KYUNGPOOK Math. J. 53(2013), 625-638
http://dx.doi.org/10.5666/KMJ.2013.53.4.625

Extreme Bilinear Forms of $\mathcal{L}(^2d_*(1,w)^2)$

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Abstract. First we present the explicit formula for the norm of a bilinear form on the 2-dimensional real predual of the Lorentz sequence space $d_*(1,w)^2$. Using this formula, we classify the extreme points of the unit ball of $\mathcal{L}(^2d_*(1,w)^2)$.

1. Introduction

Let $n \in \mathbb{N}$. We write B_E and S_E for the closed unit ball and sphere of a real Banach space E respectively and the dual space of E is denoted by E^*. A unit vector x in E is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. We denote by $extB_E$ the sets of all the extreme points of B_E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \cdots, x_n)|$. A n-linear form T is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation σ on $\{1, 2, \ldots, n\}$. We denote by $\mathcal{L}_s(^nE)$ the Banach space of all continuous symmetric n-linear forms on E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists $T \in \mathcal{L}_s(^nE)$ such that $P(x) = T(x, \cdots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7]. We will denote by $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$ and $P(x, y) = ax^2 + by^2 + cxy$ a bilinear form and a 2-homogeneous polynomial on a real Banach space of dimension 2 respectively.

Since 1998, many authors have been developing the problem of characterizing extreme points of the unit balls of $\mathcal{P}(^nE)$ for some classical real Banach spaces. Choi, Ki and the author [2, Theorem 2.4] showed that a sufficient and necessary

Received November 12, 2012; Revised March 25, 2013; accepted June 20, 2013.
2010 Mathematics Subject Classification: Primary 46A22.
Key words and phrases: extreme bilinear forms, the 2-dimensional real predual of the Lorentz sequence space.
This research was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A2057788).

condition on the coefficients a, b and c for $P(x, y)$ defined on the real space l_1^2 to have norm 1, is,

(i) ($|a| = 1$ or $|b| = 1$) and $|c| \leq 2$

or

(ii) $|a| < 1,\ |b| < 1,\ 2 < |c| \leq 4$ and $4|c| - c^2 = 4(|a + b| - ab)$.

It was also proved in [2, Theorem 2.6] that $P \in extB_{\mathcal{P}(^2l_1^2)}$ if and only if

$$(|a| = |b| = 1,\ |c| = 2) \text{ or } a = -b,\ 2 < |c| \leq 4,\ 4a^2 = 4|c| - c^2.$$

Choi and the author [3, Theorem 2.2] showed that $P \in extB_{\mathcal{P}(^2l_2^2)}$ if and only if

$$(|a| = |b| = 1,\ |c| = 0) \text{ or } a = -b,\ 0 < |c| \leq 2,\ 4a^2 = 4 - c^2.$$

Later, B. Grecu [9] classified the sets $extB_{\mathcal{P}(^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. We denote the 2-dimensional real predual of the Lorentz sequence space with a positive weight $0 < w < 1$ by

$$d_*(1, w)^2 := \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_{d_*} := \max\{|x|,\ |y|,\ \frac{|x| + |y|}{1 + w}\ \}\ \}.$$

Recently, the author [13] characterize the extreme points of the unit ball of $\mathcal{P}(^2d_*(1, w)^2)$. In fact, we show that the extreme points of the unit ball of $\mathcal{P}(^2d_*(1, w)^2)$ are

$$\pm x^2,\ \pm y^2,\ \pm \frac{1}{1 + w^2}(x^2 + y^2),\ \pm \frac{1}{(1 + w)^2}(x^2 + y^2 \pm 2xy),$$

$$\pm\{ax^2 - ay^2\ \pm\ 2\sqrt{a(1 - a)}xy\ \}(\forall \frac{1}{1 + w^2} \leq a \leq 1),$$

$$\pm[ax^2 - ay^2 \pm \{\frac{2}{(1 + w)^2} + 2\sqrt{\frac{1}{(1 + w)^4} - a^2}\}xy](\forall 0 \leq a \leq \frac{1 - w}{(1 + w)(1 + w^2)}).$$

Notice that $\mathcal{P}(^nE)$ and $\mathcal{L}(^nE)$ are not isometric in general. It is natural to ask the following question: what are extreme points of the unit ball of $\mathcal{L}(^nE)$?

In 2009, the author [12] started the study of characterizing extreme points of the unit balls of $\mathcal{L}_s(^nE)$ and classified the extreme points of the unit ball of $\mathcal{L}_s(^2l_\infty^2)$. Very recently, the author [14] characterize the extreme points of the unit ball of $\mathcal{L}_s(^2d_*(1, w)^2)$.

We refer to ([1–6], [8–20] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

Continuing the problem of characterizing extreme points of the unit balls of $\mathcal{L}(^nE)$, in this paper, we focus on the space $\mathcal{L}(^2d_*(1, w)^2)$. First we present the explicit formula for the norm of a bilinear form in $\mathcal{L}(^2d_*(1, w)^2)$. Using this formula, we can classify the extreme points of the unit ball of $\mathcal{L}(^2d_*(1, w)^2)$ by the method of step by step.

