

Zeros and Uniqueness of Difference Polynomials of Meromorphic Functions

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ABSTRACT. This research is a continuation of a recent paper due to the first author in [9]. Different from previous results, we investigate the value distribution of difference polynomials of meromorphic functions in this paper. In particular, we are interested in the existence of zeros of $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m) - a$, where f is a meromorphic function, n, m are two non-negative integers, and λ, μ are non-zero complex numbers. However, the proof here is obviously different to the one in [9]. We also study difference polynomials of entire functions sharing a common value, which improves the result in [10, 13].

1. Introduction

A meromorphic function $f(z)$ means meromorphic in the whole complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [12]. As usual, the abbreviation CM stands for counting multiplicities, while IM means ignoring multiplicities. We use $\sigma(f)$ to denote the order of $f(z)$ and $N_p(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f - a$, where an m -fold zero is counted m times if $m \leq p$ and p times if $m > p$.

Recently, a number of papers focus on complex difference equations and differences analogues of Nevanlinna's theory. Among these papers, some of them are devoted to the value distribution and uniqueness of complex difference polynomials, which can be viewed as difference analogues of corresponding results of differential

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polynomials.

In this paper, we investigate the value distributions of difference polynomials of meromorphic functions, which are supplements of previous results. We also study the value sharing problem of difference polynomial $f^n(f-1)f(z+c)$, which improves the result in [10, 13].

2. Zeros of Difference Polynomials of Meromorphic Functions

Many mathematicians were interested in the value distribution of different expressions of a meromorphic function and obtained a lot of fruitful results. Here, we recall a question posed by Hayman. Let $f(z)$ be a transcendental meromorphic function, and let n be a positive integer. Hayman [4, Corollary to Theorem 9] proved that $f^n f'$ takes every non-zero complex value infinitely often provided that $n \geq 3$. Mues [8, Satz 3] proved that $f^2 f' - 1$ has infinitely many zeros. Later on, Bergweiler and Eremenko [1, Theorem 2] showed that $f f' - 1$ has infinitely many zeros also. For an analogue of the above results in difference polynomials, Laine and Yang [6, Theorem 2] proved:

Theorem A. *Let f be a transcendental entire function with finite order, and let c be a non-zero complex constant. Then, for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often.*

Some improvements of Theorem A can be found in [7]. Recently, we studied the value distribution of $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$ in [9], where n, m are non-negative integers, and λ, μ are non-zero complex numbers. We obtain the following result which generalizes some theorems in [6, 7].

Theorem B[9, Theorem 1.2]. *Let f be a transcendental entire function with finite order, c be a non-zero constant, n and m be integers satisfying $n \geq m > 0$, and let λ, μ be two complex numbers such that $|\lambda| + |\mu| \neq 0$. If $n \geq 2$, then either $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often or $f(z) = e^{\frac{\log t}{c} z} g(z)$, where $t = \left(-\frac{\mu}{\lambda}\right)^{\frac{1}{m}}$, and $g(z)$ is periodic function with period c .*

In [9], we also considered the value distribution of $f(z)^n + \mu f(z+c)^m$, where $m \neq n$.

Theorem C[9, Theorem 1.3]. *Let f be a transcendental entire function with finite order, μ and c be non-zero constants, and let $a(z)$ be a non-zero small function to f . Suppose that n, m are positive integers such that $n > m + 1$ (or $m > n + 1$). Then the difference polynomial $f(z)^n + \mu f(z+c)^m - a(z)$ has infinitely many zeros.*

In the present paper, we improve Theorem B and Theorem C on the condition that f is a meromorphic function and get the following results.

Theorem 2.1. *Let f be a transcendental meromorphic function with finite order $\sigma(f)$, c be a non-zero constant, n and m be non-negative integers, and let λ, μ be two*

complex numbers such that $\lambda f(z+c)^m + \mu f(z)^m \neq 0$. If the exponent of convergence of the zeros and poles of f satisfies $\max\{\lambda(f), \lambda(\frac{1}{f})\} < \sigma(f)$ and $n \geq m + 1$, then $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m) - a(z)$ has infinitely many zeros, where $a(z)$ is a non-zero small function to f .

