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Ricci Semi-Symmetric Lightlike Hypersurfaces of an Indefinite Cosymplectic Space Form

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Abstract. This paper is devoted to study Ricci semi-symmetric lightlike hypersurfaces of an indefinite cosymplectic space form with structure vector field tangent to hypersurface. The condition for Ricci tensor of lightlike hypersurface of indefinite cosymplectic space form to be semi-symmetric and parallel have been obtained. An example of non-totally geodesic Ricci semi-symmetric lightlike hypersurface in R_2^7 have been given.

1. Introduction

A semi-Riemannian manifold is called semi-symmetric if $R(X,Y)\cdot R = 0$, where $R(X,Y)$ is the curvature operator act as a derivative on R. Semi-symmetric hypersurfaces of Euclidean spaces were classified by Nomizu [9] and a general study of semi-symmetric Riemannian manifolds was made by Szabo [13]. A semi-Riemannian manifold is said to be Ricci semi-symmetric [4], if the following condition is satisfied: $R(X,Y)\cdot Ric = 0$.

It is clear that every semi-symmetric manifold is Ricci semi-symmetric; however the converse is not true in general. P. J. Ryan [11] raised the following question for hypersurfaces of Euclidean spaces in 1972; "Are the conditions $R(X,Y)\cdot R = 0$ and $R(X,Y)\cdot Ric = 0$ equivalent for hypersurfaces of Euclidean spaces?". In his papers [10, 11], Ryan proved that the two conditions are equivalent for hypersurfaces in spheres and hyperbolic spaces and for hypersurfaces of Euclidean space with non-negative scalar curvature. The explicit example of Ricci-symmetric but not semi-symmetric hypersurfaces in Euclidean space $E^{n+1}(n \geq 4)$ is given in [1, 4]. Also it is proved in [3] that Ricci-semisymmetric and semi-symmetric conditions are equivalent for hypersurfaces of 5-dimensional semi-Riemannian space of constant curvature. The geometry of lightlike hypersurfaces of semi-Riemannian manifolds was studied in [6]. The lightlike hypersurfaces of semi-Euclidean spaces satisfying curvature conditions of semi-symmetry type was studied in [12].

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The purpose of the present paper is to study the Ricci semi-symmetric lightlike hypersurface of indefinite cosymplectic space form with structure vector field ξ tangent to hypersurface.

In Section 2, We have collected the formulae and information which are useful in our subsequent sections. Section 3, is devoted to study the Ricci semi-symmetric lightlike hypersurfaces of an indefinite cosymplectic space form. Also, we have given an example of non-totally geodesic Ricci semi-symmetric lightlike hypersurface in R_2^7.

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, \xi, \eta, \overline{g}\}$, where ϕ is a (1,1) tensor field, ξ a vector field, η a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$(2.1)\qquad \begin{cases} \phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0, \qquad \phi\xi = 0, \quad \eta(\xi) = 1 \\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), \quad \overline{g}(X, \xi) = \eta(X) \end{cases}$$

for any $X, Y \in \Gamma(T\overline{M})$, where $\Gamma(T\overline{M})$ denotes the Lie algebra of vector fields on $\overline{M}$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite cosymplectic manifold if [7],

$$(2.2)\qquad (\overline{\nabla}_X \phi)Y = 0, \quad \textit{and} \quad \overline{\nabla}_X \xi = 0$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denote the Levi-Civita connection on $\overline{M}$.

An indefinite almost contact metric manifold $\{\overline{M}, \phi, \xi, \eta, \overline{g}\}$ is called an indefinite cosymplectic space form $\overline{M}(c)$ if it satisfies [7]

$$(2.3)\qquad \begin{aligned} \overline{R}(X, Y)Z = \tfrac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ &+2g(X, \phi Y)\phi Z + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

We write as follows:

$$(2.4)\qquad \overline{R}(X, Y, Z, W) = \overline{g}(\overline{R}(X, Y)Z, W)$$

$$(2.5)\qquad Ric(X, Y) = trace\{Z \to \overline{R}(X, Z)Y\}$$

where Ric denotes the Ricci tensor on $\overline{M}$ for $X, Y, Z, W \in \Gamma(T\overline{M})$.

