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FILTERS OF BE-ALGEBRAS BASED ON FUZZY SET THEORY

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Abstract. The fuzzification of filters in BE-algebras is discussed. Characterizations of a fuzzy filter of BE-algebras are given. An answer to the following question is provided.
Question. Let Λ be a subset of $[0, 1]$ containing a maximal element $m > 0$ and let $\mathscr{F} := \{F_\alpha \mid \alpha \in \Lambda\}$ be a decreasing chain of filters of a BE-algebra X. Then does there exist a fuzzy filter μ of X such that $\mu(X) = \Lambda$ and $\mathscr{F}_\mu = \mathscr{F}$?

1. Introduction

As a generalization of a BCK-algebra, Kim et al. [2] introduced the notion of a BE-algebra, and using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. Ahn et al. [1] dealt with ideal theory in BE-algebras. Walendziak [3] investigated the relationship between BE-algebras, implicative algebras, and J-algebras. He defined commutative BE-algebras, and showed that commutative BE-algebras are equivalent to the commutative dual BCK-algebras.

The goal of this paper is to consider the fuzzification of filters in BE-algebras. We discuss characterizations of fuzzy filter, and provide conditions for a fuzzy set to be a fuzzy filter. We pose a question and answer to this question.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BE-algebra* we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms hold (see [2]):

(2.1) $$(\forall x \in X)\,(x * x = 1);$$

(2.2) $$(\forall x \in X)\,(x * 1 = 1);$$

(2.3) $$(\forall x \in X)\,(1 * x = x);$$

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$$(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)) \quad \text{(exchange)}. \tag{2.4}$$

A relation "$\leq$" on a BE-algebra X is defined by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 1). \tag{2.5}$$

A BE-algebra $(X; *, 1)$ is said to be *transitive* (see [1]) if it satisfies:

$$(\forall x, y, z \in X)\,(y * z \leq (x * y) * (x * z)). \tag{2.6}$$

A BE-algebra $(X; *, 1)$ is said to be *self distributive* (see [2]) if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = (x * y) * (x * z)). \tag{2.7}$$

Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [1]).

A nonempty subset F of a BE-algebra X is called a *filter* of X (see [2]) if it satisfies:

$$1 \in F, \tag{2.8}$$

$$(\forall x \in F)(\forall y \in X)\,(x * y \in F \;\Rightarrow\; y \in F). \tag{2.9}$$

For any fuzzy set μ in a set X, the range (also called image) of μ, denoted by $\mu(X)$, is the set

$$\mu(X) = \{\mu(x) \mid x \in X\}. \tag{2.10}$$

The level sets of the fuzzy set μ are denoted by $U(\mu; \alpha)$, $\alpha \in [0, 1]$, and are given by

$$U(\mu; \alpha) = \{x \in X \mid \mu(x) \geq \alpha\} = \mu^{-1}[\alpha, 1]. \tag{2.11}$$

The collection of all level sets corresponding to the range $\mu(X)$ of μ is denoted by $\mathscr{F}_\mu$ and is given by

$$\mathscr{F}_\mu := \{U(\mu; \alpha) \mid \alpha \in \mu(X)\}. \tag{2.12}$$

Throughout the paper, a partially ordered set, *poset*, $(P, \leq)$, is a nonempty set P endowed with a reflexive, anti-symmetric and transitive relation $\leq$. A poset is often denoted by the underlying set P only. If more than one poset is considered, we denote all (generally different) ordering relations by the same symbol $\leq$. If $(P, \leq)$ and $(Q, \leq)$ are two posets, then the map $f : P \to Q$ is said to be *isotone* if it preserves the order, i.e.,

$$(\forall p, q \in P)(p \leq q \;\Rightarrow\; f(p) \leq f(q)). \tag{2.13}$$

The map $f : P \to Q$ is said to be *anti-isotone* if

$$(\forall p, q \in P)(p \leq q \;\Rightarrow\; f(q) \leq f(p)). \tag{2.14}$$

$(P, \leq)$ and $(Q, \leq)$ are said to be *isomorphic* if there is a bijection $f : P \to Q$ such that f and f^{-1} are isotone. If f is a bijection and f and f^{-1} are anti-isotone, then P and Q are said to be *anti-isomorphic*.

