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Majorization Properties for Subclasses of Analytic p-Valent Functions Defined by Convolution

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Abstract. The object of the present paper is to investigate the majorization properties of certain subclasses of analytic p-valent functions defined by convolution.

1. Introduction

Let f and g be analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. We say that f is majorized by g in U (see [7]) and write

$$f(z) << g(z) \quad (z \in U), \tag{1.1}$$

if there exists a function φ, analytic in U such that

$$|\varphi(z)| < 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in U). \tag{1.2}$$

It may be noted that (1.1) is closely related to the concept of quasi-subordination between analytic functions.

If f and g are analytic functions in U, we say that f is subordinate to g, written symbolically as $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)), z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (see [8, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

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Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.3}$$

which are analytic and p-valent in the open unit disc U. Also let $A = A(1)$.

For functions $f_i \in A(p)$ $(i = 1, 2)$ given by

$$f_i(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,i} z^k \quad (i = 1, 2; p \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{1.4}$$

For $h(z)$ given by

$$h(z) = z^p + \sum_{k=p+1}^{\infty} \mu_k z^k \quad (\mu_k \geq 0), \tag{1.5}$$

then a function $f \in A(p)$ is said to be in the class $S_p^j(h;\gamma)$ of p-valent starlike functions of complex order $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ in U, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z(f*h)^{(j+1)}(z)}{(f*h)^{(j)}(z)} - p + j\right)\right\} > 0$$

$$(z \in U; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \gamma \in \mathbb{C}^*; |2\gamma - p + j| < (p - j)). \tag{1.6}$$

Furthermore, a function $f \in A(p)$ is said to be in the class $K_p^j(h,\gamma)$ of p-valent convex functions of complex order $\gamma \in \mathbb{C}^*$ in U, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z(f*h)^{(j+2)}(z)}{(f*h)^{(j+1)}(z)} - p + j + 1\right)\right\} > 0$$

$$(z \in U; p \in \mathbb{N}; j \in \mathbb{N}_0; \gamma \in \mathbb{C}^*; |\gamma - p + j| < (p - j)). \tag{1.7}$$

From (1.6) and (1.7) it follows that:

(i) $S_1^0(\frac{z}{1-z};\gamma) = S(\gamma)$ and $K_1^0(\frac{z}{1-z};\gamma) = K(\gamma)$ $(\gamma \in \mathbb{C}^*)$.

The class $S(\gamma)$ was introduced by Nasr and Aouf [9], and the class $K(\gamma)$ was introduced and studied by Nasr and Aouf [9] and Wiatrowski [13].

(ii) $S_p^0(\frac{z^p}{1-z};p-\alpha) = S_p^*(\alpha)$ and $K_p^0(\frac{z^p}{1-z};p-\alpha) = K_p(\alpha)$ $(0 \leq \alpha < p)$.

The class $S_p^*(\alpha)$ was introduced by Patil and Thakare [12], and the class $K_p(\alpha)$was introduced by Owa [11]. We note that

$$S_p^*(\alpha) \subseteq S_p^*(0) = S_p^* \textit{ and } K_p(\alpha) \subseteq K_p(0) = K_p \quad (0 \leq \alpha < p),$$

where S_p^* and K_p denote the subclasses of $A(p)$ consisting of functions which are p-valent starlike in U and p-valent convex in U, respectively (see Goodman [5]).

Definition 1.1. Let $-1 \leq B < A \leq 1, p \in \mathbb{N}, j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*, |\gamma(A-B) + (p-j)\,B| < (p-j), f \in A(p)$ and $h(z)$ is given by (1.5). Then $f \in S_p^j\left(h;\gamma;A,B\right)$, the class of p-valent functions of complex order γ in U if and only if

$$\left\{1 + \frac{1}{\gamma}\left(\frac{z(f*h)^{(j+1)}(z)}{(f*h)^{(j)}(z)} - p + j\right)\right\} \prec \frac{1+Az}{1+Bz}. \tag{1.8}$$

Clearly, we have the following relationships:
(i) $S_p^j\left(h;\gamma;1,-1\right) = S_p^j\left(h;\gamma\right)$;
(ii) $S_p^j\left(\frac{z^p}{1-z};\gamma\right) = S_p^j\left(\gamma\right)$.

We shall need the following lemma.

