \otimes \mathrm{vec}C \otimes \mathrm{vec}D).
\end{aligned}$$

[3]

$$\begin{aligned}
&\mathrm{vec}(A \otimes B \otimes C \otimes D)\\
= \ &(I_n \otimes I_{n,n^3} \otimes I_{n^3})(I_{n^4} \otimes I_{n^2,n} \otimes I_n)(I_{n^3} \otimes I_{n,n} \otimes I_{n^3})\\
&\cdot(\mathrm{vec}A \otimes \mathrm{vec}B \otimes \mathrm{vec}C \otimes \mathrm{vec}D).
\end{aligned}$$

[4]

$$\begin{aligned} & \mathrm{vec}(A \otimes B \otimes C \otimes D) \\ = \; & (I_{n^3} \otimes I_{n^3,n} \otimes I_n)(I_n \otimes I_{n,n^2} \otimes I_{n^4})(I_{n^3} \otimes I_{n,n} \otimes I_{n^3}) \\ & \cdot(\mathrm{vec}A \otimes \mathrm{vec}B \otimes \mathrm{vec}C \otimes \mathrm{vec}D). \end{aligned}$$

[5]

$$\begin{aligned} & \mathrm{vec}(A \otimes B \otimes C \otimes D) \\ = \; & (I_{n^3} \otimes I_{n^3,n} \otimes I_n)(I_{n^2} \otimes I_{n^2,n} \otimes I_{n^3})(I_n \otimes I_{n,n} \otimes I_{n^5}) \\ & \cdot(\mathrm{vec}A \otimes \mathrm{vec}B \otimes \mathrm{vec}C \otimes \mathrm{vec}D). \end{aligned}$$

[6]

$$\begin{aligned} & \mathrm{vec}(A \otimes B \otimes C \otimes D) \\ = \; & (I_{n^3} \otimes I_{n^3,n} \otimes I_n)(I_{n^2} \otimes I_{n^2,n} \otimes I_{n^3})(I_n \otimes I_{n,n} \otimes I_{n^5}) \\ & \cdot(\mathrm{vec}A \otimes \mathrm{vec}B \otimes \mathrm{vec}C \otimes \mathrm{vec}D). \end{aligned}$$

[7]

$$\begin{aligned} & \mathrm{vec}(A \otimes B \otimes C \otimes D) \\ = \; & (I_{n^2} \otimes I_{n^2,n^2} \otimes I_{n^2})(I_n \otimes I_{n,n} \otimes I_{n^2} \otimes I_{n,n} \otimes I_n) \\ & \cdot(\mathrm{vec}A \otimes \mathrm{vec}B \otimes \mathrm{vec}C \otimes \mathrm{vec}D). \end{aligned}$$

□

Corollary 3.7. *The following equalities hold.*

$$(I_n \otimes I_{n,n^3} \otimes I_{n^3})(I_{n^3} \otimes I_{n,n^2} \otimes I_{n^2}) = (I_{n^2} \otimes I_{n^2,n^2} \otimes I_{n^2})(I_n \otimes I_{n,n} \otimes I_{n^5})$$

$$(I_n \otimes I_{n,n^3} \otimes I_{n^3})(I_{n^4} \otimes I_{n^2,n} \otimes I_n) = (I_{n^3} \otimes I_{n^3,n} \otimes I_n)(I_n \otimes I_{n,n^2} \otimes I_{n^4})$$

$$(I_{n^2} \otimes I_{n^2,n^2} \otimes I_{n^2})(I_{n^5} \otimes I_{n,n} \otimes I_n) = (I_{n^3} \otimes I_{n^3,n} \otimes I_n)(I_{n^2} \otimes I_{n^2,n} \otimes I_{n^3})$$

$$(I_{n^5} \otimes I_{n,n} \otimes I_n)(I_n \otimes I_{n,n} \otimes I_{n^5}) = I_n \otimes I_{n,n} \otimes I_{n^2} \otimes I_{n,n} \otimes I_n$$

Proof. Substituting $m = p = q = r = s = u = v = n$ and applying Theorem 3.3, we can prove Corollary 3.7. □

4. Conclusions

There are a lot of unsolved problems or issues regarding Kronecker products, vec-operator and Commutation matrices. For example, there is no exact formula of the general moment of matrix quadratic form. In fact, only the third moment of matrix quadratic form has been solved. The equalities proven in this paper will contribute to study in this area.

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Yong Sik Yun
Department of Mathematics
Jeju National University
102 Jejudaehakno Jeju-si
Jeju Special Self-Governing Province, KOREA. 690-756
E-mail address: yunys@jejunu.ac.kr

Chul Kang
Department of Applied Mathematics
Hankyong National University
327 Chungang-no Anseong-si
Kyonggi-do, KOREA. 456-749
E-mail address: stat@hknu.ac.kr