2. Main Results

If $T \in \mathcal{L}(^2d_*(1,w)^2)$, then $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$ for some reals a,b,c,d.

Theorem 2.1. *Let* $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2d_*(1,w)^2)$*. Then there exists (unique)* $T^{'}((x_1,y_1),\ (x_2,y_2)) = a^*x_1x_2 + b^*y_1y_2 + c^*x_1y_2 + d^*x_2y_1 \in \mathcal{L}(^2d_*(1,w)^2)$ *such that* $a^*, b^*, c^*, d^* \in \{\pm a, \pm b, \pm c, \pm d\}$ *with* $a^* \geq b^* \geq 0, c^* \geq |d^*|$ *and* $\|T\| = \|T^{'}\|$ *and that* T *is extreme if and only if* $T^{'}$ *is extreme.*

Proof. If $a < 0$, taking $-T$, we assume $a \geq 0$.
Case 1: $|b| > a$

$$\begin{aligned} \text{Let } T_1^{'}((x_1,y_1),\ (x_2,y_2)) &:= T((y_1, sign(b)x_1),(y_2,x_2)) \\ &= |b|x_1x_2 + |a|y_1y_2 + sign(b)dx_1y_2 + cx_2y_1. \end{aligned}$$

Then $\|T_1^{'}\| = \|T\|$ and T is extreme if and only if $T_1^{'}$ is extreme. If $sign(b)d \geq |c|$, then the bilinear form $T_1^{'}$ satisfies the conditions of the the theorem. Suppose that $sign(b)d < |c|$.

Subcase 1: $c \geq 0$

If $sign(b)d = |d|$ or $(sign(b)d = -|d|, |d| \leq |c|)$,

$$\begin{aligned} \text{let } T_2^{'}((x_1,y_1),\ (x_2,y_2)) &:= T_1^{'}((x_2,y_2),(x_1,y_1)) \\ &= |b|x_1x_2 + |a|y_1y_2 + |c|x_1y_2 + sign(b)dx_2y_1. \end{aligned}$$

Then $\|T_2^{'}\| = \|T\|$ and T is extreme if and only if $T_2^{'}$ is extreme. Hence, the bilinear form $T_2^{'}$ satisfies the conditions of the theorem. If $sign(b)d = -|d|, |d| > |c|$,

$$\begin{aligned} \text{let } T_2^{'}((x_1,y_1),\ (x_2,y_2)) &:= T_1^{'}((x_2,-y_2),(x_1,-y_1)) \\ &= |b|x_1x_2 + |a|y_1y_2 + |sign(b)d|x_1y_2 - |c|x_2y_1. \end{aligned}$$

Then $\|T_2^{'}\| = \|T\|$ and T is extreme if and only if $T_2^{'}$ is extreme. Hence, the bilinear form $T_2^{'}$ satisfies the conditions of the the theorem.

Subcase 2: $c < 0$

$$\begin{aligned} \text{Let } T_3^{'}((x_1,y_1),\ (x_2,y_2)) &:= T_1^{'}((-x_1,y_1),(-x_2,y_2)) \\ &= |b|x_1x_2 + |a|y_1y_2 - sign(b)dx_1y_2 + |c|x_2y_1. \end{aligned}$$

Applying Subcase 1 to $T_3^{'}$, we can find a bilinear form $T^{'}$ satisfying the conditions of the theorem.

Case 2: $|b| \leq a$

$$\begin{aligned} \text{Let } T_4^{'}((x_1,y_1),\ (x_2,y_2)) &:= T((x_1,y_1),(x_2, sign(b)y_2)) \\ &= ax_1x_2 + |b|y_1y_2 + sign(b)cx_1y_2 + dx_2y_1. \end{aligned}$$

Applying Case 1 to $T_4^{'}$, we can find a bilinear form $T^{'}$ satisfying the conditions of the theorem. □

Theorem 2.2. *Let* $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2d_*(1,w)^2)$ *with* $a \geq |b|, c \geq |d|$. *Then* $\|T\| = \max\{a + bw^2 + (c+d)w, a - bw^2 + (c-d)w, (a+b)w + c + dw^2, (a-b)w + c - dw^2\}$.

Proof. Since $\{(\pm 1,\ \pm w),\ (\pm w,\ \pm 1)\}$ is the set of all extreme points of the unit ball of $d_*(1,w)^2$ and T is bilinear,

$$\begin{aligned}\|T\| &= \max\{|T((\pm 1,\ \pm w),(\pm 1,\ \pm w))|, |T((\pm 1,\ \pm w),(\pm w,\ \pm 1))|,\\ &\qquad |T((\pm w,\ \pm 1),(\pm 1,\ \pm w))|, |T((\pm w,\ \pm 1),(\pm w,\ \pm 1))|\}.\end{aligned}$$