Lemma 1.2([1]). *Let $\gamma \in \mathbb{C}^*$ and $f \in K_p^j(\gamma)$. Then $f \in S_p^j(\frac{1}{2}\gamma)$, that is,*

$$K_p^j(\gamma) \subset S_p^j(\frac{1}{2}\gamma) \quad (\gamma \in \mathbb{C}^*). \tag{1.9}$$

A majorization problem for the classes $S(\gamma)$ and $K(\gamma)$ has recently been investigated by Altintas et al. [2]. Also, majorization problem for the classes $S^* = S^*(0)$ and $K = K(0)$ has been investigated by MacGregor [7]. In this paper we investigate majorization problem for the class $S_p^j\left(h,\gamma;A,B\right)$ and some related subclasses.

2. Main Results

Unless otherwise mentioned we shall assume throughout the paper that, $-1 \leq B < A \leq 1, \gamma \in \mathbb{C}^*, p \in \mathbb{N}, j \in \mathbb{N}_0$ and $h(z)$ is given by (1.5).

Theorem 2.1. *Let the function $f \in A(p)$ and suppose that $(g*h) \in S_p^j(h,\gamma;A,B)$. If $(f*h)^{(j)}(z)$ is majorized by $(g*h)^{(j)}(z)$ in U, then*

$$\left|(f*h)^{(j+1)}(z)\right| \leq \left|(g*h)^{(j+1)}(z)\right| \quad (|z| < r_0), \tag{2.1}$$

where $r_0 = r_0(p,\gamma,j,A,B)$ is the smallest positive root of the equation

$$|\gamma(A-B) + (p-j)B|\, r^3 - [2\,|B| + (p-j)]\, r^2 -$$

$$[2 + |\gamma(A-B) + (p-j)\,B|]\, r + (p-j) = 0. \tag{2.2}$$

Proof. Since $(g*h)\in S_p^j(h,\gamma;A,B)$, we find from (1.8) that

$$1+\frac{1}{\gamma}\left(\frac{z(g*h)^{(j+1)}(z)}{(g*h)^{(j)}(z)}-p+j\right)=\frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$. From (2.3), we have

$$\frac{z(g*h)^{(j+1)}(z)}{(g*h)^{(j)}(z)}=\frac{(p-j)+(\gamma(A-B)+(p-j)B)w(z)}{1+Bw(z)}. \tag{2.4}$$

From (2.4), we have

$$\left|(g*h)^{(j)}(z)\right|\le\frac{(1+|B|\,|z|)\,|z|}{(p-j)-|\gamma(A-B)+(p-j)\,B|\,|z|}\left|(g*h)^{(j+1)}(z)\right|. \tag{2.5}$$

Next, since $(f*h)^{(j)}(z)$ is majorized by $(g*h)^{(j)}(z)$ in U, from (1.2), we have

$$(f*h)^{(j)}(z)=\varphi(z)(g*h)^{(j)}(z). \tag{2.6}$$

Differentiating (2.6) with respect to z, we have

$$(f*h)^{(j+1)}(z)=\varphi^{'}(z)(g*h)^{(j)}(z)+\varphi(z)(g*h)^{(j+1)}(z). \tag{2.7}$$

Thus, by noting that $\varphi(z)$ satisfies the inequality (see [10]),

$$\left|\varphi^{'}(z)\right|\le\frac{1-|\varphi(z)|^2}{1-|z|^2}\quad(z\in U), \tag{2.8}$$

using (2.5) and (2.8), in (2.7), we have

$$\left|(f*h)^{(j+1)}(z)\right|\le \left(|\varphi(z)|+\frac{1-|\varphi(z)|^2}{1-|z|^2}.\frac{(1+|B|\,|z|)\,|z|}{(p-j)-|\gamma(A-B)+(p-j)\,B|\,|z|}\right)\left|(g*h)^{(j+1)}(z)\right|, \tag{2.9}$$

which upon setting

$$|z|=r\ \text{ and }\ |\varphi(z)|=\rho\ \ (0\le\rho\le1),$$

leads us to the inequality

$$\left|(f*h)^{(j+1)}(z)\right|\le \frac{\Theta(\rho)}{(1-r^2)((p-j)-|\gamma(A-B)+(p-j)\,B|\,r)}\left|(g*h)^{(j+1)}(z)\right|,$$