3. Fuzzy Filters

In what follows, let X denote a BE-algebra unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called a *fuzzy filter* of X if it satisfies:

$$(\forall x \in X)\,(\mu(1) \geq \mu(x)), \tag{3.1}$$

$$(\forall x, y \in X)\,(\mu(y) \geq \min\{\mu(x * y), \mu(x)\}). \tag{3.2}$$

Theorem 3.2. *Let μ be a fuzzy set in X. Then μ is a fuzzy filter of X if and only if it satisfies:*

$$(\forall \alpha \in [0, 1])(U(\mu; \alpha) \neq \emptyset \;\Rightarrow\; U(\mu; \alpha) \text{ is a filter of } X). \tag{3.3}$$

We call $U(\mu; \alpha)$ a *level filter.*

Proof. Assume that μ is a fuzzy filter of X. Let $\alpha \in [0, 1]$ be such that $U(\mu; \alpha) \neq \emptyset$. Then there exists $a \in U(\mu; \alpha)$, and so $\mu(1) \geq \mu(a) \geq \alpha$ by (3.1). Hence $1 \in U(\mu; \alpha)$. Let $x, y \in X$ be such that $x * y \in U(\mu; \alpha)$ and $x \in U(\mu; \alpha)$. Then $\mu(x * y) \geq \alpha$ and $\mu(x) \geq \alpha$. It follows from (3.2) that

$$\mu(y) \geq \min\{\mu(x * y), \mu(y)\} \geq \alpha$$

so that $y \in U(\mu; \alpha)$. Therefore $U(\mu; \alpha)$ is a filter of X.

Conversely, suppose μ satisfies (3.3). If μ does not satisfy (3.1), then $\mu(1) < \mu(a)$ for some $a \in X$. It follows that there exist $\alpha_0 \in (0, 1)$ such that $\mu(1) < \alpha_0 \leq \mu(a)$ so that $a \in U(\mu; \alpha_0)$ and $1 \notin U(\mu; \alpha_0)$. Hence $U(\mu; \alpha_0)$ is a filter of X which does not contain 1. This is a contradiction. Therefore (3.1) is valid. Now suppose that there exist $a, b \in X$ such that $\mu(b) < \min\{\mu(a * b), \mu(a)\}$. Then $\mu(b) < \beta \leq \min\{\mu(a * b), \mu(a)\}$ for some $\beta \in (0, 1)$, and so $a * b \in U(\mu; \beta)$ and $a \in U(\mu; \beta)$, but $b \notin U(\mu; \beta)$. This is a contradiction. Therefore (3.2) is true. Thus μ is a fuzzy filter of X. □

Example 3.3. Let $X = \{1, a, b, c, d, 0\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra (see [2]).

(1) Let μ be a fuzzy set in X defined by

$$\mu(x) := \begin{cases} 0.7 & \text{if } x \in \{1, a, b\}, \\ 0.2 & \text{if } x \in \{c, d, 0\}. \end{cases}$$

Then

$$U(\mu; \alpha) = \begin{cases} \emptyset & \text{if } \alpha \in (0.7, 1], \\ \{1, a, b\} & \text{if } \alpha \in (0.2, 0.7], \\ X & \text{if } \alpha \in [0, 0.2]. \end{cases}$$

Note that $\{1, a, b\}$ and X are filters of X, and so μ is a fuzzy filter of X.

(2) Let ν be a fuzzy set in X defined by

$$\nu(x) := \begin{cases} 0.6 & \text{if } x \in \{1, a\}, \\ 0.4 & \text{if } x \in \{b, c, d, 0\}. \end{cases}$$

Then

$$U(\nu; \beta) = \begin{cases} \emptyset & \text{if } \beta \in (0.6, 1], \\ \{1, a\} & \text{if } \beta \in (0.4, 0.6], \\ X & \text{if } \alpha \in [0, 0.4]. \end{cases}$$

Note that $\{1, a\}$ is not a filter of X since $a * b \in \{1, a\}$ and $a \in \{1, a\}$, but $b \notin \{1, a\}$. Hence ν is not a fuzzy filter of X.

For any $a, b \in X$, the set

$$A(a, b) := \{x \in X \mid a * (b * x) = 1\}$$

is called the *upper set* of a and b (see [2]). Clearly, $1, a, b \in A(a, b)$ for all $a, b \in X$ (see [2]). Note that $A(a, b)$ is not a filter of X in general (see [2]).

