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EMBEDDING PROPERTIES IN NEAR-RINGS

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Abstract. In this paper, we initiate the study of zero symmetric and constant parts of near-rings, and then apply these to self map near-rings.

Next, we investigate that every near-ring can be embedded into some self map near-ring, and every zero symmetric near-ring can be embedded into some zero symmetric self map near-ring.

1. Introduction

Throughout this paper, a near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations, say $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all a, b, c in R. We note that obviously, $a0 = 0$ and $a(-b) = -ab$ for all a, b in R, but in general, $0a \neq 0$ and $(-a)b \neq -ab$.

If R has a unity 1, then R is called *unitary*. An element d in R is called *distributive* if $(a + b)d = ad + bd$ for all a and b in R.

We consider the following substructures of near-rings: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ which is called the *zero symmetric part* of R,

$$R_c = \{a \in R \mid 0a = a\} = \{a \in R \mid ra = a,\ for\ all\ r \in R\} = \{0a \mid a \in R\}$$

which is called the *constant part* of R, and $R_d = \{a \in R \mid a\ is\ distributive\}$ which is called the *distributive part* of R.

We note that R_0 and R_c are subnear-rings of R, but R_d is not a subnear-ring of R. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is said to be *zero symmetric,* also, in case $R = R_c$, R is called a *constant* near-ring, and in case $R = R_d$, R is called a *distributive* near-ring.

Let $(G, +)$ be a group (not necessarily abelian). We may obtain some examples of near-rings as following:

In the set

$$M(G) = \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f + g$ of any two mappings f, g in $M(G)$ by the rule $x(f + g) = xf + xg$ for all $x \in G$ and the product $f \circ g$ by

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the rule $x(f \circ g) = (xf)g$ for all $x \in G$, then $(M(G), +, \circ)$ becomes a near-ring. It is called the *self map near-ring* on the group G. Also, we can define the substructures of $(M(G), +, \circ)$ as following: $M_0(G) = \{f \in M(G) \mid 0f = 0\}$ and $M_c(G) = \{f \in M(G) \mid f \textit{ is constant}\}$.

For the remainder basic concepts and results on near-rings, we refer to [3].

2. Some embedding Properties in near-rings

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $(a+b)\theta = a\theta + b\theta$, (ii) $(ab)\theta = a\theta b\theta$. Obviously, $R\theta < S$ and $T\theta^{-1} < R$ for any $T < S$.

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation* of R on G, we may write that xr (right scalar multiplication in R) for $x(r\theta)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus a (unitary) R-group is an additive group G satisfying (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (if R has a unity 1), for all $x \in G$ and $a,\ b \in R$.

Evidently, every near-ring R can be given the structure of an R-group (unitary, if R is unitary) by right multiplication in R. Moreover, every group G has a natural $M(G)$-group structure, from the representation of $M(G)$ on G by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

Proposition 2.1. *Let $(G,\ +)$ be a group and let Φ a subset of endomorphisms of $(G,\ +)$ containing zero endomorphism ζ. Then the set*

$$M_\Phi(G) = \{f \in M(G) | f \circ \phi = \phi \circ f, \forall \phi \in \Phi\}$$

is a unitary zero symmetric subnear-ring of $M(G)$.

Proof. Let $f, g \in M_\Phi(G)$. Then $f \circ \phi = \phi \circ f$ and $g \circ \phi = \phi \circ g$ for any $\phi \in \Phi$, and so we have, since ϕ is an endomorphism,

$$x[(f-g) \circ \phi] = (x(f-g))\phi = (xf - xg)\phi$$
$$= (xf)\phi - (xg)\phi = x(f \circ \phi) - x(g \circ \phi) = x(\phi \circ f) - x(\phi \circ g)$$
$$= (x\phi)f - (x\phi)g = (x\phi)(f-g) = x[\phi \circ (f-g)],$$

for all $x \in G$. Hence $(f-g) \circ \phi = \phi \circ (f-g)$, and so $f - g \in M_\Phi(G)$. This implies that $(M_\Phi(G),\ +)$ is a subgroup of $(M(G),\ +)$.

Next, for any $f, g \in M_\Phi(G)$, we have that

$$(f \circ g) \circ \phi = f \circ (g \circ \phi) = f \circ (\phi \circ g) = (\phi \circ f) \circ g = \phi \circ (f \circ g),$$

for any $\phi \in \Phi$. Hence $f \circ g \in M_\Phi(G)$. Consequently, $(M_\Phi(G),\ +,\ \circ)$ is a subnear-ring of $(M(G),\ +,\ \circ)$.

Finally, let $f \in M_\Phi(G)$. Since $\zeta \in \Phi$, we see that $\zeta \circ f = f \circ \zeta = \zeta$. Therefore $M_\Phi(G)$ is zero symmetric with identity 1_G. $\square$

Corollary 2.2. *For any group* $(G, +)$*,* $(M_0(G), +, \circ)$ *is a zero symmetric subnear-ring of* $(M(G), +, \circ)$*. Moreover,* $M_0(G) = M(G)_0$*, where* $M(G)_0$ *is a zero symmetric part of* $M(G)$*.*

Proof. The first paragraph is immediately from the Proposition 2.1.