For a (0, k)-tensor field T on $\overline{M}$, $k \geq 1$, the (0, $k+2$) tensor field $\overline{R} \cdot T = 0$ is called curvature conditions of semi-symmetry type [4] and given by

$$(2.6)\qquad \begin{aligned} (\overline{R}.T)(X_1, ..., X_k, X, Y) = &-T(\overline{R}(X, Y)X_1, X_2, ..., X_k) \\ &-... - T(X_1, ..., X_{k-1}, \overline{R}(X, Y)X_k) \end{aligned}$$

for $X, Y, X_1, X_k \in \Gamma(T\overline{M})$.

A semi-Riemannian space form $\overline{M}$ is said to be Ricci semi-symmetric if $\overline{R}.Ric = 0$, i.e.,

(2.7) $$(\overline{R}(X,Y).Ric)(X_1,X_2) = -Ric(\overline{R}(X,Y)X_1,X_2) - Ric(X_1,\overline{R}(X,Y)X_2) = 0$$

for any $X, Y, X_1, X_2 \in \Gamma(T\overline{M})$.

Let (M, g) be a hypersurface of a $(2m+1)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ with index s, $0 < s < 2m+1$ and $g = \overline{g}_{|M}$. Then M is lightlike hypersurface of $\overline{M}$ if g is of constant rank $(2m-1)$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M [6]. A non-degenerate complementary distribution $S(TM)$ of rank $(2m-1)$ to $TM^{\perp}$ in TM, that is, $TM = TM^{\perp} \perp S(TM)$, is called screen distribution. The following result (cf. [6], Theorem 1.1, page 79) has an important role in studying the geometry of lightlike hypersurface.

Theorem A. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\overline{M}$. Then, there exists a unique vector bundle $tr(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighbourhood $U \subset M$, there exists a unique section N of $tr(TM)$ on U satisfying $\overline{g}(N,E) = 1$ and $\overline{g}(N,N) = \overline{g}(N,W) = 0$, $\forall\, W \in \Gamma(S(TM)|_u)$.*
Then, we have the following decomposition:

(2.8) $$TM = S(TM) \perp TM^{\perp}, \qquad T\overline{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)).$$

Throughout this paper, all manifolds are supposed to be paracompact and smooth. We denote by $\Gamma(E)$ the smooth sections of the vector bundle E, by $\perp$ and $\oplus$ the orthogonal and the non-orthogonal direct sum of two vector bundles, respectively.

Let $\overline{\nabla}$, ∇ and ∇^t denote the linear connections on $\overline{M}$, M and vector bundle $tr(TM)$, respectively. Then, the Gauss and Weingarten formulae are given by

(2.9) $$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \forall X, Y \in \Gamma(TM)$$

(2.10) $$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \ \ \forall V \in \Gamma(tr(TM))$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively and A_V is the shape operator of M with respect to V. Moreover, in view of decomposition (2.9), equations (2.10) and (2.11) take the form

(2.11) $$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N$$

(2.12) $$\overline{\nabla}_X N = -A_N X + \tau(X)N$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$, where $B(X,Y)$ and $\tau(X)$ are local second fundamental form and a 1-form on U, respectively. It follows that

$$B(X,Y) = \overline{g}(\overline{\nabla}_X Y, E) = \overline{g}(h(X,Y),E), B(X,E) = 0, \text{ and}$$

$$\tau(X) = \overline{g}(\nabla^t_X N, E).$$

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ and ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and $STM^\perp$, respectively. Then from the decomposition of tangent bundle of lightlike hypersurface, we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY) \tag{2.13}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$ and $E \in \Gamma(TM^\perp)$, where h^*, A^* are the second fundamental form and the shape operator of distribution $S(TM)$ respectively.