For every $a, b \in X$, let μ_a^b be a fuzzy set in X defined by

$$\mu_a^b(x) := \begin{cases} \alpha & \text{if } a * (b * x) = 1, \\ \beta & \text{otherwise} \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$.

The following example shows that there exist $a, b \in X$ such that μ_a^b is not a fuzzy filter of X.

Example 3.4. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.3. Then μ_1^a is not a fuzzy filter of X since

$$U(\mu_1^a;\delta) = \begin{cases} \emptyset & \text{if } \delta \in (\alpha, 1], \\ \{1, a\} & \text{if } \delta \in (\beta, \alpha], \\ X & \text{if } \delta \in [0, \beta] \end{cases}$$

in which $\{1, a\}$ is not a filter of X.

We provide conditions for a fuzzy set μ_a^b in X to be a fuzzy filter of X.

Theorem 3.5. *Let X be a BE-algebra. If $b \in X$ satisfies $b * c = 1$ for all $c \in X$, then μ_a^b and μ_b^a are fuzzy filters of X for all $a \in X$.*

Proof. Note that $a * (b * x) = b * (a * x) = 1$ for all $a, x \in X$. Hence it is clear thatμ_a^b and μ_b^a are fuzzy filters of X for all $a \in X$. □

Theorem 3.6. *If X is self distributive, then the fuzzy set μ_a^b in X is a fuzzy filter of X for all $a, b \in X$.*

Proof. Since $a * (b * 1) = 1$ for all $a, b \in X$, we have $\mu_a^b(1) = \alpha \geq \mu_a^b(x)$ for all $x \in X$. Let $x, y \in X$. If $a * (b * (x * y)) \neq 1$ or $a * (b * x) \neq 1$, then $\mu_a^b(x * y) = \beta = \mu_a^b(x)$, and so

$$\mu_a^b(y) \geq \beta = \min\{\mu_a^b(x * y), \mu_a^b(x)\}.$$

Assume that $a * (b * (x * y)) = 1$ and $a * (b * x) = 1$. Then $\mu_a^b(x * y) = \alpha = \mu_a^b(x)$. Using (2.3) and the self distributivity of X, we get

$$\begin{aligned} a * (b * y) &= 1 * (a * (b * y)) = (a * (b * x)) * (a * (b * y)) \\ &= a * ((b * x) * (b * y)) = a * (b * (x * y)) = 1. \end{aligned}$$

Hence $\mu_a^b(y) = \alpha = \min\{\mu_a^b(x * y), \mu_a^b(x)\}$, and therefore μ_a^b is a fuzzy filter of X for all $a, b \in X$. □

We pose a question as follows:

Question. Does any fuzzy set μ of X satisfy the following assertion?

(3.4) $$(\forall a, b \in X)(\forall \alpha \in [0, 1])(U(\mu;\alpha) \neq \emptyset \Rightarrow A(a, b) \subseteq U(\mu;\alpha)).$$

The following example gives a negative answer to the above question.

Example 3.7. Consider $X = \{1, a, b, c, d, 0\}$ and ν as in Example 3.3(2). Since $\nu(b) = 0.4 < 0.6 = \nu(a) = \min\{\nu(a * b), \nu(a)\}$, ν is not a fuzzy filter of X. Put $\alpha := 0.6$. Then $A(a, b) = \{1, a, b\}$ and $U(\nu; 0.6) = \{1, a\}$. But $A(a, b) \nsubseteq U(\nu; 0.6)$.

Theorem 3.8. *Let μ be a fuzzy set in X. Then μ is a fuzzy filter of X if and only if it satisfies the following assertion:*

$$(\forall a, b \in X)\,(\forall \alpha \in [0,1])(a, b \in U(\mu;\alpha) \;\Rightarrow\; A(a,b) \subseteq U(\mu;\alpha)). \tag{3.5}$$

Proof. Assume that μ is a fuzzy filter of X and let $a, b \in U(\mu;\alpha)$. Then $\mu(a) \geq \alpha$ and $\mu(b) \geq \alpha$. Let $x \in A(a,b)$. Then $a * (b * x) = 1$. Hence

$$\begin{aligned} \mu(x) &\geq \min\{\mu(b * x), \mu(b)\} \\ &\geq \min\{\min\{\mu(a * (b * x)), \mu(a)\}, \mu(b)\} \\ &= \min\{\mu(1), \mu(a), \mu(b)\} \\ &= \min\{\mu(a), \mu(b)\} \;\geq\; \alpha. \end{aligned}$$

Thus $x \in U(\mu;\alpha)$, and so $A(a,b) \subseteq U(\mu;\alpha)$.