where

$$\Theta(\rho) = -\,r\,(1+|B|\,r)\,\rho^2 + (1-r^2)\,[(p-j) - |\gamma(A-B)+(p-j)\,B|\,r]\,\rho + r\,(1+|B|\,r)\,, \tag{2.10}$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(p,j,\gamma,A,B)$, where $r_0(p,j,\gamma,A,B)$ is the smallest positive root of (2.2). Therefore the function $\Phi(\rho)$ defined by

$$\Phi(\rho) = -\,\sigma\,(1+|B|\,\sigma)\,\rho^2 + (1-\sigma^2)\,[(p-j) - |\gamma(A-B)+(p-j)\,B|\,\sigma]\,\rho + \sigma\,(1+|B|\,\sigma) \tag{2.11}$$

is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\Phi(\rho) \le \Phi(1) = (1-\sigma^2)\,[(p-j) - |\gamma(A-B)+(p-j)\,B|\,\sigma] \quad (0 \le \rho \le 1; 0 \le \sigma \le r_0(p,j,\gamma,A,B))\,. \tag{2.12}$$

Hence upon setting $\rho = 1$ in (2.11), we conclude that (2.1) holds true for $|z| \le r_0 = r_0(p,\gamma,j,A,B)$, where $r_0(p,\gamma,j,A,B)$, is the smallest positive root of (2.2). This completes the proof of Theorem 2.1. □

Putting $A = 1$ and $B = -1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let the function $f \in A(p)$ and suppose that $g \in S_p^j(h;\gamma)$. If $(f * h)^{(j)}(z)$ is majorized by $(g * h)^{(j)}(z)$ in U, then*

$$\left|(f*h)^{(j+1)}(z)\right| \le \left|(g*h)^{(j+1)}(z)\right| \quad (|z| < r_0)\,,$$

where $r_0 = r_0(p,j,\gamma)$ is given by

$$r_0 = r_0(p,j,\gamma) = \frac{k - \sqrt{k^2 - 4\,|2\gamma-(p-j)|\,(p-j)}}{2\,|2\gamma-(p-j)|},$$

where $(k = 2 + (p-j) + |2\gamma-(p-j)|\,, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Remark 2.3. Putting $h(z) = \frac{z^p}{1-z}$ in Corollary 2.2, we obtain the result obtained by Altintas and Srivastava [1, Theorem 1].

Putting $h(z) = \frac{z^p}{1-z}, A = 1, B = -1$ and $j = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.4. *Let the function $f \in A(p)$ and suppose that $g \in S_p(\gamma)(\gamma \in \mathbb{C}^*)$. If $f(z)$ is majorized by $g(z)$ in U, then*

$$|f'(z)| \le |g'(z)| \quad (|z| < r_0)\,,$$

where $r_0 = r_0(p;\gamma)$ is given by

$$r_0 = r_0(p;\gamma) = \frac{k - \sqrt{k^2 - 4p\,|2\gamma-p|}}{2\,|2\gamma-p|},$$

where $(k = 2 + p + |2\gamma - p|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Putting $h(z) = \frac{z^p}{1-z}, A = 1, B = -1, p = 1$ and $j = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.5([2,6]). *Let the function* $f \in A$ *and suppose that* $g \in S(\gamma)(\gamma \in \mathbb{C}^*)$. *If* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0),$$

where $r_0 = r_0(\gamma)$ *is given by*

$$r_0 = r_0(\gamma) = \frac{k - \sqrt{k^2 - 4|2\gamma - 1|}}{2|2\gamma - 1|},$$

where $(k = 3 + |2\gamma - 1|, \gamma \in \mathbb{C}^*)$.

Putting $\gamma = 1$ in Corollary 2.5, we obtain the following result.

Corollary 2.6([6,7]) *Let the function* $f \in A$ *and suppose that* $g \in S^*$. *If* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0),$$

where r_0 *is given by*

$$r_0 = 2 - \sqrt{3}.$$

Using Lemma 2.3 it is easy to prove the following lemma.

Lemma 2.7. *Let* $\gamma \in \mathbb{C}^*$ *and* $f \in K_p^j(h;\gamma)$. *Then* $f \in S_p^j(h; \frac{1}{2}\gamma)$, *that is,*

$$K_p^j(h;\gamma) \subset S_p^j(h; \frac{1}{2}\gamma) \ (\gamma \in \mathbb{C}^*). \tag{2.13}$$

By using Lemma 2.7 and Corollary 2.2, we can prove the following theorem.