By direct calculations using Gauss-Weingarten formulae, (2.14) and (2.15), we find

$$g(A_N Y, PW) = \overline{g}(N, h^*(Y, PW)); \qquad \overline{g}(A_N Y, N) = 0, \tag{2.15}$$

$$g(A^*_E X, PY) = \overline{g}(E, h(X, PY); \qquad \overline{g}(A^*_E X, N) = 0, \tag{2.16}$$

for any $X, Y, W \in \Gamma(TM)$, $E \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$.

Locally, we define on U

$$C(X, PY) = \overline{g}(h^*(X, PY), N), \qquad \lambda(X) = \overline{g}(\nabla^{*t}_X E, N). \tag{2.17}$$

Hence,

$$h^*(X, PY) = C(X, PY)E, \qquad \nabla^{*t}_X E = \lambda(X)E. \tag{2.18}$$

On the other hand, by using (2.12), (2.13), (2.15) and (2.18), we obtain

$$\lambda(X) = \overline{g}(\nabla_X E, N) = \overline{g}(\overline{\nabla}_X E, N) = -\overline{g}(E, \overline{\nabla}_X N) = -\tau(X).$$

Thus, locally (2.14) and (2.15) become

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \qquad \nabla_X E = -A^*_E X - \tau(X)E. \tag{2.19}$$

Finally, (2.16) and (2.17), locally become

$$g(A_N Y, PW) = C(Y, PW); \qquad \overline{g}(A_N Y, N) = 0, \tag{2.20}$$

$$g(A^*_E X, PY) = B(X, PY); \qquad \overline{g}(A^*_E X, N) = 0. \tag{2.21}$$

We note that second equation of (2.21) implies that $A_N X \in \Gamma(S(TM))$ for $X \in \Gamma(TM)$, i.e. A_N is $\Gamma(S(TM))$ valued. On the other hand, from $\overline{g}(\overline{\nabla}_X E, E) = 0$, we have

$$B(X, E) = 0. \tag{2.22}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, we have

$$0 = (\overline{\nabla}_X \overline{g})(Y, Z) = X(\overline{g}(Y, Z)) - \overline{g}(\overline{\nabla}_X Y, Z) - \overline{g}(Y, \overline{\nabla}_X Z).$$

By using (2.12) in this equation, we obtain

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \quad X, Y \in \Gamma(S(TM)|_u), \tag{2.23}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) = \overline{g}(N, \cdot)$.

If $\overline{R}$ and R are the curvature tensors of $\overline{M}$ and M, then using (2.12) in the equation $\overline{R}(X, Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z$, we obtain

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+\{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N \end{aligned} \tag{2.24}$$

$$(\nabla_X B)(Y, Z) = XB(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{2.25}$$

3. Ricci Semi-symmetric Lightlike Hypersurfaces in Indefinite Cosymplectic Space Form

In this section, we consider Ricci semi-symmetric lightlike hypersurfaces M in an indefinite cosymplectic space form $\overline{M}(c)$.

For $X \in \Gamma(TM)$, we write

$$\phi X = tX + \beta(X)N \tag{3.1}$$

where tX is the tangential parts of ϕX and β is the one form on M.

We have following :

Lemma 3.1. *Let M be a lightlike hypersurface of a $(2m + 1)$-dimensional indefinite cosymplectic space form $\overline{M}(c)$. Then the Gauss equation of M is given by*

$$\begin{aligned} R(X, Y)Z = B(Y, Z)A_N X - B(X, Z)A_N Y + \tfrac{c}{4}\{g(Y, Z)X - g(X, Z)Y \\ +g(X, \phi Z)tY - g(Y, \phi Z)tX + 2g(X, \phi Y)tZ + \eta(X)\eta(Z)Y \\ -\eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}. \end{aligned} \tag{3.2}$$

Proof. From (2.3), (2.25), (3.1) and comparing the tangential part, we obtain (3.2). □