Remark 1. If $m = 0$ and $\max\{\lambda(f), \lambda(\frac{1}{f})\} = \sigma(f)$, then the claim of Theorem 2.1 fails. Indeed, let $f = e^z + 1$ and $n = 1$, it is easy to see that $(\lambda + \mu)f - (\lambda + \mu)$ has no zeros, where $\lambda + \mu \neq 0$.

If we delate the assumption that $\max\{\lambda(f), \lambda(\frac{1}{f})\} < \sigma(f)$ in Theorem 2.1, then we get the following result.

Theorem 2.2. Let f be a transcendental meromorphic function with finite order, c be a non-zero constant, n and m be no-negative integers, and let λ, μ be two complex numbers such that $\lambda f(z+c)^m + \mu f(z)^m \neq 0$. If $n \geq 4m + 5$, then $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m) - a(z)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.

The reasoning used in proving Theorem C, yields the following result, however, the proof is different in details.

Theorem 2.3. Let f be a transcendental meromorphic function with finite order $\sigma(f)$, μ and c be non-zero constants, and let $a(z)$ be a non-zero small function to f . Suppose that n, m are positive integers such that $n > m + 1$ (or $m > n + 1$) and the exponent of convergence of the poles of f satisfies $\lambda(\frac{1}{f}) < \sigma(f)$. Then the difference polynomial $f(z)^n + \mu f(z+c)^m - a(z)$ has infinitely many zeros.

In case of delating the assumption that $\lambda(\frac{1}{f}) < \sigma(f)$ in Theorem 2.3, we get:

Theorem 2.4. Let f be a transcendental meromorphic function with finite order, μ and c be non-zero constants, and let $a(z)$ be a non-zero small function to f . Suppose that n, m are positive integers such that $n \geq 2m + 5$ (or $m \geq 2n + 5$). Then the difference polynomial $f(z)^n + \mu f(z+c)^m - a(z)$ has infinitely many zeros.

Remark 2. Using the same way of Theorem 2.4, we know if $a(z) = 0$ in Theorem 2.4 and $n \geq m + 6$ (or $m \geq n + 6$), then Theorem 2.4 holds as well.

Lemma 2.5([3, Theorem 2.1]). Let f be a meromorphic function with finite order, and let $c \in \mathbb{C}$ and $\delta \in (0, 1)$. Then

$$m \left(r, \frac{f(z+c)}{f(z)} \right) + m \left(r, \frac{f(z)}{f(z+c)} \right) = o \left(\frac{T(r, f)}{r^\delta} \right) = S(r, f).$$

Remark 3. Lemma 2.5 is a difference analogue of the logarithmic derivative lemma, given by Halburd-Korhonen [3]. Chiang and Feng have obtained similar estimates for the logarithmic difference[2, Corollary 2.5], and this work is independent from [3]. The following lemma is essentially in our proof, due to Heittokangas et al, see [5, Theorems 6 & 7].

Lemma 2.6 *Let f be a non-constant meromorphic function with finite order, $c \in \mathbb{C}$. Then*

$$N(r, \frac{1}{f(z+c)}) \leq N(r, \frac{1}{f}) + S(r, f), \quad N(r, f(z+c)) \leq N(r, f) + S(r, f),$$

outside of a possible exceptional set E with finite logarithmic measure.

Remark 4. From Lemma 2.5 and Lemma 2.6, we know that $T(r, f(z+c)) = T(r, f) + S(r, f)$ for a meromorphic function of finite order.

Lemma 2.7 ([2, Theorem 2.1]). *Let f be a non-constant meromorphic function with finite order $\sigma(f)$, and let c be a non-zero constant. Then, for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r).$$

Lemma 2.8 ([12, Theorem 1.17 and 1.18]). *Let f and g be two non-constant meromorphic functions in the complex plane with $\sigma(f)$ as the order of f and $\mu(g)$ as the lower order of g . If $\sigma(f) < \mu(g)$, then*

$$\mu(fg) = \mu(g),$$

and

$$T(r, f) = o(T(r, g)), \quad (r \rightarrow \infty).$$

Proof of Theorem 2.1 Set $F(z) = f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$. By Remark 4, it follows that

$$\begin{aligned} T(r, F) &\leq T(r, f(z)^n) + T(r, f(z)^m) + T(r, f(z+c)^m) + S(r, f) \\ &\leq (n+2m)T(r, f) + S(r, f). \end{aligned}$$