It follows that

$$\begin{aligned}\|T\| &= \max\{|T((1,w),\ (1,w))|, |T((1,w),\ (1,-w))|, |T((1,-w),\ (1,-w))|,\\ &\quad |T((1,-w),\ (1,w))|, |T((1,\ w),\ (w,\ 1))|,\ |T((1,\ w),\ (w,\ -1))|,\\ &\quad |T((1,\ -w),\ (w,\ 1))|, |T((1,\ -w),\ (w,\ -1))|,\ |T((w,\ 1),\ (1,\ w))|,\\ &\quad |T((w,\ -1),\ (1,\ w))|, |T((w,\ 1),\ (1,\ -w))|,\ |T((w,\ -1),(1,\ -w))|,\\ &\quad |T((w,\ 1),\ (w,\ 1))|, |T((w,\ 1),\ (w,\ -1))|, |T((w,\ -1),\ (w,\ 1))|\\ &\quad |T((w,\ -1),\ (w,\ -1))|\}\\ &= \max\{a + bw^2 + (c+d)w, a - bw^2 + (c-d)w, (a+b)w + c + dw^2,\\ &\quad (a-b)w + c - dw^2\}.\end{aligned}$$

□

By Theorem 2.2, notice that if $\|T\| = 1$ for some $T \in \mathcal{L}(^2d_*(1,w)^2)$, then $|a| \leq 1,\ |b| \leq 1,\ |c| \leq 1,\ |d| \leq 1$.

Theorem 2.3. [14, *Theorem* 2.3] *Let* $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s(^2d_*(1,w)^2)$. *Then*

(a) *Let* $w < \sqrt{2} - 1$. *Then* T *is extreme if and only if*

$$\begin{aligned}T \quad &\in \{\pm x_1x_2, \pm y_1y_2, \pm\frac{1}{1+w^2}(x_1x_2 + y_1y_2),\\ &\pm\frac{1}{(1+w)^2}[x_1x_2 + y_1y_2 \pm (x_1y_2 + x_2y_1)],\\ &\pm\frac{1}{1+w^2}[x_1x_2 - y_1y_2 \pm w(x_1y_2 + x_2y_1)],\\ &\pm\frac{1}{1+w^2}[wx_1x_2 - wy_1y_2 \pm (x_1y_2 + x_2y_1)],\\ &\pm\frac{1}{1+2w-w^2}[x_1x_2 - y_1y_2 \pm (x_1y_2 + x_2y_1)],\end{aligned}$$

$$\pm\frac{1}{(1+w)^2(1-w)}[(1-w-w^2)x_1x_2 - wy_1y_2 \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{(1+w)^2(1-w)}[wx_1x_2 - (1-w-w^2)y_1y_2 \pm (x_1y_2+x_2y_1)]\}.$$

(b) *Let* $w=\sqrt{2}-1$. *Then* T *is extreme if and only if*

$$T \in \{\pm x_1x_2, \pm y_1y_2, \pm\frac{2+\sqrt{2}}{4}(x_1x_2+y_1y_2), \pm\frac{1}{2}[x_1x_2+y_1y_2 \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{\sqrt{2}}{4}[x_1x_2+y_1y_2 \pm (\sqrt{2}+1)(x_1y_2+x_2y_1)],$$
$$\pm\frac{\sqrt{2}}{4}[(\sqrt{2}+1)(x_1y_2-x_2y_1) \pm (x_1y_2+x_2y_1)]\}.$$

(c) *Let* $w>\sqrt{2}-1$. *Then* T *is extreme if and only if*

$$T \in \{\pm x_1x_2, \pm y_1y_2, \pm\frac{1}{1+w^2}(x_1x_2+y_1y_2),$$
$$\pm\frac{1}{(1+w)^2}[x_1x_2+y_1y_2 \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{1+2w-w^2}[x_1x_2-y_1y_2 \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{1+w^2}[x_1x_2-y_1y_2 \pm \frac{1-w}{1+w}(x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{1+w^2}[\frac{1-w}{1+w}(x_1x_2-y_1y_2) \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{2+2w}[(2+w)x_1x_2 - \frac{1}{w}y_1y_2 \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{2+2w}[\frac{1}{w}x_1x_2 - (2+w)y_1y_2 \pm (x_1y_2+x_2y_1)]\}.$$

It is obvious that if a symmetric bilinear form $T \notin extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$, then $T \notin extB_{\mathcal{L}(^2d_*(1,w)^2)}$.

Theorem 2.4. *Let* $S((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2d_*(1,w)^2)$ *with* $a \geq b \geq 0, c \geq |d|$. *Then*
(a) *Let* $w<\sqrt{2}-1$. S *is extreme if and only if*

$$S \in \{x_1x_2, x_1y_2, \frac{1}{1+w}(x_1x_2+x_1y_2), \frac{1}{(1+w)^2}(x_1x_2+y_1y_2+x_1y_2+x_2y_1),$$
$$\frac{1}{1+w^2}(x_1x_2+y_1y_2+wx_1y_2-wx_2y_1), \frac{1}{1+w^2}(wx_1x_2+wy_1y_2+x_1y_2-x_2y_1),$$
$$\frac{1}{1+2w-w^2}(x_1x_2+y_1y_2+x_1y_2-x_2y_1),$$

$$\frac{1}{(1+w)^2(1-w)}(x_1x_2 + y_1y_2 + (1-w-w^2)x_1y_2 - wx_2y_1),$$
$$\frac{1}{(1+w)^2(1-w)}((1-w-w^2)x_1x_2 + wy_1y_2 + x_1y_2 - x_2y_1)\}.$$

