

Some Identities Involving Euler Polynomials Arising from a Non-linear Differential Equation

SEOG-HOON RIM, JOOHEE JEONG* AND JIN-WOO PARK
Department of Mathematics Education, Kyungpook National University, Daegu, 702-701, Korea
e-mail: shrim@knu.ac.kr, jhjeong@knu.ac.kr and a0417001.paper@gmail.com

ABSTRACT. We derive a family of non-linear differential equations from the generating functions of the Euler polynomials and study the solutions of these differential equations. Then we give some new and interesting identities and formulas for the Euler polynomials of higher order by using our non-linear differential equations.

1. Introduction

The Euler polynomials $E_n(x)$ are defined by generating functions as follows:

$$(1.1) \quad \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (\text{see [2, 4, 5]})$$

In the special case $x = 0$, $E_n(0) = E_n$ for $n = 0, 1, \dots$ are called the n -th Euler numbers. (see [4])

By (1.1), we get

$$(1.2) \quad E_n(x) = \sum_{\ell=0}^{\infty} \binom{n}{\ell} x^{n-\ell} E_{\ell}, \quad \text{for } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.$$

Thus, by (1.1) and (1.2), we get the recursive relation for E_n 's as follows:

$$(1.3) \quad E_0 = 1, \\ (E + 1)^n + E_n = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

* Corresponding Author.

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with the usual convention of replacing E^k 's in the binomial expansion of $(E + 1)^n$ by E_k 's.

In [11], Nielson obtained a formula for the product of two Euler polynomials $E_n(x)$ and $E_m(x)$ as follows:

$$E_m(x)E_n(x) = 2 \sum_{r=1}^m \binom{m}{r} E_r \frac{B_{m+n-r+1}(x)}{m+n-r+1} + 2 \sum_{s=1}^n \binom{n}{s} E_s \frac{B_{m+n-s+1}(x)}{m+n-s+1} + (-1)^{n+1} 2 \frac{m!n!}{(m+n+1)!} E_{m+n+1},$$

where $B_m(x)$ is the well-known Bernoulli polynomials. (See [2, 4, 5, 12]).

For $r \in \mathbb{N}$, the Euler polynomials $E_n^{(r)}(x)$ of order r are defined by generating functions as follows:

$$(1.4) \quad \underbrace{\left(\frac{2}{e^t+1}\right) \cdots \left(\frac{2}{e^t+1}\right)}_{r\text{-times}} e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}.$$

In the special case $x = 0$, $E_n^{(r)}(0) = E_n^{(r)}$ are called the n -th Euler numbers of order r . (see [1–15, 17])

In this paper, using the idea of T. Kim [10], we derive a family of non-linear differential equations from (1.1) and study the solutions of these differential equations. Then we give some new and interesting identities and formulas for the Euler polynomials of higher order by using our non-linear differential equations.

2. The Non-linear Differential Equations

Throughout this paper, we put

$$(2.1) \quad F = F(t) = \frac{1}{e^t + 1}.$$

Then we get

$$(2.2) \quad F'(t) = -\frac{e^t}{(e^t + 1)^2} = \frac{1}{(e^t + 1)^2} - \frac{1}{e^t + 1} = F^2 - F.$$

We rewrite (2.2) as

$$(2.3) \quad F^2 = F' + F,$$

and differentiate both sides with respect to t to get

$$(2.4) \quad 2FF' = F'' + F'.$$

Now, by (2.2), we may replace F' in the left hand side of (2.4) by $F^2 - F$ to obtain

$$(2.5) \quad 2F(F^2 - F) = F'' + F',$$

and consequently

$$(2.6) \quad 2F^3 = 2F^2 + F'' + F'.$$

From (2.3) and (2.6), we get

$$(2.7) \quad \begin{aligned} 2F^3 &= 2(F' + F) + F'' + F' \\ &= F'' + 3F' + 2F. \end{aligned}$$

Now we repeat the process from (2.4) to (2.7). We start from differentiating both sides of (2.7).