Thus, we get $S(r, F) = o(T(r, f)) = S(r, f)$. As the poles of $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$ come from the poles of $f(z)$ and $f(z+c)$, we get

$$\begin{aligned} N(r, f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)) &\leq N(r, f^n) + N(r, f^m) + N(r, f(z+c)^m) + S(r, f) \\ &\leq (n+2m)N(r, f) + S(r, f), \end{aligned}$$

by Lemma 2.6. From above inequality, we know $\lambda(\frac{1}{F}) \leq \lambda(\frac{1}{f}) < \sigma(f)$. In addition, from Lemma 2.5 and Lemma 2.7, we have

$$\begin{aligned} (2.1) \quad (n+m)T(r, f) &= T(r, \frac{1}{f^{n+m}}) + S(r, f) \\ &= m(r, \frac{1}{f^{n+m}}) + O(r^{\lambda(f)+\varepsilon}) + S(r, f) \\ &\leq m(r, \frac{F}{f^{n+m}}) + m(r, \frac{1}{F}) + O(r^{\lambda(f)+\varepsilon}) + S(r, f) \\ &\leq T(r, F) + O(r^{\lambda(f)+\varepsilon}) + S(r, f). \end{aligned}$$

The second main theorem yields

$$\begin{aligned}
 (2.2) \quad T(r, F) &\leq \bar{N}(r, F) + \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F-a}) + S(r, F) \\
 &\leq \bar{N}(r, \frac{1}{F-a}) + \bar{N}(r, \frac{1}{\lambda f(z+c)^m + \mu f(z)^m}) \\
 &\quad + O(r^{\lambda(f)+\varepsilon}) + O(r^{\lambda(\frac{1}{f})+\varepsilon}) + S(r, f) \\
 &\leq \bar{N}(r, \frac{1}{F-a}) + 2mT(r, f) \\
 &\quad + O(r^{\lambda(f)+\varepsilon}) + O(r^{\lambda(\frac{1}{f})+\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + S(r, f).
 \end{aligned}$$

Combining (2.1) and (2.2), we have

$$(n-m)T(r, f) \leq \bar{N}(r, \frac{1}{F-a}) + O(r^{\lambda(\frac{1}{f})+\varepsilon}) + O(r^{\lambda(f)+\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + S(r, f),$$

which is a contradiction to the fact that f is with order $\sigma(f)$, if $F-a$ has finitely many zeros. The conclusion follows. \square

Proof of Theorem 2.2. Set $F(z) = f^n(\lambda f(z+c)^m + \mu f(z)^m) - a(z)$, then from Remark 4, we know

$$\begin{aligned}
 nT(r, f) = T(r, f^n) &= T(r, \frac{F+a}{\lambda f(z+c)^m + \mu f(z)^m}) + S(r, f) \\
 &\leq T(r, F) + T(r, f^m) + T(r, f(z+c)^m) + S(r, f) \\
 &= T(r, F) + 2mT(r, f) + S(r, f).
 \end{aligned}$$

That is

$$(2.3) \quad T(r, F) \geq (n-2m)T(r, f) + S(r, f).$$

As we know, the poles of $\lambda f(z+c)^m + \mu f(z)^m$ come from the poles $f(z)$ and $f(z+c)$, therefore

$$(2.4) \quad \bar{N}(r, \lambda f(z+c)^m + \mu f(z)^m) \leq \bar{N}(r, f) + \bar{N}(r, f(z+c)) + S(r, f).$$

By the second main theorem and (2.3), (2.4), we get

$$\begin{aligned}
 (n - 2m)T(r, f) &\leq T(r, F) + S(r, f) \\
 &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F+a}) + \bar{N}(r, F) + S(r, F) \\
 &\leq \bar{N}(r, \frac{1}{f}) + \bar{N}(\frac{1}{\lambda f(z+c)^m + \mu f(z)^m}) + \bar{N}(r, f) \\
 &\quad + \bar{N}(r, \lambda f(z+c)^m + \mu f(z)^m) + \bar{N}(r, \frac{1}{F}) + S(r, F) \\
 &\leq 2T(r, f) + T(r, \lambda f(z+c)^m + \mu f(z)^m) + \bar{N}(r, f) \\
 &\quad + \bar{N}(r, f(z+c)) + \bar{N}(r, \frac{1}{F}) + S(r, F) \\
 &\leq (2m + 4)T(r, f) + \bar{N}(r, \frac{1}{F}) + S(r, f)
 \end{aligned}$$

