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LOCALIZATION OF INJECTIVE MODULES OVER w-NOETHERIAN RINGS

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Abstract. We give some characterizations of injective modules over w-Noetherian rings. It is also shown that each localization of a GV-torsion-free injective module over a w-Noetherian ring is injective.

1. Introduction

One of bad properties of injective modules is that they do not preserve localization in general. However, there are some positive results as follows. If a (commutative) ring R is Noetherian or hereditary, then localizations of injective R-modules are also injective ([10, Theorem 4.88] or [1, Proposition 2]). Recently Couchot investigated localizations of injective modules over valuation rings ([1]) and arithmetical rings ([2]). The purpose of this article is to provide another positive result if R is a w-Noetherian ring.

We first introduce some definitions and notation from [12, 16]. Throughout we let R be a commutative ring with identity. For an R-module M, the dual module $\mathsf{Hom}_R(M, R)$ of M is denoted by $M^\flat$. Following [16, Definition 1.1], an ideal J of a commutative ring R is called a *Glaz-Vasconcelos ideal* or a *GV-ideal*, denoted by $J \in \mathsf{GV}(R)$, if J is finitely generated and the natural homomorphism $\phi : R \to J^\flat$ ($\phi(r)(a) = ra$ for all $r \in R$ and $a \in J$) is an isomorphism. Recall from [16, Definition 1.3] that an R-module M is called a *GV-torsion-free module* if whenever $Jx = 0$ for some $J \in \mathsf{GV}(R)$ and $x \in M$, then $x = 0$. Then it is clear that R is a GV-torsion-free R-module, and that every submodule of a GV-torsion-free module is GV-torsion-free. It is also introduced in [16, Definition 2.1] that a GV-torsion-free R-module M is said to be a *w-module* if, for any $J \in \mathsf{GV}(R)$, $\mathsf{Ext}^1_R(R/J, M) = 0$. Then it is clear that R is a w-module, and that for a GV-torsion-free R-module M, $E(M)$, the injective envelope of M, is a w-module. Note also that the concept of w-modules over commutative rings generalizes that of w-modules over integral

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domains in [7, 13]. Let w-Max(R) denote the set of w-ideals of R maximal among proper w-ideals of R and we call $\mathfrak{m} \in w$-Max(R) a maximal w-ideal of R. Then by [16, Proposition 3.8] every maximal w-ideal is prime. Let M be a GV-torsion-free R-module. Then the *w-envelope* of M is defined by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathsf{GV}(R)\}.$$

It follows from [16, Theorem 2.2] that a GV-torsion-free module M is a w-module if and only if $M_w = M$. So M is a w-ideal when M is an ideal of R with $M_w = M$. We say that a GV-torsion-free module M is said to be *of finite type* if $M_w = N_w$ for some finitely generated submodule N of M. In [15] Wang and Zhang generalized the notion of w-Noetherian modules over commutative rings: A w-module M is called a *w-Noetherian module* if M has the ascending chain condition on w-submodules of M and R is said to be *w-Noetherian* if R itself is a w-Noetherian module. It was shown in [16] that a w-module M is w-Noetherian if and only if every submodule of M is of finite type. If R is an integral domain, then the notion of w-Noetherian rings is the same as that of SM (strong Mori) domains introduced by Wang and McCasland in [14]. It is shown in [16, Corollary 4.4] that if R is a w-Noetherian ring, then $R_{\mathfrak{p}}$ is Noetherian for each prime w-ideal $\mathfrak{p}$ of R. Any undefined terminology is standard, as in [4, 5, 10].

2. Main result

We begin with this section by giving a Baer-like characterization for a w-module.

Theorem 2.1. *Let R be a commutative ring and let E be a w-module over R. Then the following statements are equivalent*:

(1) *E is an injective R-module*;
(2) $\mathsf{Ext}_R^1(R/I, E) = 0$ *for any w-ideal I of R*;
(3) *for any w-ideal I of R, a homomorphism $f : I \to E$ can be extended to R*;
(4) *for any w-submodule A of a w-module B, a homomorphism $f : A \to E$ can be extended to B*;
(5) *for any w-submodule A of a w-module B,* $\mathsf{Ext}_R^1(B/A, E) = 0$;
(6) *for any GV-torsion-free module C,* $\mathsf{Ext}_R^1(C, E) = 0$.