$$(9') \quad 3!F^2F' = F^{(3)} + 3F^{(2)} + 2F^{(1)},$$

$$(10') \quad 3!F^2(F^2 - F) = F^{(3)} + 3F^{(2)} + 2F^{(1)},$$

$$(11') \quad 3!F^4 = 3!F^3 + (F^{(3)} + 3F^{(2)} + 2F^{(1)}),$$

$$3!F^4 = 3(F'' + 3F' + 2F) + F^{(3)} + 3F^{(2)} + 2F^{(1)}, \quad \text{from (2.7) and (11')}$$

$$(12') \quad 3!F^4 = F^{(3)} + 6F^{(2)} + 11F^{(1)} + 6F.$$

We want to generalize the pattern (2.3), (2.7) and (12') to obtain

$$(2.8) \quad (N-1)!F^N = \sum_{k=0}^{N-1} a_k(N)F^{(k)}, \quad \text{for all } N \in \mathbb{N}$$

for some suitable coefficients $a_k(N)$'s. By letting $N = 1$ in (2.8), it is easy to get

$$(2.9) \quad a_0(1) = 1.$$

Taking the derivative of (2.8), we have

$$(2.10) \quad N!F^{N-1}F' = \sum_{k=0}^{N-1} a_k(N)F^{(k+1)} = \sum_{k=1}^N a_{k-1}(N)F^{(k)}.$$

But

$$\begin{aligned} N!F^{N-1}F' &= N!F^{N-1}(F^2 - F) \quad (\text{by (2.2)}) \\ &= N!F^{N+1} - N!F^N. \end{aligned}$$

Thus

$$\sum_{k=1}^N a_{k-1}(N)F^{(k)} = N!F^{N+1} - N!F^N$$

and hence

$$\begin{aligned}
 N!F^{N+1} &= N!F^N + \sum_{k=1}^N a_{k-1}(N)F^{(k)} \\
 &= N(N-1)!F^N + \sum_{k=1}^N a_{k-1}(N)F^{(k)} \\
 (2.11) \quad &= N \sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^N a_{k-1}(N)F^{(k)}, \quad (\text{by (2.8)}).
 \end{aligned}$$

By replacing N by $N+1$, (2.8) may be rewritten as

$$(2.12) \quad N!F^{N+1} = \sum_{k=0}^N a_k(N+1)F^{(k)}.$$

Combining (2.12) and (2.11), we get

$$(2.13) \quad \sum_{k=0}^N a_k(N+1)F^{(k)} = N \sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^N a_{k-1}(N)F^{(k)}.$$

By comparing the coefficients of $F^{(k)}$'s in the left and right side of above identity, we obtain the following recursive formulas:

$$(2.14) \quad a_0(N+1) = Na_0(N),$$

$$(2.15) \quad a_k(N+1) = Na_k(N) + a_{k-1}(N), \quad (1 \leq k \leq N-1)$$

$$(2.16) \quad a_N(N+1) = a_{N-1}(N),$$

where we set

$$(2.17) \quad a_k(N) = 0 \text{ for } k \geq N \text{ or } k < 0.$$

From (2.9), (2.14), (2.15), (2.16) and (2.17), T. Kim was able to obtain an explicit formula for $a_k(N)$ as follows. (see [10])

$$(2.18) \quad a_k(N) = \frac{N!}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}}.$$

Therefore (2.8) now becomes

$$(N-1)!F^N = \sum_{k=0}^{N-1} \frac{N!}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} F^{(k)},$$

and consequently we obtain the following theorem.

Theorem 2.1. For each $N \in \mathbb{N}$, the following non-linear differential equation with respect to t :

$$(2.19) \quad F^N(t) = N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1+\dots+\ell_{k+1}=N} \frac{1}{\ell_1\ell_2\cdots\ell_{k+1}} F^{(k)}(t)$$

where $F^{(k)}(t) = \frac{d^k F(t)}{dt^k}$ and $F^N(t) = \underbrace{F(t) \times \cdots \times F(t)}_{N\text{-times}}$ has a solution

$$F(t) = \frac{1}{e^t + 1}.$$

3. Some Identities Concerning Euler Polynomials of Order r

We set

$$(3.1) \quad F^N(t, x) = \underbrace{F(t) \times F(t) \times \cdots \times F(t)}_{N\text{-times}} e^{xt}, \quad \text{for } N \in \mathbb{N},$$

$$(3.2) \quad F^{(k)}(t, x) = F^{(k)}(t)e^{xt}, \quad \text{for } k \in \mathbb{Z}_+.$$

Then, by multiplying e^{xt} to both sides of (2.19), we have the following corollary.

Corollary 3.1. For each $n \in \mathbb{N}$, the differential equation

$$(3.3) \quad F^N(t, x) = N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1+\dots+\ell_{k+1}=N} \frac{1}{\ell_1\ell_2\cdots\ell_{k+1}} F^{(k)}(t, x)$$

has a solution $F(t, x) = \frac{e^{xt}}{e^t + 1}$.