Hence, the assumption that $n \geq 4m + 5$ implies that F have infinitely many zeros, completing the proof. □

Proof of Theorem 2.3. Let $F(z) = f^n + \mu f(z+c)^m - a$, then using the same way above, we know that

$$|n - m|T(r, f) + S(r, f) \leq T(r, F) \leq (m + n)T(r, f) + S(r, f).$$

From the assumption that $m \neq n$, we know

$$(2.5) \quad \sigma(F) = \sigma(f), \quad \mu(F) = \mu(f),$$

where $\mu(f)$ is the lower order of f . Assume that $F(z)$ admits finitely many zeros only. From the Hadamard factorization theorem, there exist $p(z)$, $q(z)$ and $A(z)$, where $p(z)$ and $q(z)$ are polynomials and $A(z)$ is the canonical product formed with the poles of $F(z)$, so that

$$(2.6) \quad f(z)^n + \mu f(z+c)^m - a(z) = \frac{q(z)}{A(z)} e^{p(z)} = H(z) e^{p(z)}.$$

By (2.6), we get $\sigma(H) \leq \max\{\sigma(q), \sigma(A)\}$. If $\sigma(A) \leq \sigma(q)$, then we get $\sigma(H) = \sigma(q)$. As $q(z)$ is a polynomial, we know $T(r, H) = S(r, f)$. Furthermore, by (2.5) and (2.6), we conclude that $p(z)$ is a non-constant polynomial. Hence, $T(r, H) = S(r, e^{p(z)})$.

It remains to consider the case $\sigma(A) > \sigma(q)$. From the assumption $\lambda(\frac{1}{f}) < \sigma(f)$, we get $\sigma(H) = \sigma(A) = \lambda(A) = \lambda(\frac{1}{f}) < \sigma(f)$. Moreover, we know from (2.6)

$$(2.7) \quad \sigma(F) = \sigma(e^{p(z)}) = \mu(e^{p(z)}) = \sigma(f) > \sigma(H).$$

From Lemma 2.8 and (2.5), (2.7), we obtain that

$$\mu(f) = \mu(F) = \mu(e^{p(z)}) > \sigma(H).$$

From above equation and Lemma 2.8, we get $T(r, H) = S(r, f)$ and $T(r, H) = S(r, e^{p(z)})$. The assertion now follows by case as Theorem 3 in [9, p. 184-185], we have

$$T\left(r, nf'(z) - \left(p'(z) + \frac{H'(z)}{H(z)}\right)f\right) = S(r, f)$$

and

$$T\left(r, f\left(nf'(z) - \left(p'(z) + \frac{H'(z)}{H(z)}\right)f\right)\right) = S(r, f).$$

Hence

$$T(r, f) = S(r, f),$$

which is a contradiction, the conclusion holds. □

Proof of Theorem 2.4. Case 1. $n \geq 2m + 5$. Suppose

$$(2.8) \quad \phi(z) = \frac{\mu f(z+c)^m - a(z)}{f^n(z)},$$

then from Remark 4, we get

$$\begin{aligned} T(r, f^n) &= T\left(r, \frac{1}{f^n}\right) + O(1) = T\left(r, \phi \frac{1}{\mu f(z+c)^m - a(z)}\right) + O(1) \\ &\leq T(r, \phi) + T(r, f(z+c)^m) + S(r, f) \\ &\leq T(r, \phi) + mT(r, f) + S(r, f). \end{aligned}$$

From above equation, we get that

$$(2.9) \quad T(r, \phi) \geq (n - m)T(r, f) + S(r, f).$$

Concerning to the zeros and poles of ϕ , we obtain that

$$(2.10) \quad \begin{aligned} \bar{N}(r, \phi) &\leq \bar{N}(r, f(z+c)) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq 2T(r, f) + S(r, f). \end{aligned}$$