Theorem 3.1. *Let M be a lightlike hypersurface of a $(2m + 1)$-dimensional indefinite cosymplectic space form $\overline{M}(c)$. Then, we have*

$$\begin{aligned} Ric(X, Y) = \tfrac{c}{4}\{-(2m - 1)g(X, Y) - g(tX, \phi Y) + 2g(\phi X, tY) \\ +g(E, \phi Y)g(tX, N) - 2g(X, \phi E)g(tY, N) + g(X, Y) \\ +(2m - 2)\eta(X)\eta(Y)\} + \textstyle\sum_{i=1}^{2m-1} \epsilon_i B(w_i, Y)C(X, w_i) \\ -\alpha B(X, Y) \end{aligned} \tag{3.3}$$

where $\{(w_i), i = 1, 2, ..., (2m-1)\}$ *is the orthogonal basis of* $S(TM)$ *and* $\alpha = \sum_{i=1}^{2m-1} \epsilon_i C(W_i, W_i)$.

Proof. By the definition of Ricci curvature

$$Ric(X,Y) = \sum_{i=1}^{2m-1} \epsilon_i g(R(X,w_i)Y,w_i) + \overline{g}(R(X,E)Y,N)$$

From (3.2), we have

$$\begin{aligned} Ric(X,Y) = \tfrac{c}{4}\{&-(2m-1)g(X,Y) - g(tX,\phi Y) + 2g(\phi X, tY) \\ &+g(E,\phi Y)g(tX,N) - 2g(X,\phi E)g(tY,N) + g(X,Y) \\ &+(2m-2)\eta(X)\eta(Y)\} + \textstyle\sum_{i=1}^{2m-1} \epsilon_i\{B(W_i,Y)C(X,W_i) \\ &-B(X,Y)C(W_i,W_i)\}. \end{aligned} \tag{3.4}$$

Since $\sum_{i=1}^{2m-1} \epsilon_i C(W_i, W_i) = \alpha$, hence (3.3) follows from (3.4). □

From theorem 3.1, we have

Proposition 3.1. *The Ricci tensor of a lightlike hypersurface in a* $(2m+1)$*-dimensional indefinite cosymplectic space form* $\overline{M}(c)$ *is degenerate if* $c = 0$.

Proposition 3.2. *The Ricci tensor of a lightlike hypersurface in a* $(2m+1)$*-dimensional indefinite cosymplectic space form* $\overline{M}(c)$ *is symmetric if* $c = 0$ *and the shape operator of a lightlike hypersurface of* $\overline{M}(c)$ *is symmetric with respect to the second fundamental form of* M.

Proof. For proof (cf. Proposition 3.3 [15]). □

The following corollary is similar to the Corollary 3.3 in [15]:

Corollary 3.1. *The Ricci tensor of a lightlike hypersurface in a* $(2m+1)$*-dimensional indefinite cosymplectic space form* $\overline{M}(c)$ *is symmetric if* $c = 0$ *and* $C(X, A^*_\xi Y) = C(Y, A^*_\xi X)$.

Theorem 3.2. *Let* M *be a totally geodesic lightlike hypersurface of an indefinite cosymplectic space form* $\overline{M}(c)$. *Then, the Ricci tensor of* M *is parallel with respect to* ∇ *if* $c = 0$.

Proof. The derivative of Ricci tensor is given by

$$(\nabla_Z Ric)(X,Y) = \nabla_Z Ric(X,Y) - Ric(\nabla_Z X, Y) - Ric(X, \nabla_Z Y).$$