The goal of this section is to find an explicit formula for Euler polynomials of order r using the Euler numbers. But first, we will work on Euler numbers of order r instead of Euler polynomials of order r .

By letting $x = 0$ in (1.1) and (1.4), we get

$$(3.4) \quad \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

$$(3.5) \quad \underbrace{\left(\frac{2}{e^t + 1}\right) \times \cdots \times \left(\frac{2}{e^t + 1}\right)}_{N\text{-times}} = \sum_{n=0}^{\infty} E_n^{(N)} \frac{t^n}{n!}.$$

From (2.1) and (3.4), we get

$$(3.6) \quad F(t) = \frac{1}{2} \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

and from (3.5),

$$(3.7) \quad \begin{aligned} F^N(t) &= \left(\frac{1}{e^t + 1} \right)^N = \frac{1}{2^N} \left(\frac{2}{e^t + 1} \right)^N \\ &= \frac{1}{2^N} \sum_{n=0}^{\infty} E_n^{(N)} \frac{t^n}{n!}. \end{aligned}$$

Note that we have the following by differentiating both sides of (3.6) k -times:

$$(3.8) \quad F^{(k)}(t) = \frac{1}{2} \sum_{n=0}^{\infty} E_{n+k} \frac{t^n}{n!}.$$

From (2.19) in Theorem 2.1, (3.7) and (3.8), we get

$$(3.9) \quad \begin{aligned} N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} F^{(k)}(t) \\ = N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} \frac{1}{2} \sum_{n=0}^{\infty} E_{n+k} \frac{t^n}{n!} \\ = \frac{1}{2} N \sum_{n=0}^{\infty} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{E_{n+k}}{\ell_1 \ell_2 \dots \ell_{k+1}} \frac{t^n}{n!} \end{aligned}$$

$$(3.10) \quad = \frac{1}{2^N} \sum_{n=0}^{\infty} E_n^{(N)} \frac{t^n}{n!}.$$

Finally, from (3.9) and (3.10) we obtain the following theorem.

Theorem 3.2. For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, the n -th Euler number of order N is given by

$$(3.11) \quad E_n^{(N)} = N 2^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{E_{n+k}}{\ell_1 \ell_2 \dots \ell_{k+1}}.$$

From (3.7) and (3.4), we can derive the following equation:

$$\begin{aligned}
 \sum_{n=0}^{\infty} E_n^{(N)} \frac{t^n}{n!} &= \left(\frac{2}{e^t + 1} \right)^N \\
 &= \left(\sum_{\ell_1=0}^{\infty} E_{\ell_1} \frac{t^{\ell_1}}{\ell_1!} \right) \times \cdots \times \left(\sum_{\ell_N=0}^{\infty} E_{\ell_N} \frac{t^{\ell_N}}{\ell_N!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{\ell_1+\cdots+\ell_N=n} \frac{E_{\ell_1} \cdots E_{\ell_N} n!}{\ell_1! \cdots \ell_N!} \right) \frac{t^n}{n!} \\
 (3.12) \quad &= \sum_{n=0}^{\infty} \left(\sum_{\ell_1+\cdots+\ell_N=n} \binom{n}{\ell_1 \cdots \ell_N} E_{\ell_1} \cdots E_{\ell_N} \right) \frac{t^n}{n!}.
 \end{aligned}$$

From (3.11) and (3.12) we get the following corollary.

Corollary 3.3.

$$(3.13) \quad \sum_{\ell_1+\cdots+\ell_N=n} \binom{n}{\ell_1 \cdots \ell_N} E_{\ell_1} \cdots E_{\ell_N} = N 2^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1+\cdots+\ell_{k+1}=N} \frac{E_{n+k}}{\ell_1 \ell_2 \cdots \ell_{k+1}}.$$

We obtained Theorem 3.2, a formula for the n -th Euler numbers of order N , using Theorem 2.1. If we use Corollary 3.1 instead of Theorem 2.1, then we can obtain Theorem 3.4, a formula for the n -th Euler polynomials of order N .