Conversely, suppose μ satisfies (3.5). Note that $1 \in A(a,b) \subseteq U(\mu;\alpha)$ for all $a, b \in X$ and $\alpha \in [0,1]$. Let $x, y \in X$ be such that $x * y \in U(\mu;\alpha)$ and $x \in U(\mu;\alpha)$. Since $(x * y) * (x * y) = 1$, we have

$$y \in A(x * y, x) \subseteq U(\mu;\alpha).$$

Hence $U(\mu;\alpha)$ is a filter of X, and so μ is a fuzzy filter of X by Theorem 3.2. □

Corollary 3.9. *Let μ be a fuzzy set in X such that*

$$(\forall a, b \in X)\,(\forall \alpha \in [0,1])(A(a,b) \subseteq U(\mu;\alpha)). \tag{3.6}$$

Then μ is a fuzzy filter of X.

Proof. Since $a, b \in A(a,b) \subseteq U(\mu;\alpha)$ for all $a, b \in X$, it follows from Theorem 3.8. □

Theorem 3.10. *If μ is a fuzzy filter of X, then*

$$(\forall \alpha \in [0,1])\left(U(\mu;\alpha) \neq \emptyset \;\Rightarrow\; U(\mu;\alpha) = \bigcup_{a,b \in U(\mu;\alpha)} A(a,b)\right). \tag{3.7}$$

Proof. Let $\alpha \in [0,1]$ be such that $U(\mu;\alpha) \neq \emptyset$. Since $1 \in U(\mu;\alpha)$, we have

$$U(\mu;\alpha) \subseteq \bigcup_{a \in U(\mu;\alpha)} A(a,1) \subseteq \bigcup_{a,b \in U(\mu;\alpha)} A(a,b).$$

Now, let $x \in \bigcup_{a,b \in U(\mu;\alpha)} A(a,b)$. Then there exist $u, v \in U(\mu;\alpha)$ such that $x \in A(u,v) \subseteq U(\mu;\alpha)$ by Theorem 3.8. Thus $\bigcup_{a,b \in U(\mu;\alpha)} A(a,b) \subseteq U(\mu;\alpha)$. This completes the proof. □

Corollary 3.11. *If μ is a fuzzy filter of X, then*

$$(\forall \alpha \in [0,1]) \left(U(\mu;\alpha) \neq \emptyset \;\Rightarrow\; U(\mu;\alpha) = \bigcup_{a \in U(\mu;\alpha)} A(a,1) \right). \tag{3.8}$$

Proof. Straightforward. □

The collection of level filters corresponding to the range $\mu(X)$ of the fuzzy filter μ is denoted by $\mathscr{F}_\mu$.

We have the following question:

Question. Let Λ be a subset of $[0,1]$ containing a maximal element $m > 0$ and let $\mathscr{F} := \{F_\alpha \mid \alpha \in \Lambda\}$ be a decreasing chain of filters of a BE-algebra X. Then does there exist a fuzzy filter μ of X such that $\mu(X) = \Lambda$ and $\mathscr{F}_\mu = \mathscr{F}$?

We will give an answer to the above question.

Denote by $FF(X)$ the set of all fuzzy filters of X, that is,

$$FF(X) := \{\text{Fuzzy filters of } X\}. \tag{3.9}$$

Note that the collection $\mathscr{F}_\mu$ is a chain of filters in the sense that it is totally ordered by inclusion.

Example 3.12. Let $X = \{1, a, b, c, d\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	c
b	1	1	1	c	c
c	1	a	b	1	a
d	1	1	a	1	1

Then $(X; *, 1)$ is a BE-algebra. Let μ be a fuzzy set in X given by

	1	a	b	c	d
μ	0.8	0.6	0.4	0.2	0.2

Then $\mu \in FF(X)$, $\mu(X) = \{0.8, 0.6, 0.4, 0.2\}$ and

$$\mathscr{F}_\mu = \{U(\mu;0.8), U(\mu;0.6), U(\mu;0.4), U(\mu;0.2)\},$$

which is a chain because $U(\mu;0.8) \subseteq U(\mu;0.6) \subseteq U(\mu;0.4) \subseteq U(\mu;0.2)$.