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) Let I be an ideal of R and let $f : I \to E$ be a homomorphism. Since E is a w-module, $\mathsf{Ext}_R^1(I_w/I, E) = 0$ by [16, Theorem 3.6]. From the exact sequence $0 \to I_w/I \to R/I \to R/I_w \to 0$, we have the exact sequence

$$0 = \mathsf{Ext}_R^1(R/I_w, E) \to \mathsf{Ext}_R^1(R/I, E) \to \mathsf{Ext}_R^1(I_w/I, E) = 0.$$

Thus $\mathsf{Ext}_R^1(R/I, E) = 0$, and therefore, E is injective.

(2) $\Leftrightarrow$ (3) This is clear.

The proofs of the other equivalences are easy or similar to those of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). $\square$

Corollary 2.2. *Let E be a w-module. If E is not injective, then there is a w-ideal I of R and a homomorphism $f : I \to E$ such that f cannot be extended to R.*

Let R be an integral domain. Recall that an R-module M is said to be *divisible* if for every nonunit $r \in R$ and for every $m \in M$, the equation $rx = m$ admits a solution $x \in M$.

Lemma 2.3 ([4, Lemma I.7.2])**.** *For a module M over a domain R the following are equivalent*:

(1) *M is divisible*;
(2) *for every $r \in R$, every homomorphism $rR \to M$ can be extended to a homomorphism $R \to M$*;
(3) $\mathsf{Ext}^1_R(R/rR, M) = 0$ *for every $r \in R$*;
(4) $\mathsf{Ext}^1_R(R/I, M) = 0$ *for every invertible ideal I of R.*

It is well known that a domain R is a Dedekind domain if and only if every divisible R-module is injective, and that for a torsion-free module M over an integral domain, M is divisible if and only if M is injective. It is also well known that an integral domain R is a unique factorization domain if and only if every w-ideal of R is principal (cf., [6]). Thus we have the following:

Corollary 2.4. *Let R be a unique factorization domain and let E be a w-module. Then E is injective if and only if E is divisible.*

To give a Cohen-type theorem of Baer's criterion for GV-torsion-free injective modules over w-Noetherian rings, we need the following:

Lemma 2.5. *Let A be a w-submodule of the w-module M. Then for any $x \in M \setminus A$ and $r \in R$, $P_{rx} := \{r' \in R \mid r'(rx) \in A\}$ is a w-ideal of R.*

Proof. For any $r' \in (P_{rx})_w$, there exists a $J \in \mathsf{GV}(R)$ such that $Jr' \subseteq P_{rx}$. Then $Jr'(rx) \subseteq A$. Since A is a w-module, we have $r'(rx) \in A$. So $r' \in P_{rx}$, and hence P_{rx} is a w-ideal of R. $\square$

Theorem 2.6. *Let R be a w-Noetherian ring and let A be a w-submodule of the w-module M. Then for any $x \in M \setminus A$, there exists $r \in R$ such that $P_{rx} = \{r' \in R \mid r'(rx) \in A\}$ is a prime w-ideal of R.*

Proof. Let $\mathscr{C}_x := \{P_{rx} \mid rx \notin A\}$. Then $P_{1x} \in \mathscr{C}_x$, so $\mathscr{C}_x$ is nonempty. By Lemma 2.5, P_{rx} is a w-ideal of R for any $r \in R$. Since R is a w-Noetherian ring, there exists an $r \in R$ such that P_{rx} is maximal in $\mathscr{C}_x$. Since $P_{rx} \in \mathscr{C}_x$, $rx \notin A$, therefore $1 \notin P_{rx}$, that is, $P_{rx} \neq R$. For any $st \in P_{rx}$ with $s \notin P_{rx}$, we have $srx \notin A$. If $y \in P_{rx}$, then $yrx \in A$, and so $ysrx = s(yrx) \in A$. Hence $P_{srx} = P_{rx}$. But $t(srx) = st(rx) \in A$, we get $t \in P_{srx} = P_{rx}$. Therefore, P_{rx} is a prime w-ideal of R. $\square$