We start from (3.2) and (3.8) as follows:

$$\begin{aligned}
 F^{(k)}(t, x) &= F^{(k)}(t) e^{xt} = \frac{1}{2} \sum_{n=0}^{\infty} E_{n+k} \frac{t^n}{n!} e^{xt} \\
 &= \frac{1}{2} \sum_{m=0}^{\infty} E_{m+k} \frac{t^m}{m!} \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} x^\ell \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{m+\ell=n} \frac{n!}{m! \ell!} \frac{t^n}{n!} E_{m+k} x^\ell \right) \\
 (3.14) \quad &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} E_{m+k} x^{n-m} \right) \frac{t^n}{n!}.
 \end{aligned}$$

In Corollary 3.1, replace $F^{(k)}(t, x)$ by (3.14) to get

(3.15)

$$F^N(t, x) = N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} E_{m+k} x^{n-m} \right) \frac{t^n}{n!}$$

(3.16)

$$= \sum_{n=0}^{\infty} \left(\frac{N}{2} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} \sum_{m=0}^n \binom{n}{m} E_{m+k} x^{n-m} \right) \frac{t^n}{n!}.$$

But (1.4) can be written as

$$(3.17) \quad 2^N F^N(t, x) = \sum_{n=0}^{\infty} E_n^{(N)}(x) \frac{t^n}{n!}.$$

Finally, from (3.16) and (3.17), we get the following theorem.

Theorem 3.4. For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, the n -th Euler polynomial of order N is given by

(3.18)

$$E_n^{(N)}(x) = N 2^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \dots + \ell_{k+1} = N} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} \sum_{m=0}^n \binom{n}{m} E_{m+k} x^{n-m}.$$

We will conclude our paper by giving two more formulas for $E_n^{(N)}(x)$. They are given in Proposition 3.5 and Proposition 3.6.

First, multiply $2^N e^{xt}$ to both sides of (3.7). Then we have

$$(3.19) \quad 2^N F^N(t) e^{xt} = \sum_{n=0}^{\infty} E_n^{(N)} \frac{t^n}{n!} e^{xt}.$$

Then apply the usual trick of utilizing the Taylor expansion of e^{xt} to get

$$(3.20) \quad 2^N F^N(t)e^{xt} = \left(\sum_{\ell=0}^{\infty} E_{\ell}^{(N)} \frac{t^{\ell}}{\ell!} \right) \left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \right)$$

$$(3.21) \quad = \sum_{n=0}^{\infty} \left(\sum_{\ell+m=n} E_{\ell}^{(N)} \frac{n!}{\ell!m!} \frac{t^n}{n!} x^m \right)$$

$$(3.22) \quad = \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^n \binom{n}{\ell} x^{n-\ell} E_{\ell}^{(N)} \right) \frac{t^n}{n!}$$

Note that (3.17) can be written as

$$(3.23) \quad 2^N F^N(t, x) = 2^N F^N(t)e^{xt} = \sum_{n=0}^{\infty} E_n^{(N)}(x) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ in the r.h.s's of above two equations, we get

Proposition 3.5.

$$(3.24) \quad E_n^{(N)}(x) = \sum_{\ell=0}^n \binom{n}{\ell} x^{n-\ell} E_{\ell}^{(N)}.$$

Second, rewrite (1.4) as

$$\sum_{n=0}^{\infty} E_n^{(N)}(x) \frac{t^n}{n!} = 2^N F^N(t)e^{tx}.$$

Using (3.6) and the Taylor expansion for e^{tx} , above equation becomes

$$\begin{aligned} \sum_{n=0}^{\infty} E_n^{(N)}(x) \frac{t^n}{n!} &= \left(\sum_{\ell_1=0}^{\infty} \frac{E_{\ell_1}}{\ell_1!} t^{\ell_1} \right) \times \dots \times \left(\sum_{\ell_N=0}^{\infty} \frac{E_{\ell_N}}{\ell_N!} t^{\ell_N} \right) \sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{\ell_1+\dots+\ell_N+m=n} \frac{E_{\ell_1} \dots E_{\ell_N}}{\ell_1! \dots \ell_N! m!} x^m n! \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{\ell_1+\dots+\ell_N+m=n} \binom{n}{\ell_1, \dots, \ell_N, m} E_{\ell_1} \dots E_{\ell_N} x^m \right) \frac{t^n}{n!} \end{aligned}$$

By comparing the coefficients of t^n 's on both sides, we finally get

Proposition 3.6.

$$(3.25) \quad E_n^{(N)}(x) = \sum_{\ell_1 + \dots + \ell_N + m = n} \binom{n}{\ell_1, \dots, \ell_N, m} E_{\ell_1} \cdots E_{\ell_N} x^m.$$

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