Then, from (2.23) and (3.3), we have

$$
\begin{aligned}
(\nabla_Z Ric)(X,Y) = -(2m-1)[Z(\tfrac{c}{4})g(X,Y) + \tfrac{c}{4}\{B(Z,X)\theta(Y)\\
+B(Z,Y)\theta(X)\}] - Z(\tfrac{c}{4})\{g(tX,\phi Y) + 2g(\phi X,tY)\\
+g(E,\phi Y)g(tX,N) - 2g(X,\phi E)g(tY,N)\} - \tfrac{c}{4}\{B(Z,tX)\theta(\phi Y)\\
+B(Z,\phi Y)\theta(tX) + g((\nabla_Z t)X,\phi Y) + g(tX,(\nabla_Z\phi)Y)\\
+B(Z,\phi Y)g(tX,N) + g(\nabla_Z E,\phi Y)g(tX,N)\\
+g(E,(\nabla_Z\phi)Y)g(tX,N) + g(E,\phi Y)g((\nabla_Z t)X,N)\\
+g(E,\phi Y)g(tX,\nabla_Z N) - 2B(Z,X)\theta(\phi E)g(tY,N)\\
-2B(Z,\phi E)\theta(X)g(tY,N) - 2g(X,\phi E)g((\nabla_Z t)Y,N)\\
-2g(X,\phi E)g(tY,\nabla_Z N) - 2g(X,\nabla_Z\phi E)g(tY,N)\\
+g(X,Y) + (2m-2)\eta(X)\eta(Y) + B(Z,X)\theta(Y)\\
+B(Z,Y)\theta(X) + (2m-2)\{B(Z,\xi)\theta(X)\eta(Y) + B(Z,\xi)\theta(Y)\eta(X)\\
+g(X,\nabla_Z\xi)\eta(Y) + g(Y,\nabla_Z\xi)\eta(X) + B(Z,\xi)\theta(Y)\eta(X)\\
+g(Y,\nabla_Z\xi)\eta(X)\} + \textstyle\sum_{i=1}^{2m-1}\epsilon_i\{(\nabla_Z B)(W_i,Y)C(X,W_i)\\
+B(\nabla_Z W_i,Y)C(X,W_i) + B(W_i,Y)C(X,\nabla_Z W_i)\\
+B(W_i,Y)(\nabla_Z C)(X,W_i)\} - Z(\alpha)B(X,Y) - \alpha(\nabla_Z B)(X,Y).
\end{aligned}
\tag{3.5}
$$

Since, M is totally geodesic lightlike hypersurface of an indefinite cosympletic space form, therefore $B(X,Y)=0$ and $\nabla_X\xi = 0$ $\forall X,Y\in\Gamma(TM)$. Hence, from (3.5), we find

$$
\begin{aligned}
(\nabla_Z Ric)(X,Y) = -(2m-1)Z(\tfrac{c}{4})g(X,Y) - Z(\tfrac{c}{4})\{g(tX,\phi Y)\\
+2g(\phi X,tY) + g(E,\phi Y)g(tX,N) - 2g(X,\phi E)g(tY,N)\}\\
-\tfrac{c}{4}\{g((\nabla_Z t)X,\phi Y) + g(tX,(\nabla_Z\phi)Y)\\
+g(\nabla_Z E,\phi Y)g(tX,N) + g(E,(\nabla_Z\phi)Y)g(tX,N)\\
+g(E,\phi Y)g((\nabla_Z t)X,N) + g(E,\phi Y)g(tX,\nabla_Z N)\\
-2g(X,\phi E)g((\nabla_Z t)Y,N) - 2g(X,\phi E)g(tY,\nabla_Z N)\\
-2g(X,\nabla_Z\phi E)g(tY,N) + g(X,Y) + (2m-2)\eta(X)\eta(Y)\}.
\end{aligned}
\tag{3.6}
$$

From (3.6) it is obvious that $(\nabla_Z Ric)(X,Y) = 0$ if $c = 0$, which proves the Theorem. □

Theorem 3.3. *Let M be a Ricci semi-symmetric lightlike hypersurface of an $(2m+1)$-dimensional indefinite cosymplectic space form $\overline{M}(c)$. If $c=0$, then, either M is totally geodesic or $Ric(E,A_N E)=0$ for $E\in\Gamma(TM^{\perp})$, where Ric is the Ricci tensor of M and A denotes the shape operator of M.*