(b) *Let* $w = \sqrt{2}-1$. *Then* S *is extreme if and only if*

$$S \in \{x_1x_2, x_1y_2, \frac{1}{\sqrt{2}}(x_1x_2 + x_1y_2), \frac{1}{2}(x_1x_2 + y_1y_2 + x_1y_2 + x_2y_1),$$
$$\frac{\sqrt{2}}{4}((\sqrt{2}+1)(x_1x_2 + y_1y_2) + x_1y_2 - x_2y_1),$$
$$\frac{\sqrt{2}}{4}(x_1x_2 + y_1y_2 + (\sqrt{2}+1)(x_1y_2 - x_2y_1))\}.$$

(c) *Let* $w > \sqrt{2}-1$. *Then* S *is extreme if and only if*

$$S \in \{x_1x_2, x_1y_2, \frac{1}{1+w}(x_1x_2 + x_1y_2), \frac{1}{(1+w)^2}(x_1x_2 + y_1y_2 + x_1y_2 + x_2y_1),$$
$$\frac{1}{1+2w-w^2}(x_1x_2 + y_1y_2 + x_1y_2 - x_2y_1),$$
$$\frac{1}{1+w^2}(\frac{1-w}{1+w}(x_1x_2 + y_1y_2) + x_1y_2 - x_2y_1),$$
$$\frac{1}{1+w^2}(x_1x_2 + y_1y_2 + \frac{1-w}{1+w}(x_1y_2 - x_2y_1)),$$
$$\frac{1}{2+2w}(x_1x_2 + y_1y_2 + (2+w)x_1y_2 - \frac{1}{w}x_2y_1),$$
$$\frac{1}{2+2w}((2+w)x_1x_2 + \frac{1}{w}y_1y_2 + x_1y_2 - x_2y_1)\}.$$

Proof. It consists of two cases. Suppose that $S((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in extB_{\mathcal{L}}(^2d_*(1,w)^2)$ with $a \geq b \geq 0, c \geq |d|$. Then $S \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$ if and only if $S^{'}((x_1, y_1),\ (x_2, y_2)) := cx_1x_2 + dy_1y_2 + ax_1y_2 + bx_2y_1 \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$. Without loss of generality we will consider $S^{'}$ instead of S.

Case 1: $a = b$
In this case, $S^{'} \in \mathcal{L}_s(^2d_*(1,w)^2)$. Since $S^{'} \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$, $S^{'} \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$. Let $S^{'} \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$ in the list of Theorem 2.3.
Claim: $S^{'}((x_1, y_1), (x_2, y_2)) = \frac{1}{1+w^2}(x_1x_2 + y_1y_2 + x_1y_2 + x_2y_1) \notin extB_{\mathcal{L}(^2d_*(1,w)^2)}$

Let $\epsilon > 0$ such that

$$\epsilon(1+w^2) < 1, \frac{1-w^2}{1+w^2} + 2\epsilon w < 1, \frac{2w}{1+w^2} + \epsilon(1-w^2) < 1.$$

Let $R_1((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) + \epsilon(x_1y_2 - x_2y_1)$ and $R_2((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) - \epsilon(x_1y_2 - x_2y_1)$. By Theorem 2.2, $\|R_1\| = 1 = \|R_2\|, S^{'} = \frac{1}{2}(R_1 + R_2)$. Since $R_1 \neq R_2$, $S^{'}$ is not extreme.
Claim: $S^{'}((x_1,y_1),\ (x_2,y_2)) = \frac{1}{(1+w)^2}(x_1x_2+y_1y_2+x_1y_2+x_2y_1) \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$

Notice that

$$\begin{aligned} 1 &= |S^{'}((1,w),(1,w))| = |S^{'}((w,1),(w,1))| \\ &= |S^{'}((1,w),(w,1))| = |S^{'}((w,1),(1,w))|. \end{aligned}$$

Let $R_1((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) + (\epsilon x_1x_2 + \delta y_1y_2 + \gamma x_1y_2 + \delta x_2y_1)$ and $R_2((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) - (\epsilon x_1x_2 + \delta y_1y_2 + \gamma x_1y_2 + \delta x_2y_1)$ with $\|R_1\| = 1 = \|R_2\|, \epsilon, \delta, \gamma, \beta \in \mathbb{R}$. Since

$$\begin{aligned} &|R_i((1,w),(1,w))| \leq 1, |R_i((w,1),(w,1))| \leq 1, \\ &|R_i((1,w),(w,1))| \leq 1, |R_i((w,1),(1,w))| \leq 1, \end{aligned}$$

we have

$$\begin{aligned} 0 &= \epsilon + \delta w^2 + \gamma w + \beta w \\ 0 &= \epsilon w^2 + \delta + \gamma w + \beta w \\ 0 &= \epsilon w + \delta w + \gamma + \beta w^2 \\ 0 &= \epsilon w + \delta w + \gamma w^2 + \beta, \end{aligned}$$

which imply that $0 = \epsilon = \delta = \gamma = \beta$. Therefore, $R_1 = S^{'} = R_2$ and $S^{'}$ is extreme.
Claim: if $w \neq \sqrt{2} - 1$, then $S^{'}((x_1,y_1),\ (x_2,y_2)) = \frac{1}{1+2w-w^2}(x_1x_2 + y_1y_2 + x_1y_2 - x_2y_1) \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$