Theorem 2.7. *Let R be a w-Noetherian ring and let E be a w-module over R. Then the following statements are equivalent*:

(1) *E is injective*;
(2) *for any prime w-ideal $\mathfrak{p}$ of R, every R-module homomorphism $f:\mathfrak{p}\to E$ can be extended to R*;
(3) *for any prime w-ideal $\mathfrak{p}$ of R, $\mathsf{Ext}_R^1(R/\mathfrak{p},E)=0$.*

Proof. (1) $\Rightarrow$ (2) By Baer's criterion.

(2) $\Rightarrow$ (1) Let A be a w-submodule of a w-module B. Consider the following diagram:

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"] & B \end{tikzcd}$$

Let $\mathscr{C}$ consist of all pairs (A',g'), where $A\subseteq A'\subseteq B$, A' is a w-submodule of B, and $g':A'\to E$ extends f. Note that $\mathscr{C}\neq\emptyset$ for $(A,f)\in\mathscr{C}$. Define a partial order on $\mathscr{C}$ by

$$(A_1,g_1)\le(A_2,g_2) \text{ if and only if } A_1\subseteq A_2 \text{ and } g_2 \text{ extends } g_1.$$

By Zorn's Lemma, there exists a maximal pair (A_0,g_0) in $\mathscr{C}$. If $A_0=B$, then we are done. Now assume that $A_0\neq B$. Then by Theorem 2.6, there exists an $x\in B\setminus A_0$ such that P_x is a prime w-ideal of R. Define $h:P_x\to E$ by $h(r)=g_0(rx)$ for $r\in P_x$. By hypothesis, h can be extended to R. Define $g:A_0+Rx\to E$ by $g(a+rx)=g_0(a)+h(r)$, $a\in A_0$ and $r\in R$. It is routine to verify that g is well-defined. Obviously, g extends g_0. So $(A_0+Rx,g)\in\mathscr{C}$, a contradiction. Therefore E is an injective module.

(2) $\Leftrightarrow$ (3) It is straightforward from the following exact sequence: $0\to\mathsf{Hom}_R(R/\mathfrak{p},E)\to\mathsf{Hom}_R(R,E)\to\mathsf{Hom}_R(\mathfrak{p},E)\to\mathsf{Ext}_R^1(R/\mathfrak{p},E)\to 0$. $\square$

In order to give another proof of Theorem 2.7, which is suggested by the referee, we first introduce notation we need.

Let M be a module and N a submodule of M. Set $N_{(w,M)}=\{x\in M\mid Jx\in N \text{ for some } J\in GV(R)\}$. If $N_{(w,M)}=N$, then N is called *a relative w-submodule* of M. It is clear that if N is a relative w-submodule of M, then M/N is GV-torsion-free, and if $N_{(w,M)}=M$, then M/N is GV-torsion. Moreover, a relative w-submodule of M is not a w-module in general, for example, the (total) GV-torsion submodule of a module M is a relative w-submodule, but is not a w-submodule. When M is a w-module, the relative w-submodules of M are actually w-submodules of M.

Lemma 2.8. *Let M be a finite type module. Then there is a finitely generated submodule N such that $N_{(w,M)}=M$.*

Proof. Since M is of finite type, there is a finitely generated submodule N of M such that M/N is GV-torsion. It is clear that $N_{(w,M)}=M$. $\square$

The following result can be easily obtained.

Theorem 2.9. *Let M be a module. Then M is a w-Noetherian module if and only if M has ACC on relative w-submodules of M.*

Let M be an R-module. A prime ideal $\mathfrak{p}$ of R is called an *associated* prime ideal of M if there exists $x \in M \setminus \{0\}$ such that $\mathfrak{p}$ is a prime ideal minimal over $ann(x)$. As in the domain case, we have that a commutative ring R is a w-Noetherian ring if and only if every finite type w-module is a w-Noetherian module ([15]). We can rewrite [15, Theorem 3.18] as follows:

Theorem 2.10. *Let R be a w-Noetherian ring and let M be a nonzero finite type GV-torsion-free module. Then there is a finite ascending chain of relative w-submodules of M:*

$$0 = N_0 \subset (N_1)_{(w,M)} \subset (N_2)_{(w,M)} \subset \cdots \subset (N_{n-1})_{(w,M)} \subset (N_n)_{(w,M)} = M$$

such that $(N_{i-1})_{(w,M)} \subset N_i$ and $N_i/(N_{i-1})_{(w,M)} \cong R/\mathfrak{p}_i$ for some prime w-ideal $\mathfrak{p}_i$ of R, $i = 1, \ldots, n$.