Proof. Suppose M is Ricci semi-symmetric, then from (2.8), we have

$$
0 = -Ric(R(X,Y)X_1,X_2) - Ric(X_1,R(X,Y)X_2) \tag{3.7}
$$

Using (3.3) in (3.7), we find

$$
\begin{aligned}
0 = & -B(Y,X_1)Ric(A_NX,X_2)+B(X,X_1)Ric(A_NY,X_2)\\
& -\tfrac{c}{4}\{g(Y,X_1)Ric(X,X_2)-g(X,X_1)Ric(Y,X_2)\\
& -g(X,\phi X_1)Ric(tY,X_2)-g(Y,\phi X_1)Ric(tX,X_2)\\
& +2g(X,\phi Y)Ric(tX_1,X_2)-\eta(X)\eta(X_1)Ric(Y,X_2)\\
& -\eta(Y)\eta(X_1)Ric(X,X_2)+g(X,X_1)\eta(Y)Ric(\xi,X_2)\\
& -g(Y,X_1)\eta(X)Ric(\xi,X_2)-g(Y,X_2)Ric(X_1,X)\\
& -g(X,X_2)Ric(X_1,Y)-g(X,\phi X_2)Ric(X_1,tY)\\
& -g(Y,\phi X_2)Ric(X_1,tX)+2g(X,\phi Y)Ric(X_1,tX_2)\\
& -\eta(X)\eta(X_2)Ric(X_1,Y)-\eta(Y)\eta(X_2)Ric(X_1,X)\\
& +g(X,X_2)\eta(Y)Ric(X_1,\xi)-g(Y,X_2)\eta(X)Ric(X_1,\xi)\}\\
& -B(Y,X_2)Ric(X_1,A_NX)+B(X,X_2)Ric(X_1,A_NY).
\end{aligned}
\tag{3.8}
$$

Putting $X_1 = E$ in (3.8) and using (2.22), we obtain;

$$
\begin{aligned}
0 = & -\tfrac{c}{4}\{g(X,\phi E)Ric(tY,X_2)-g(Y,\phi E)Ric(tX,X_2)\\
& +2g(X,\phi Y)Ric(tE,X_2)-g(Y,X_2)Ric(E,X)\\
& -g(X,X_2)Ric(E,Y)-g(X,\phi X_2)Ric(E,tY)\\
& -g(Y,\phi X_2)Ric(E,tX)+2g(X,\phi Y)Ric(E,tX_2)\\
& -\eta(X)\eta(X_2)Ric(E,Y)-\eta(Y)\eta(X_2)Ric(E,X)\\
& +g(X,X_2)\eta(Y)Ric(E,\xi)-g(Y,X_2)\eta(X)Ric(E,\xi)\}\\
& -B(Y,X_2)Ric(E,A_NX)+B(X,X_2)Ric(E,A_NY).
\end{aligned}
\tag{3.9}
$$

Putting $Y = E$ in (3.9), we get;

$$
\begin{aligned}
0 = & -\tfrac{c}{4}\{3g(X,\phi E)Ric(tE,X_2)+g(X,\phi X_2)Ric(E,tE)\\
& -g(E,\phi X_2)Ric(E,tX)+2g(X,\phi E)Ric(E,tX_2)\}\\
& +B(X,X_2)Ric(E,A_NE).
\end{aligned}
\tag{3.10}
$$

If $c = 0$ then from (3.10), we have

$$B(X,X_2)Ric(E,A_NE)=0.$$

So, if $B(X,X_2) = 0$ for any $X, X_2 \in \Gamma(TM)$, then M is totally geodesic. If M is not totally geodesic, it follows that $Ric(E,A_NE) = 0$. □

Hereafter, $(R_q^{2m+1}, \overline{\phi}_0, \xi, \eta, \overline{g}_0)$ will denote the manifold R_q^{2m+1} with its cosymplectic structure given by

$$
\begin{cases}
\eta = dz, \qquad \xi = \partial z,\\
\overline{g_0} = \eta\bigotimes\eta-\sum_{i=1}^{q}dx^i\bigotimes dx^i+\sum_{i=q+1}^{2m}dx^i\bigotimes dx^i,\\
\overline{\phi}_0(\sum_{i=1}^{2m}X_i\partial x^i+Z\partial z)=(-X_2,X_1,-X_4,X_3,...,-X_{2m},X_{2m-1},0),
\end{cases}
\tag{3.11}
$$

where (x^i, z) are the Cartesian coordinates.