Notice that

$$\begin{aligned} 1 &= |S^{'}((1,-w),(1,w))| = |S^{'}((w,1),(w,-1))| \\ &= |S^{'}((1,w),(w,1))| = |S^{'}((w,-1),(1,-w))|. \end{aligned}$$

Let $R_1((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) + (\epsilon x_1x_2 + \delta y_1y_2 + \gamma x_1y_2 + \delta x_2y_1)$ and $R_2((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) - (\epsilon x_1x_2 + \delta y_1y_2 + \gamma x_1y_2 + \delta x_2y_1)$ with $\|R_1\| = 1 = \|R_2\|, \epsilon, \delta, \gamma, \beta \in \mathbb{R}$. Since

$$\begin{aligned} &|R_i((1,-w),(1,w))| \leq 1, |R_i((w,1),(w,-1))| \leq 1, \\ &|R_i((1,w),(w,1))| \leq 1, |R_i((w,-1),(1,-w))| \leq 1, \end{aligned}$$

we have

$$\begin{aligned} 0 &= \epsilon - \delta w^2 + \gamma w - \beta w \\ 0 &= \epsilon w^2 - \delta - \gamma w + \beta w \\ 0 &= \epsilon w + \delta w + \gamma + \beta w^2 \\ 0 &= \epsilon w + \delta w - \gamma w^2 - \beta, \end{aligned}$$

which imply that $0=\epsilon=\delta=\gamma=\beta$. Therefore, $R_1=S^{'}=R_2$ and $S^{'}$ is extreme.
Claim: $S^{'}((x_1,y_1),\ (x_2,y_2)) = \frac{1}{(1+w)^2(1-w)}(x_1x_2+y_1y_2+(1-w-w^2)x_1y_2 - wx_2y_1) \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$ if $w<\sqrt{2}-1$

Notice that

$$\begin{aligned} 1 &= |S^{'}((1,w),(1,w))| = |S^{'}((1,w),(w,1))| \\ &= |S^{'}((w,1),(w,1))| = |S^{'}((w,1),(w,-1))|. \end{aligned}$$

Let $R_1((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) + (\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ and $R_2((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) - (\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ with $\|R_1\|=1=\|R_2\|, \epsilon,\delta,\gamma,\beta\in\mathbb{R}$. Since

$$\begin{aligned} &|R_i((1,w),(1,w))|\le 1, |R_i((1,w),(w,1))|\le 1,\\ &|R_i((w,1),(w,1))|\le 1, |R_i((w,1),(w,-1))|\le 1, \end{aligned}$$

we have

$$\begin{aligned} 0 &= \epsilon+\delta w^2+\gamma w+\beta w \\ 0 &= \epsilon w+\delta w+\gamma+\beta w^2 \\ 0 &= \epsilon w^2+\delta+\gamma w+\beta w \\ 0 &= \epsilon w^2-\delta-\gamma w+\beta w, \end{aligned}$$

which imply that $0=\epsilon=\delta=\gamma=\beta$. Therefore, $R_1=S^{'}=R_2$ and $S^{'}$ is extreme.
Claim: $S^{'}((x_1,y_1),\ (x_2,y_2)) = \frac{\sqrt{2}}{4}((\sqrt{2}+1)(x_1x_2+y_1y_2)+x_1y_2-x_2y_1) \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$ if $w=\sqrt{2}-1$

Notice that

$$\begin{aligned} 1 &= |S^{'}((1,w),(1,w))| = |S^{'}((1,w),(w,1))| \\ &= |S^{'}((1,-w),(-1,w))| = |S^{'}((w,1),(w,1))|. \end{aligned}$$

Let $R_1((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) + (\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ and $R_2((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) - (\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ with $\|R_1\|=1=\|R_2\|, \epsilon,\delta,\gamma,\beta\in\mathbb{R}$. Since

$$\begin{aligned} &|R_i((1,w),(1,w))|\le 1, |R_i((1,w),(w,1))|\le 1,\\ &|R_i((1,-w),(-1,w))|\le 1, |R_i((w,1),(w,1))|\le 1, \end{aligned}$$

we have

$$\begin{aligned} 0 &= \epsilon+\delta w^2+\gamma w+\beta w \\ 0 &= \epsilon w+\delta w+\gamma+\beta w^2 \\ 0 &= -\epsilon-\delta w^2+\gamma w+\beta w \\ 0 &= \epsilon w^2+\delta+\gamma w+\beta w, \end{aligned}$$

which imply that $0=\epsilon=\delta=\gamma=\beta$. Therefore, $R_1=S^{'}=R_2$ and $S^{'}$ is extreme.
Claim: $S^{'}((x_1,y_1),\ (x_2,y_2)) = \frac{1}{1+w^2}(\frac{1-w}{1+w}(x_1x_2+y_1y_2)+x_1y_2-x_2y_1) \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$ if $w>\sqrt{2}-1$