Proof. Choose an associated prime ideal $\mathfrak{p}_1$ of M. Then $\mathfrak{p}_1$ is a prime w-ideal of R and $\mathfrak{p}_1 = ann(x_1)$ for some $x_1 \in M$. Set $N_1 := Rx_1 \subseteq M$. Then $N_1 = R/\mathfrak{p}_1$. If $(N_1)_{(w,M)} = M$, then we have $0 = N_0 \subset (N_1)_{(w,M)} = M$, as desired. If $(N_1)_{(w,M)} \neq M$, then $M/(N_1)_{(w,M)}$ is a nonzero finite type GV-torsion-free R-module. Take $x_2 \in M \setminus (N_1)_{(w,M)}$ such that $((N_1)_{(w,M)} + Rx_2)/(N_1)_{(w,M)} \cong R/\mathfrak{p}_2$ for some prime w-ideal $\mathfrak{p}_2$ of R. Set $N_2 := (N_1)_{(w,M)} + Rx_2$. If $(N_2)_{(w,M)} = M$, then we have $0 = N_0 \subset (N_1)_{(w,M)} \subset (N_2)_{(w,M)} = M$, as desired. If $(N_2)_{(w,M)} \neq M$, then .... Continuing this process, Theorem 2.9 gives an ascending chain as in the statement of the theorem. $\square$

It was shown in [7, Corollary 3.3] that for a w-module M over a domain R, M is injective if and only if $\mathsf{Ext}^1_R(R/I, M) = 0$ for any w-ideal I of R. This result readily extends to any commutative ring by using [16, Theorem 3.6], and can be strengthened as follows.

Theorem 2.7′. Let R be a w-Noetherian ring and let E be a w-module. Then E is injective if (and only if) $\mathsf{Ext}^1_R(R/\mathfrak{p}, E) = 0$ for any prime w-ideal $\mathfrak{p}$ of R.

Proof. Let M be a GV-torsion-free finitely generated R-module. By Theorem 2.10, there is a finite ascending chain of relative w-submodules of M:

$$0 = N_0 \subset (N_1)_{(w,M)} \subset (N_2)_{(w,M)} \subset \cdots \subset (N_{n-1})_{(w,M)} \subset (N_n)_{(w,M)} = M$$

such that $(N_{i-1})_{(w,M)} \subset N_i$ and $N_i/(N_{i-1})_{(w,M)} \cong R/\mathfrak{p}_i$ for some prime w-ideal $\mathfrak{p}_i$ of R, $i = 1, \ldots, n$. Thus we have $\mathsf{Ext}^1_R(N_1, E) = 0$. Set $M_i := (N_i)_{(w,M)}$. Since $0 \to N_1 \to M_1 \to M_1/N_1 \to 0$ is exact and M_1/N_1 is GV-torsion, we have $\mathsf{Ext}^1_R(M_1, E) = 0$. Since $0 \to M_1 \to N_2 \to N_2/M_1 \to 0$ is exact and $N_2/M_1 \cong R/\mathfrak{p}_2$, we have $\mathsf{Ext}^1_R(N_2, E) = 0$. As $0 \to N_2 \to M_2 \to M_2/N_2 \to 0$ is exact and M_2/N_2 is GV-torsion, we have also $\mathsf{Ext}^1_R(M_2, E) = 0$.

Continuing this process, we have that $\mathsf{Ext}^1_R(M,E)=0$. Therefore E is injective. $\square$

Recall from [12, Definition 1.2] that a sequence $A \to B \to C$ of R-modules and homomorphisms is said to be *w-exact* if the sequence $A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is exact for any maximal w-ideal $\mathfrak{m}$ of R. The following result comes from [12, Theorem 3.4].