Example. Let $\overline{M} = (R_2^7, \overline{g}_0)$ be a 7-dimensional indefinite cosymplectic manifold of index 2 with signature (-, -, +, +, +, +, +) of the canonical basis $\{\partial x_1, \partial x_2, ..., \partial x_6, \partial z\}$. Consider a lightlike hypersurface M of R_2^7 given by

$$X(u_1, u_2, u_3, u_4, u_5, t) = (u_1 + \sqrt{2}\sqrt{u_2^2 + u_3^2}, u_1, u_2, u_3, u_4, u_5, t).$$

Then $\mathrm{Rad}(TM) = span\{E = \sqrt{x_3^2 + x_4^2}\partial x_1 - \sqrt{x_3^2 + x_4^2}\partial x_2 + \sqrt{2}x_3\partial x_3 + \sqrt{2}x_4\partial x_4\}$ and $tr(TM) = span\{N = \frac{1}{4(x_3^2+x_4^2)}(-\sqrt{x_3^2 + x_4^2}\partial x_1 + \sqrt{x_3^2 + x_4^2}\partial x_2 + \sqrt{2}x_3\partial x_3 + \sqrt{2}x_4\partial x_4)\}$. The screen distribution $S(TM)$ is spanned by

$$\{Z_1 = \partial x_1 + \partial x_2, Z_2 = -x_4\partial x_3 + x_3\partial x_4, Z_3 = \partial x_5, Z_4 = \partial x_6\}\perp\{\xi = \partial z\}.$$

Now, we have

$$\overline{\phi}_0 E = \sqrt{x_3^2 + x_4^2} Z_1 + Z_2 \in S(TM),$$

$$\overline{\phi}_0 N = \frac{1}{4(x_3^2+x_4^2)}(-\sqrt{x_3^2 + x_4^2} Z_1 + Z_2) \in S(TM)$$

and $D_0 = span\{Z_3, Z_4\}$. By direct computations, we obtain

$$\overline{\nabla}_X Z_1 = \overline{\nabla}_{Z_1} X = 0, \overline{\nabla}_E E = \sqrt{2}E, \overline{\nabla}_{Z_2} E = \overline{\nabla}_E Z_2 = \sqrt{2}Z_2, \overline{\nabla}_X \xi = \overline{\nabla}_\xi X = 0,$$

$$\overline{\nabla}_{Z_2} Z_2 = -x_3\partial x_3 - x_4\partial x_4, \overline{\nabla}_X Z_3 = \overline{\nabla}_{Z_2} X = 0, \overline{\nabla}_X Z_4 = \overline{\nabla}_{Z_4} X = 0,$$

$$\overline{\nabla}_E N = \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{x_3^2+x_4^2}}\partial x_1 - \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{x_3^2+x_4^2}}\partial x_2 - \frac{1}{2}\frac{x_3}{(x_3^2+x_4^2)}\partial x_3 - \frac{1}{2}\frac{x_4}{(x_3^2+x_4^2)}\partial x_4,$$

$$\overline{\nabla}_{Z_1} N = \overline{\nabla}_{Z_3} N = \overline{\nabla}_{Z_4} N = \overline{\nabla}_\xi N = 0,$$

and

$$\overline{\nabla}_{Z_2} N = -\frac{x_4}{2\sqrt{2}(x_3^2+x_4^2)}\partial x_3 + \frac{x_3}{2\sqrt{2}(x_3^2+x_4^2)}\partial x_4,$$

for any $X \in \Gamma(TM)$. Thus, from the Weingarten formulae, we have

$$A_N E = 0, A_N Z_1 = A_N Z_3 = A_N Z_4 = A_N \xi = 0, A_N Z_2 = \frac{1}{2\sqrt{2}(x_3^2+x_4^2)} Z_2,$$

$$A_E^* Z_1 = 0, A_E^* Z_2 = -\sqrt{2}Z_2, A_E^* Z_3 = A_E^* Z_4 = A_E^* \xi = 0.$$

It is easy to see that M is non-totally geodesic Ricci semi-symmetric lightlike hypersurface.

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