Notice that

$$\begin{aligned}1 &= |S^{'}((1,w),(-w,1))| = |S^{'}((1,-w),(w,1))| = |S^{'}((w,1),(-1,w))|\\ &= |S^{'}((w,-1),(1,w))| = |S^{'}((1,-w),(1,w))|.\end{aligned}$$

Let $R_1((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) + (\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ and $R_2((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) - (\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ with $\|R_1\|=1=\|R_2\|, \epsilon,\delta,\gamma,\beta\in\mathbb{R}$. Since

$$\begin{aligned}&|R_i((1,w),(-w,1))|\le 1, |R_i((1,-w),(w,1))|\le 1,\\ &|R_i((w,1),(-1,w))|\le 1, |R_i((w,-1),(1,w))|\le 1,\\ &|R_i((1,-w),(1,w))|\le 1,\end{aligned}$$

we have

$$\begin{aligned}0 &= -\epsilon w+\delta w+\gamma-\beta w^2\\ 0 &= \epsilon w-\delta w+\gamma-\beta w^2\\ 0 &= -\epsilon w+\delta w+\gamma w^2-\beta\\ 0 &= \epsilon w-\delta w+\gamma w^2-\beta\\ 0 &= \epsilon-\delta w^2+\gamma w-\beta w\end{aligned}$$

which imply that $0=\epsilon=\delta=\gamma=\beta$. Therefore, $R_1=S^{'}=R_2$ and $S^{'}$ is extreme.
Claim: $S^{'}((x_1,y_1),\ (x_2,y_2)) = \frac{1}{2+2w}((x_1x_2+y_1y_2)+(2+w)x_1y_2-\frac{1}{w}x_2y_1) \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$ if $w>\sqrt{2}-1$

Notice that

$$\begin{aligned}1 &= |S^{'}((1,-w),(1,w))| = |S^{'}((1,w),(w,1))|\\ &= |S^{'}((1,w),(-w,1))| = |S^{'}((w,1),(-w,1))|.\end{aligned}$$

Let $R_1((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) + (\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ and $R_2((x_1,y_1),\ (x_2,y_2)) = S^{'}((x_1,y_1),\ (x_2,y_2)) - (\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ with $\|R_1\|=1=\|R_2\|, \epsilon,\delta,\gamma,\beta\in\mathbb{R}$. Since

$$\begin{aligned}&|R_i((1,-w),(1,w))|\le 1, |R_i((1,w),(w,1))|\le 1,\\ &|R_i((1,w),(-w,1))|\le 1, |R_i((w,1),(-w,1))|\le 1,\end{aligned}$$

we have

$$\begin{aligned} 0 &= \epsilon - \delta w^2 + \gamma w - \beta w \\ 0 &= \epsilon w + \delta w + \gamma + \beta w^2 \\ 0 &= -\epsilon w + \delta w + \gamma - \beta w^2 \\ 0 &= -\epsilon w^2 + \delta + \gamma w - \beta w, \end{aligned}$$

which imply that $0 = \epsilon = \delta = \gamma = \beta$. Therefore, $R_1 = S^{'} = R_2$ and $S^{'}$ is extreme.
Claim: $S^{'}((x_1, y_1),\ (x_2, y_2)) = \frac{1}{1+w^2}(w(x_1x_2+y_1y_2)+x_1y_2-x_2y_1) \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$ if $w < \sqrt{2}-1$

Notice that

$$\begin{aligned} 1 &= |S^{'}((1,-w),(w,1))| = |S^{'}((w,1),(1,-w))| \\ &= |S^{'}((w,-1),(1,w))| = |S^{'}((w,-1),(1,-w))|. \end{aligned}$$

Let $R_1((x_1, y_1),\ (x_2, y_2)) = S^{'}((x_1, y_1),\ (x_2, y_2)) + (\epsilon x_1x_2 + \delta y_1y_2 + \gamma x_1y_2 + \delta x_2y_1)$ and $R_2((x_1, y_1),\ (x_2, y_2)) = S^{'}((x_1, y_1),\ (x_2, y_2)) - (\epsilon x_1x_2 + \delta y_1y_2 + \gamma x_1y_2 + \delta x_2y_1)$ with $\|R_1\| = 1 = \|R_2\|, \epsilon, \delta, \gamma, \beta \in \mathbb{R}$. Since

$$\begin{aligned} &|R_i((1,-w),(w,1))| \leq 1, |R_i((w,1),(1,-w))| \leq 1, \\ &|R_i((w,-1),(1,w))| \leq 1, |R_i((w,-1),(1,-w))| \leq 1, \end{aligned}$$

we have

$$\begin{aligned} 0 &= \epsilon w - \delta w + \gamma - \beta w^2 \\ 0 &= \epsilon w - \delta w - \gamma w^2 + \beta \\ 0 &= \epsilon w - \delta w + \gamma w^2 - \beta \\ 0 &= \epsilon w + \delta w - \gamma w^2 - \beta, \end{aligned}$$

which imply that $0 = \epsilon = \delta = \gamma = \beta$. Therefore, $R_1 = S^{'} = R_2$ and $S^{'}$ is extreme.