Lemma 2.11. *Let $A \to B \to C \to 0$ be a w-exact sequence and let N be a w-module. Then*

$$0 \to \mathsf{Hom}_R(C,N) \to \mathsf{Hom}_R(B,N) \to \mathsf{Hom}_R(A,N)$$

is exact.

Recall from [12] that a GV-torsion-free R-module M is of finite type if and only if there exists a w-exact sequence $F_0 \to M \to 0$, while a GV-torsion-free R-module M is said to be *of finitely presented type* if there is an w-exact sequence $F_1 \to F_0 \to M \to 0$, where F_0 and F_1 are finitely generated free R-modules. A commutative ring R is called a *w-coherent ring* if every finite type ideal of R is of finitely presented type. Then it is shown in [12, Corollary 3.3] that every w-Noetherian ring is w-coherent.

Lemma 2.12. *Let $\mathfrak{p}$ be a prime w-ideal of R. Suppose M is a finitely presented type module and N is a w-module. Then we have*:

(a) *The natural homomorphism*

$$\theta : \mathsf{Hom}_R(M,N)_{\mathfrak{p}} \to \mathsf{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})$$

is an isomorphism.

(b) *If R is w-Noetherian and M is finitely generated, then the induced homomorphism*

$$\bar{\theta} : \mathsf{Ext}^1_R(M,N)_{\mathfrak{p}} \to \mathsf{Ext}^1_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})$$

is an isomorphism.

Proof. (a) Because M is a finitely presented type R-module, there exist finitely generated free R-modules F_0 and F_1 such that $F_1 \to F_0 \to M \to 0$ is a w-exact sequence. Then

$$(F_1)_{\mathfrak{p}} \to (F_0)_{\mathfrak{p}} \to M_{\mathfrak{p}} \to 0$$

is exact. By Lemma 2.11, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_R(M,N)_{\mathfrak{p}} \arrow[r] \arrow[d, "\theta"] & \mathsf{Hom}_R(F_0,N)_{\mathfrak{p}} \arrow[r] \arrow[d, "\cong"] & \mathsf{Hom}_R(F_1,N)_{\mathfrak{p}} \arrow[d, "\cong"] \\
0 \arrow[r] & \mathsf{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}) \arrow[r] & \mathsf{Hom}_{R_{\mathfrak{p}}}((F_0)_{\mathfrak{p}},N_{\mathfrak{p}}) \arrow[r] & \mathsf{Hom}_{R_{\mathfrak{p}}}((F_1)_{\mathfrak{p}},N_{\mathfrak{p}}).
\end{tikzcd}$$

So θ is an isomorphism by the Five Lemma. From the argument above we remark that if M is of finite type, then θ is a monomorphism.

For (b), we assume first that M is finitely generated. Let $0 \to A \to F \to M \to 0$ be exact, where F is finitely generated free. Thus A is of finite type. Since R is w-Noetherian, A is of finitely presented. Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathsf{Hom}_R(F,N)_\mathfrak{p} \arrow[r] \arrow[d, "\cong"] & \mathsf{Hom}_R(A,N)_\mathfrak{p} \arrow[r] \arrow[d, "\theta_A"] & \mathsf{Ext}^1_R(M,N)_\mathfrak{p} \arrow[r] \arrow[d, "\bar{\theta}"] & 0 \\
\mathsf{Hom}_{R_\mathfrak{p}}(F_\mathfrak{p},N_\mathfrak{p}) \arrow[r] & \mathsf{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p},N_\mathfrak{p}) \arrow[r] & \mathsf{Ext}^1_{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p}) \arrow[r] & 0
\end{tikzcd}$$

Note that θ_A is an isomorphism. Hence $\bar{\theta}$ is an isomorphism. □

Theorem 2.13. *Let R be a w-Noetherian ring. If E is a GV-torsion-free injective R-module, then $E_\mathfrak{p}$ is an injective $R_\mathfrak{p}$-module for any maximal w-ideal $\mathfrak{p}$ of R.*