Next, clearly, $M_0(G) \subseteq M(G)_0$, because every element of $M_0(G)$ is zero symmetric by Proposition 2.1.

Conversely, let $f \in M(G)_0$. Then $\zeta \circ f = \zeta$, that is, for any $x \in G$, $(x\zeta)f = x\zeta$. This implies that $0f = 0$. Hence, $f \in M_0(G)$. □

Remark 1. Let G be an additive group. Then we see that $M_c(G) = M(G)_c$, where $M(G)_c$ is the constant part of $M(G)$.

Indeed, if $f \in M_c(G)$, then f is constant, say, $f = c$. From this,

$$x(\zeta \circ f) = (x\zeta)f = 0f = c = xf,$$

for all $x \in G$. Hence $\zeta \circ f = \zeta$, and so $f \in M(G)_c$.

Conversely, if $f \in M(G)_c$, then $\zeta \circ f = f$, that is, $0f = (x\zeta)f = xf$, for all $x \in G$. Hence f is constant, so that $f \in M_c(G)$.

Note that from Corollary 2.2 and Remark 1, $M_0(G)$ is a zero symmetric near-ring and $M_C(G)$ is a constant near-ring.

Lemma 2.3. *Let* $f : R \longrightarrow S$ *be a near-ring homomorphism. Then the following conditions are true. (1)* $R_0 f \subseteq S_0$*. (2)* $R_c f \subseteq S_c$*.*

Proof. (1) Let $y \in R_0 f$ which is in S. Then there exists $a \in R_0$ such that $y = af$, where $0a = 0$. Thus

$$0y = 0(af) = 0faf = (0a)f = 0f = 0.$$

Hence $y \in S_0$.
(2) Let $y \in R_c f$ which is in S. Then there exists $a \in R_c$ such that $y = af$, where $0a = a$. Thus

$$0y = 0(af) = 0faf = (0a)f = af = y.$$

Hence $y \in S_c$. □

Let $f : R \longrightarrow S$ be a near-ring monomorphism. We know that Rf is a subnear-ring of S, and so $f : R \longrightarrow Rf$ is a near-ring isomorphism. Thus S has an isomorphic copy of R as a subnear-ring. We say that R is embedded into S, and f is an embedding.

Proposition 2.4. *Let* $(R, +, \cdot)$ *be a near-ring and* $(G, +)$ *a group containing* $(R, +)$ *as a proper subgroup. Then* $(M(G), +, \circ)$ *is a unitary near-ring and has a subnear-ring isomorphic to* $(R, +, \cdot)$*. That is, every near-ring can be embedded into a near-ring with identity.*

Proof. For any $a \in R$, we may define a map $f_a \in M(G)$ by

$$xf_a = a,\ if\ x \notin R,\ and\ = xa,\ if\ x \in R.$$

Thus we may obtained the map $\Psi : R \longrightarrow M(G)$ which is defined by $a\Psi = f_a$. We will proceed to show that Ψ is a near-ring monomorphism. For any $a, b \in R$, since $(a+b)\Psi = f_{(a+b)}$ we have that

$$xf_{(a+b)} = a+b,\ if\ x \notin R,\ and\ = x(a+b) = xa + xb,\ if\ x \in R.$$

Also, we have that

$$x(f_a + f_b) = xf_a + xf_b = a+b,\ if\ x \notin R,\ and\ = xa + xb,\ if\ x \in R.$$

Thus

$$(a+b)\Psi = f_{(a+b)} = f_a + f_b = a\Psi + b\Psi.$$

On the other hand, from $(ab)\Psi = f_{(ab)}$ we have that

$$xf_{(ab)} = ab,\ if\ x \notin R,\ and\ = x(ab) = (xa)b,\ if\ x \in R.$$

Also, we have that

$$x(f_a \circ f_b) = (xf_a)f_b = ab,\ if\ x \notin R,\ and\ = (xa)f_b = (xa)b,\ if\ x \in R.$$

Thus

$$(ab)\Psi = f_{(ab)} = f_a \circ f_b = a\Psi \circ b\Psi.$$

Hence Ψ is a near-ring homomorphism from R into $M(G)$.

Next, it remains to show that Ψ is injective. Let $a, b \in R$ with $a \neq b$. We want to show that $a\Psi = f_a \neq f_b = b\Psi$. For $x \notin R$, we know that $xf_a = a \neq b = xf_b$. Consequently, we obtain what we wanted. □

Corollary 2.5. *For any group* $(G,\ +)$*,* $R_0\Psi \subseteq M_0(G)$ *and* $R_c\Psi \subseteq M_c(G) = M(G)_c$*.*

From Corollary 2.5, we obtain the following important result as in ring theory.

Proposition 2.6. *If R is any zero symmetric near-ring, then R can be embedded into some zero symmetric near-ring with identity.*

References

[1] Y. U. Cho *Quotient substructures of R-groups*, J Applied Mathematics and Informatics 28 No 1-2, (2010), 345-349.

[2] J.D.P. Meldrum *Upper faithful D.G. near-rings*, Proc. Edinburgh Math. Soc. 26, (1983), 361-370.

[3] G. Pilz *Near-rings*, , North Holland Publishing Company, Amsterdam, New York, Oxford 1983.

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