Case 2: $a > b$

We claim that $b = 0$. Otherwise. By Theorem 2.2, $0 < a < 1$. If $d = 0$, then

$$\begin{aligned} &1 = \|S^{'}\| = a + bw^2 + cw = (a+b)w + c, \\ &a - bw^2 + cw < 1, (a-b)w + c < 1. \end{aligned}$$

If $c > 0$, then we can find $\epsilon > 0$ such that $\|R_j\| = 1$ for $j = 1, 2$, where
$R_1((x_1, y_1),\ (x_2, y_2)) = S^{'}((x_1, y_1),\ (x_2, y_2)) + \epsilon(x_1x_2 - \frac{1}{w}y_1y_2 + x_1y_2 - \frac{1}{w}x_2y_1)$ and $R_2((x_1, y_1),\ (x_2, y_2)) = S^{'}((x_1, y_1),\ (x_2, y_2)) - \epsilon(x_1x_2 - \frac{1}{w}y_1y_2 + x_1y_2 - \frac{1}{w}x_2y_1)$,

which shows that $S^{'}$ is not extreme and we have a contradiction. If $c = 0$, then $a = \frac{1}{1+w} < \frac{1}{w(1+w)} = b$, which is impossible. Therefore, $d \neq 0$. If $d > 0$, then

$$1 = \|S^{'}\| = a + bw^2 + (c+d)w = (a+b)w + c + dw^2,$$
$$a - bw^2 + (c-d)w < 1, (a-b)w + c - dw^2 < 1,$$

which shows that $S^{'}$ is not extreme and we have a contradiction. If $d < 0$, then $a > b > 0, c \geq |d| = -d$. Since $S^{'}$ is extreme, it follows that

$$1 = a+bw^2+(c+d)w = a-bw^2+(c-d)w = (a+b)w+c+dw^2 = (a-b)w+c-dw^2.$$

Then $a = c = \frac{1}{1+w}, 0 = b = d$, which is a contradiction. We have shown that $b = 0$.

If $a = 1$, then $0 = c = d$. Hence $S^{'}((x_1, y_1),\ (x_2, y_2)) = x_1x_2$.

Claim: $x_1x_2 \in extB_{\mathcal{L}(^2d_*(1,w)^2)}$

Notice that

$$\begin{aligned} 1 &= |S^{'}((1,w),(1,w))| = |S^{'}((1,w),(1,-w))| \\ &= |S^{'}((1,-w),(1,w))| = |S^{'}((1,-w),(1,-w))|. \end{aligned}$$

Let $R_1((x_1, y_1),\ (x_2, y_2)) = x_1x_2 + (\epsilon x_1x_2 + \delta y_1y_2 + \gamma x_1y_2 + \delta x_2y_1)$ and $R_2((x_1, y_1), (x_2, y_2)) = x_1x_2 - (\epsilon x_1x_2 + \delta y_1y_2 + \gamma x_1y_2 + \delta x_2y_1)$ with $\|R_1\| = 1 = \|R_2\|, \epsilon, \delta, \gamma, \beta \in \mathbb{R}$. Since

$$|R_i((1,w),(1,w))| \leq 1, |R_i((1,w),(1,-w))| \leq 1,$$
$$|R_i((1,-w),(1,w))| \leq 1, |R_i((1,-w),(1,-w))| \leq 1,$$

we have

$$\begin{aligned} 0 &= \epsilon + \delta w^2 + \gamma w + \beta w \\ 0 &= \epsilon - \delta w^2 - \gamma w + \beta w \\ 0 &= \epsilon - \delta w^2 + \gamma w - \beta w \\ 0 &= \epsilon + \delta w^2 - \gamma w - \beta w, \end{aligned}$$

which imply that $0 = \epsilon = \delta = \gamma = \beta$. Therefore, $R_1 = x_1x_2 = R_2$ and x_1x_2 is extreme.

Suppose that $0 < a < 1, d \neq 0$. If $d > 0$, then

$$1 = \|S^{'}\| = a + (c+d)w = aw + c + dw^2,$$
$$a + (c-d)w < 1, aw + c - dw^2 < 1.$$

Notice that if $c > |d| = d$, then $S^{'}$ is not extreme and we have a contradiction. If $c = d$, then $S^{'}((x_1, y_1),\ (x_2, y_2)) = \frac{1-w}{1+w}x_1x_2 + \frac{1}{1+w}x_1y_2 + \frac{1}{1+w}x_2y_1$. It is not difficult to show that $S^{'}$ is not extreme and we have a contradiction.