Proof. Suppose E is injective over R. Let X be a finitely generated $R_\mathfrak{p}$-module. Then we can write $X = M_\mathfrak{p}$ for some finitely generated R-module M. Since R is w-Noetherian, M is of finitely presented type. By Lemma 2.12, we have that

$$\mathsf{Ext}^1_{R_\mathfrak{p}}(X, E_\mathfrak{p}) = \mathsf{Ext}^1_{R_\mathfrak{p}}(M_\mathfrak{p}, E_\mathfrak{p}) = \mathsf{Ext}^1_R(M, E)_\mathfrak{p} = 0.$$

Hence $E_\mathfrak{p}$ is injective over $R_\mathfrak{p}$. □

In the previous version of this paper it is shown that each localization of a GV-torsion-free injective module over a coherent SM domain is injective by using the same arguments as those of [2, Lemma 9 and Theorem 10]. As the referee suggests, Theorem 2.13 and the result just mentioned above can be better improved by proving that each localization of any GV-torsion-free injective R-module is injective if R is w-Noetherian. First it is necessary to show the following result, which is the w-theoretic analogue of the Bass-Matlis-Papp Theorem for Noetherian rings (cf., [9]):

Theorem 2.14. *The following conditions are equivalent for a commutative ring R :*

(1) *R is a w-Noetherian ring*;
(2) *each direct sum of GV-torsion-free injective R-modules is injective*;
(3) *each GV-torsion-free injective R-module is a direct sum of indecomposable GV-torsion-free injective R-modules.*

Proof. (1) ⇔ (2) [15, Theorem 4.4] (or [8, Theorem 2.9]).

(1) ⇒ (3) [15, Theorem 4.5(4)] (or [8, Corollary 2.8]).

(3) ⇒ (2) This follows by an easy modification of the proof of [11, Theorem 2]. However, for the sake of completeness we give its proof here. Let $\{E_i\}_{i\in I}$ be a family of GV-torsion-free injective R-modules, where I is an index set and let $F := E(\bigoplus_{i\in I} E_i)$. Then by hypothesis, $F = \bigoplus_{j\in J} F_j$, where each F_j is an indecomposable GV-torsion-free injective R-module and J is an index set.

Since $\bigoplus_{i\in I} E_i$ is an essential extension of F, it has a nonzero intersection with each F_j. Hence $B_j := F_j \cap (E_{i_1} \oplus \cdots \oplus E_{i_n}) \neq 0$ for some $i_1, \ldots, i_n$ (depending on j). But F_j is indecomposable, so $F_j = E(B_j) \subseteq E_{i_1} \oplus \cdots \oplus E_{i_n}$. It follows that $\bigoplus_{i\in I} E_i = F$ is injective. $\square$

There is an example of a non-Noetherian (coherent) SM domain: Let K be a field and $\{X_\alpha\}$ be a countably infinite set of indeterminates. Then the polynomial ring $R := K[\{X_\alpha\}]$ is a (coherent) SM domain which is not Noetherian. It follows from [3, Corollary 17] that the localization E_S of any injective R-module E is an injective R_S-module. Thus the following result is a generalization of the Noetherian case, and shows that one aspect of the previous example carries over to the most general case.

Theorem 2.15. *Let R be a w-Noetherian ring. Then each localization of any GV-torsion-free injective R-module is injective.*

Proof. Let E' be an injective w-module over R and S a multiplicative subset of R. Since R_S is flat over R, a module over R_S is injective if and only if it is injective over R. So, since E' is a direct sum of indecomposable modules of the form $E(R/\mathfrak{p})$ where $\mathfrak{p}$ is a prime w-ideal (by Theorem 2.14 and [15, Theorem 4.5(1)]), it is enough to show that E_S is injective over R if $E := E(R/\mathfrak{p})$. Since E is a module over $R_\mathfrak{p}$, if S' is the image of S by the natural map $R \to R_\mathfrak{p}$, then $E_S \cong E_{S'}$. Since $R_\mathfrak{p}$ is Noetherian, $E_{S'}$ is injective over $R_\mathfrak{p}$, whence E_S is injective over R. $\square$

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