Theorem 3.13. *If $\mu \in FF(X)$, then $(\mathscr{F}_\mu, \subseteq)$ and $(\mu(X), \leq)$ are anti-isomorphic.*

Proof. Let $f : \mu(X) \to \mathscr{F}_\mu$ be defined by $f(\alpha) = U(\mu;\alpha)$ for all $\alpha \in \mu(X)$. Obviously, f is a bijection, and f and f^{-1} are anti-isotone. Hence $(\mathscr{F}_\mu, \subseteq)$ and $(\mu(X), \leq)$ are anti-isomorphic. □

Let $\mu \in FF(X)$. Since $\mu(1) \geq \mu(x)$ for all $x \in X$, we have

$$\bigcap_{\alpha\in\mu(X)} U(\mu;\alpha) = U(\mu;\mu(1)) \in \mathscr{F}_\mu. \tag{3.10}$$

Note that $X \in \mathscr{F}_\mu$ if and only if $\inf(\mu(X)) \in \mu(X)$, in this case we obtain

$$X = \bigcup_{\alpha\in\mu(X)} U(\mu;\alpha) = U(\mu;\inf(\mu(X))) \in \mathscr{F}_\mu. \tag{3.11}$$

Lemma 3.14. *For any* $\mu \in FF(X)$ *and* $\alpha \in \mu(X)$, *we have*

$$\bigcup_{\beta\in(\alpha,1]\cap\mu(X)} U(\mu;\beta) \subsetneq U(\mu;\alpha). \tag{3.12}$$

Proof. Clearly, we have

$$\bigcup_{\beta\in(\alpha,1]\cap\mu(X)} U(\mu;\beta) \subseteq U(\mu;\alpha). \tag{3.13}$$

Since $\alpha \in \mu(X)$ and $U(\mu;\alpha) = \mu^{-1}[\alpha, 1]$, there exists $x \in U(\mu;\alpha)$ such that $\mu(x) = \alpha$. If $\beta \in (\alpha, 1]$, then obviously $x \notin U(\mu;\beta) = \mu^{-1}[\beta, 1]$. Hence (3.12) is valid. □

The following example shows that there exist a BE-algebra with a decreasing chain of filters (indexed by subsets of $[0, 1]$) not satisfying the equality in (3.12).

Example 3.15. Consider $X = \{1, a, b, c, d\}$ and μ as in Example 3.12. Put $\alpha := 0.4$. Since $\beta \in (0.4, 1] \cap \mu(X)$, we have $\beta \in \{0.6, 0.8\}$. Hence $\bigcup_{\beta\in(0.4,1]\cap\mu(X)} U(\mu;\beta) = \{1, a\}$. But $U(\mu; 0.4) = \{1, a, b\}$. Therefore $\bigcup_{\beta\in(0.4,1]\cap\mu(X)} U(\mu;\beta) \subsetneq U(\mu; 0.4)$.

Lemma 3.16. *Let* Λ *be a subset of* $[0, 1]$ *containing a maximal element* $m > 0$ *and let* $\mathscr{F} := \{F_\alpha \mid \alpha \in \Lambda\}$ *be a decreasing chain of filters of* X. *For any* $\mu \in FF(X)$ *such that* $\mu(X) = \Lambda$ *and* $\mathscr{F}_\mu = \mathscr{F}$, *we have*

$$\bigcup_{\gamma\in(\alpha,1]\cap\Lambda} U(\mu;\gamma) = \bigcup_{\gamma\in(\beta,1]\cap\Lambda} F_\gamma \tag{3.14}$$

whenever $U(\mu;\alpha) = F_\beta$ *for some* $\alpha, \beta \in \Lambda$, *and*

$$\{\mu^{-1}(\alpha) \mid \alpha \in \Lambda\} = \left\{ F_\delta \setminus \bigcup_{\gamma\in(\delta,1]\cap\Lambda} F_\gamma \mid \delta \in \Lambda \right\}. \tag{3.15}$$