Theorem 2.8. *Let the function* $f \in A(p)$ *and suppose that* $(g * h) \in K_p^j(h,\gamma)$. *If* $(f * h)^{(j)}(z)$ *is majorized by* $(g * h)^{(j)}(z)$ *in* U, *then*

$$\left|(f * h)^{(j+1)}(z)\right| \leq \left|(g * h)^{(j+1)}(z)\right| \quad (|z| < r_0),$$

where $r_0 = r_0(p, \gamma, j)$ *is given by*

$$r_0 = r_0(p,\gamma,j) = \frac{k - \sqrt{k^2 - 4(p-j)|\gamma - (p-j)|}}{2|\gamma - (p-j)|}, \tag{2.14}$$

where $(k = 2 + (p - j) + |\gamma - (p - j)|, p \in \mathbb{N}, j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*)$.

Proof. In view of the inclusion property (2.13), Theorem 2.8 can be deduced as a simple consequence of Theorem 2.1 with $\gamma \to \frac{1}{2}\gamma$. □

Remark 2.9. Putting $h(z) = \frac{z^p}{1-z}$ in Theorem 2.8, we obtain the result obtained by Altintas [1, Theorem 2].

Putting $h(z) = \frac{z^p}{1-z}$ and $j = 0$ in Theorem 2.8, we obtain the following result.

Corollary 2.10. *Let the function $f \in A(p)$ and suppose that $g \in K_p(\gamma)(\gamma \in \mathbb{C}^*)$. If $f(z)$ is majorized by $g(z)$ in U, then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0)\,,$$

where $r_0 = r_0(p;\gamma)$ is given by

$$r_0 = r_0(p;\gamma) = \frac{k - \sqrt{k^2 - 4p\,|\gamma - p|}}{2\,|\gamma - p|},$$

where $(k = 2 + p + |\gamma - p|\,, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Putting $h(z) = \frac{z^p}{1-z}, p = 1$ and $j = 0$ in Theorem 2.8, we obtain the following result.

Corollary 2.11([1]). *Let the function $f \in A$ and suppose that $g \in K(\gamma)$. If $f(z)$ is majorized by $g(z)$ in U, then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0)\,,$$

where $r_0 = r_0(\gamma)$ is given by

$$r_0 = r_0(\gamma) = \frac{k - \sqrt{k^2 - 4\,|\gamma - 1|}}{2\,|\gamma - 1|},$$

where $(k = 3 + |\gamma - 1|\,, \gamma \in \mathbb{C}^*)$.

Letting $\gamma \to 1$ in Corollary 2.11, we obtain the following result.

Corollary 2.12([7]). *Let the function $f \in A$ and suppose that $g \in K$. If $f(z)$ is majorized by $g(z)$ in U, then*

$$|f'(z)| \leq |g'(z)| \quad \left(|z| \leq \frac{1}{3}\right).$$

Examples

(i) Putting in (1.6) and (1.7) $h(z) = z^p + \sum_{k=p+1}^{\infty} \Gamma_{k-p}(\alpha_1) z^k$, (see [4]), where $\Gamma_{k-p}(\alpha_1) = \frac{(\alpha_1)_{k-p} \ldots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \ldots (\beta_s)_{k-p}(1)_{k-p}}, q \leq s+1, q, s \in \mathbb{N}_0, \alpha_i \in \mathbb{C}(i = 1, 2, \ldots, q)$ and

$\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-(j = 1, 2, ..., s)$, then $(f * h)(z) = H_{p,q,s}(\alpha_1)f(z), S_p^j(h;\gamma) = S_{p,q,s}^j(\alpha_1;\gamma)$ and $K_p^j(h;\gamma) = K_{p,q,s}^j(\alpha_1;\gamma)$. Then we obtain the following results.

Let the function $f \in A(p)$ and $g \in S_{p,q,s}^j(\alpha_1;\gamma)(|2\gamma - \alpha_1| < |\alpha_1|)$. If $(H_{p,q,s}(\alpha_1)f(z))^{(j)}$ is majorized by $(H_{p,q,s}(\alpha_1)g(z))^{(j)}$ in U, then

$$\left|(H_{p,q,s}(\alpha_1+1)f(z))^{(j)}\right| \leq \left|(H_{p,q,s}(\alpha_1+1)g(z))^{(j)}\right| \quad (|z| < r_1)\,,$$

where $r_1 = r_1(\gamma, \alpha_1)$ is given by

$$r_1 = r_1(\gamma, \alpha_1) = \frac{k - \sqrt{k^2 - 4\,|2\gamma - \alpha_1|\,|\alpha_1|}}{2\,|2\gamma - \alpha_1|},$$

where $(k = 2 + |\alpha_1| + |2\gamma - \alpha_1|\,; \gamma, \alpha_1 \in \mathbb{C}^*)$.