Similarly, if $d<0$, then

$$1=\|S^{'}\|=a+(c-d)w=aw+c-dw^2,$$
$$a+(c+d)w<1, aw+c+dw^2<1.$$

Notice that if $c>|d|=-d$, then $S^{'}$ is not extreme and we have a contradiction. If $c=-d$, then $S^{'}((x_1,y_1),\ (x_2,y_2))=\frac{1-w}{1+w}x_1x_2+\frac{1}{1+w}x_1y_2-\frac{1}{1+w}x_2y_1$. It is not difficult to show that $S^{'}$ is not extreme and we have a contradiction. Therefore, $d=0$ and $1=a+cw=aw+c$, so $a=c=\frac{1}{1+w}$, so $S^{'}((x_1,y_1),\ (x_2,y_2))=\frac{1}{1+w}(x_1x_2+x_1y_2)$. We will show that $S^{'}$ is extreme. Indeed,

$$\begin{aligned}1 &= |S^{'}((1,w),(1,w))|=|S^{'}((1,-w),(1,w))|\\ &= |S^{'}((1,w),(w,1))|=|S^{'}((1,-w),(w,1))|.\end{aligned}$$

Let $R_1((x_1,y_1),\ (x_2,y_2))=S^{'}((x_1,y_1),\ (x_2,y_2))+(\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$ and $R_2((x_1,y_1),\ (x_2,y_2))=S^{'}((x_1,y_1),\ (x_2,y_2))-(\epsilon x_1x_2+\delta y_1y_2+\gamma x_1y_2+\delta x_2y_1)$with $\|R_1\|=1=\|R_2\|,\epsilon,\delta,\gamma,\beta\in\mathbb{R}$. Since

$$|R_i((1,w),(1,w))|\le 1, |R_i((1,-w),(1,w))|\le 1,$$
$$|R_i((1,w),(w,1))|\le 1, |R_i((1,-w),(w,1))|\le 1,$$

we have

$$\begin{aligned}0 &= \epsilon+\delta w^2+\gamma w+\beta w\\ 0 &= \epsilon-\delta w^2+\gamma w-\beta w\\ 0 &= \epsilon w+\delta w+\gamma+\beta w^2\\ 0 &= \epsilon w-\delta w+\gamma-\beta w^2,\end{aligned}$$

which imply that $0=\epsilon=\delta=\gamma=\beta$. Hence, $R_1=S^{'}=R_2$ and $S^{'}$ is extreme. Therefore, we complete the proof. □

Using Theorems 2.1 and 2.4, we can classify the extreme bilinear forms of the unit ball of $\mathcal{L}(^2d_*(1,w)^2)$ as follows:

Theorem 2.5. *$T\in extB_{\mathcal{L}(^2d_*(1,w)^2)}$ if and only if there exist $n\in\mathbb{N}$ and $S((x_1,y_1),(x_2,y_2))=ax_1x_2+by_1y_2+cx_1y_2+dx_2y_1\in extB_{\mathcal{L}(^2d_*(1,w)^2)}$ with $a\ge|b|, c\ge|d|$ such that $T((x_1,y_1),(x_2,y_2)):=S((u_1^{(n)},v_1^{(n)}),(u_2^{(n)},v_2^{(n)}))\circ\cdots\circ((u_1^{(1)},v_1^{(1)}),(u_2^{(1)},v_2^{(1)}))$, where*

for $j=1,\ldots,n,((u_1^{(j)},v_1^{(j)}),(u_2^{(j)},v_2^{(j)}))\in\{((\pm x_1,\pm y_1),(\pm x_2,\pm y_2)),$ $((\pm x_2,\pm y_2),(\pm x_1,\pm y_1)),((\pm x_1,\pm y_1),(\pm y_2,\pm x_2)),((\pm y_2,\pm x_2),$ $(\pm x_1,\pm y_1)),((\pm y_1,\pm x_1),(\pm x_2,\pm y_2)),((\pm x_2,\pm y_2),(\pm y_1,\pm x_1)),$ $((\pm y_2,\pm x_2),(\pm y_1,\pm x_1)),((\pm y_1,\pm x_1),(\pm y_2,\pm x_2))\}.$

Proof. It follows from Theorems 2.1 and 2.4. □

Corollary 2.6. (a) $extB_{\mathcal{L}_s(^2d_*(1,w)^2)} \backslash extB_{\mathcal{L}(^2d_*(1,w)^2)} \neq \emptyset$.
(b) $extB_{\mathcal{L}(^2d_*(1,w)^2)} \backslash extB_{\mathcal{L}_s(^2d_*(1,w)^2)} \neq \emptyset$.

Proof. (a): By Theorems 2.3, 2.4 and 2.5,

$$\frac{1}{1+w^2}(x_1x_2 + y_1y_2) \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)} \backslash extB_{\mathcal{L}(^2d_*(1,w)^2)}.$$

(b): By Theorems 2.3, 2.4 and 2.5,

$$x_1y_2 \in extB_{\mathcal{L}(^2d_*(1,w)^2)} \backslash extB_{\mathcal{L}_s(^2d_*(1,w)^2)}.$$

□

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