Proof. Assume that $U(\mu;\alpha) = F_\beta$ for some $\alpha, \beta \in \Lambda$. Note that $\mathscr{F}_\mu = \mathscr{F}$ and they are decreasing chains, and so we must have $U(\mu; m) = F_m$. Since $U(\mu;\alpha) = F_\beta$ for some $\alpha, \beta \in \Lambda$, it follows that either $\alpha = \beta = m$ or $\alpha, \beta < m$. If $\alpha = \beta = m$, then

$$\{U(\mu;\gamma) \mid \gamma \in (\alpha, 1] \cap \Lambda\} = \emptyset = \{F_\gamma \mid \gamma \in (\beta, 1] \cap \Lambda\}$$

and thus (3.14) is valid. Now we assume that $U(\mu;\alpha) = F_\beta$ for some $\alpha, \beta \in \Lambda \setminus \{m\}$. Then

$$\begin{aligned}\{U(\mu;\gamma) \mid \gamma \in (\alpha, 1] \cap \Lambda\} &= \{U(\mu;\gamma) \mid \gamma \in \Lambda, U(\mu;\gamma) \subsetneq U(\mu;\alpha)\} \\ &= \{F_\gamma \mid \gamma \in \Lambda, F_\gamma \subsetneq F_\beta\} \\ &= \{F_\gamma \mid \gamma \in (\beta, 1] \cap \Lambda\},\end{aligned}$$

which induces (3.14). Since $\mathscr{F}_\mu = \mathscr{F}$, for every $\alpha \in \Lambda$ there exists $\delta \in \Lambda$ such that $U(\mu;\alpha) = F_\delta$. It follows from (3.14) that

$$\bigcup_{\gamma \in (\alpha,1] \cap \Lambda} U(\mu;\gamma) = \bigcup_{\gamma \in (\delta,1] \cap \Lambda} F_\gamma$$

so that

$$\mu^{-1}(\alpha) = U(\mu;\alpha) \setminus \bigcup_{\gamma \in (\alpha,1] \cap \Lambda} U(\mu;\gamma) = F_\delta \setminus \bigcup_{\gamma \in (\delta,1] \cap \Lambda} F_\gamma.$$

Hence

$$\{\mu^{-1}(\alpha) \mid \alpha \in \Lambda\} \subseteq \left\{ F_\delta \setminus \bigcup_{\gamma \in (\delta,1] \cap \Lambda} F_\gamma \mid \delta \in S \right\}. \tag{3.16}$$

Similarly we prove that for every $\delta \in \Lambda$, there exists $\alpha \in \Lambda$ such that

$$F_\delta \setminus \bigcup_{\gamma \in (\delta,1] \cap \Lambda} F_\gamma = \mu^{-1}(\alpha).$$

Hence

$$\left\{ F_\delta \setminus \bigcup_{\gamma \in (\delta,1] \cap \Lambda} F_\gamma \mid \delta \in \Lambda \right\} \subseteq \{\mu^{-1}(\alpha) \mid \alpha \in \Lambda\}. \tag{3.17}$$

Combining (3.16) and (3.17) induces (3.15). □

4. Conclusions

We have introduced the notion of a fuzzy filter in a BE-algebra, and have discussed its characterization. We have answered to the following question:

Question. Let Λ be a subset of $[0, 1]$ containing a maximal element $m > 0$ and let $\mathscr{F} := \{F_\alpha \mid \alpha \in \Lambda\}$ be a decreasing chain of filters of a BE-algebra X. Then does there exist a fuzzy filter μ of X such that $\mu(X) = \Lambda$ and $\mathscr{F}_\mu = \mathscr{F}$?

Based on the idea and results in this article, we will consider the following theorem:

Theorem 4.1. *Let S be a subset of $[0,1]$ containing a maximal element $m > 0$ and let $\mathscr{F} := \{F_t \mid t \in S\}$ be a decreasing chain of filters of X. Then there exists a fuzzy filter μ of X satisfying $\mu(X) = S$ and $\mathscr{F}_\mu = \mathscr{F}$ if and only if the following conditions hold:*

(1) *For every $t \in S$,*

$$\bigcup_{r\in(t,1]\cap S} F_r \subsetneq F_t. \tag{4.1}$$

(2) *The BE-algebra X is the disjoint union*

$$X = \bigcup_{t\in S}\left(F_t \setminus \bigcup_{r\in(t,1]\cap S} F_r\right). \tag{4.2}$$

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