Also, let the function $f \in A(p)$ and $g \in K_{p,q,s}^j(\alpha_1;\gamma)(|\gamma - \alpha_1| < |\alpha_1|)$. If $(H_{p,q,s}(\alpha_1)f(z))^{(j)}$ is majorized by $(H_{p,q,s}(\alpha_1)g(z))^{(j)}$ in U, then

$$\left|(H_{p,q,s}(\alpha_1+1)f(z))^{(j)}\right| \leq \left|(H_{p,q,s}(\alpha_1+1)g(z))^{(j)}\right| \quad (|z| < r_2)\,,$$

where $r_2 = r_2(\gamma, \alpha_1)$ is given by

$$r_2 = r_2(\gamma, \alpha_1) = \frac{k - \sqrt{k^2 - 4\,|\gamma - \alpha_1|\,|\alpha_1|}}{2\,|\gamma - \alpha_1|},$$

where $(k = 2 + |\alpha_1| + |\gamma - \alpha_1|\,; \gamma, \alpha_1 \in \mathbb{C}^*)$.

By specializing the parameters $p, q, s, \alpha_i (i = 1, 2, ..., q)$ and $\beta_j (j = 1, 2, ..., s)$, we obtain various results for different operators.

(ii) Putting in (1.6) and (1.7) $h(z) = z^p + \sum\limits_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m z^k$, (see, [3]), where $\ell \geq 0, \lambda > 0, m \in \mathbb{N}_0$, then $(f * h)(z) = I_p^m(\lambda,\ell)f(z), S_p^j(h;\gamma) = S_p^{m,j}(\lambda,\ell;\gamma)$ and $K_p^j(h;\gamma) = K_p^{m,j}(\lambda,\ell;\gamma)$. Also, we obtain the following results.

Let the function $f \in A(p)$ and $g \in S_p^{m,j}(\lambda,\ell;\gamma)(\left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right| < \left(\frac{p+\ell}{\lambda}\right))$. If $(I_p^m(\lambda,\ell)f(z))^{(j)}$ is majorized by $(I_p^m(\lambda,\ell)g(z))^{(j)}$ in U, then

$$\left|(I_p^{m+1}(\lambda,\ell)f(z))^{(j)}\right| \leq \left|(I_p^{m+1}(\lambda,\ell)g(z))^{(j)}\right| \quad (|z| < r_3)\,,$$

where $r_3 = r_3(p, \gamma, \lambda, \ell)$ is given by

$$r_3 = r_3(p, \gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4\left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|\left(\frac{p+\ell}{\lambda}\right)}}{2\left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|},$$

where $(k = 2 + \left(\frac{p+\ell}{\lambda}\right) + \left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*, \ell \geq 0, \lambda > 0, m \in \mathbb{N}_0)$.

Also, let the function $f \in A(p)$ and $g \in K_p^{m,j}(\lambda, \ell; \gamma)(\left|\gamma - \left(\frac{p+\ell}{\lambda}\right)\right| < \left(\frac{p+\ell}{\lambda}\right))$. If $(I_p^m(\lambda, \ell) f(z))^{(j)}$ is majorized by $(I_p^m(\lambda, \ell) g(z))^{(j)}$ in U, then

$$\left|(I_p^{m+1}(\lambda, \ell) f(z))^{(j)}\right| \leq \left|(I_p^{m+1}(\lambda, \ell) g(z))^{(j)}\right| \quad (|z| < r_4),$$

where $r_4 = r_4(p, \gamma, \lambda, \ell)$, is given by

$$r_4 = r_4(p, \gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4\left|\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|\left(\frac{p+\ell}{\lambda}\right)}}{2\left|\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|},$$

where $(k = 2 + \left(\frac{p+\ell}{\lambda}\right) + \left|\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*, \ell \geq 0, \lambda > 0)$.

By specializing the parameters p, λ, ℓ and m we obtain various results for different operators.

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