And

$$(2.11) \quad \begin{aligned} \bar{N}\left(r, \frac{1}{\phi}\right) &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f(z+c)^m - \frac{a}{\mu}}\right) + S(r, f) \\ &\leq T(r, f) + mT(r, f) + S(r, f) \leq (m + 1)T(r, f) + S(r, f). \end{aligned}$$

Using the second main theorem, (2.9)–(2.11), we have

$$\begin{aligned} (n - m)T(r, f) &\leq T(r, \phi) + S(r, f) \\ &\leq \bar{N}(r, \phi) + \bar{N}\left(r, \frac{1}{\phi}\right) + \bar{N}\left(r, \frac{1}{\phi + 1}\right) + S(r, f) \\ &\leq (3 + m)T(r, f) + \bar{N}\left(r, \frac{1}{\phi + 1}\right) + S(r, f). \end{aligned}$$

Observing $\phi(z)$, we get 1-points of $\phi(z)$ come from the poles of $f(z)$ and zeros of $f(z)^n + \mu f(z+c)^m - a(z)$. By above equation, we conclude that

$$(n-m)T(r, f) \leq (3+m)T(r, f) + \bar{N}(r, f) + N\left(r, \frac{1}{f(z)^n + \mu f(z+c)^m - a(z)}\right) + S(r, f),$$

which implies that $f(z)^n + \mu f(z+c)^m - a(z)$ has infinitely many zeros by the assumption that $n \geq 2m + 5$.

Case 2. $m \geq 2n + 5$. Set $\varphi(z) = f(z+c)$, and $\varphi(z-c) = f(z)$ follows. Therefore, we consider the value distribution problem of $\varphi(z)^m + \frac{1}{\mu}\varphi(z-c)^n - \frac{a}{\mu}$. Similarly as in Case 1, we get the conclusion, completing the proof of Theorem 2.4. \square

3. Value Sharing Problem of $f(z)^n(f(z) - 1)f(z+c)$

As a difference analogue of the value distribution of differential polynomial $f^n(f-1)f'$, Zhang [13] considered the value distribution of $f^n(f-1)f(z+c)$. Furthermore, Zhang and one of the present authors gave the following uniqueness theorem at almost the same time, however, our proofs are different.

Theorem D ([10, 13]). *Let f and g be transcendental entire functions with finite order, let c be a non-zero complex constant, and let $n \geq 7$ be an integer. If $f(z)^n(f(z) - 1)f(z+c)$ and $g(z)^n(g(z) - 1)g(z+c)$ share a CM, where a is non-zero small function to f and g , then $f(z) \equiv g(z)$.*

Now it is natural to ask what happens if the CM sharing value can be replaced by the IM sharing value in Theorem D? In this paper, we give a positive answer to the above question by proving the following result.

Theorem 3.1. *Let f and g be transcendental entire functions with finite order, let c be a non-zero complex constant, and let $n \geq 16$ be an integer. If $f(z)^n(f(z) - 1)f(z+c)$ and $g(z)^n(g(z) - 1)g(z+c)$ share a IM, where a is non-zero small function to f and g , then $f(z) \equiv g(z)$.*

Lemma 3.2 ([11, Lemma 2.3]). *Let*

$$(3.1) \quad H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

where F and G are two non-constant meromorphic functions. If F and G share 1 IM and $H \neq 0$, then

$$(3.2) \quad \begin{aligned} T(r, F) + T(r, G) &\leq 2 \left(N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r, F) + N_2(r, G) \right) \\ &+ 3 \left(\bar{N}(r, F) + \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G}\right) \right) + S(r, F) + S(r, G). \end{aligned}$$

Proof of Theorem 3.1. Let $F(z) = \frac{f(z)^n(f-1)f(z+c)}{a(z)}$ and $G(z) = \frac{g(z)^n(g-1)g(z+c)}{a(z)}$, then we get

$$\begin{aligned}(n+2)T(r, f) &= T(r, f^{n+1}(f-1)) = m(r, f^{n+1}(f-1)) \\ &\leq m(r, \frac{f^{n+1}(f-1)}{aF}) + m(r, aF) + S(r, f) \\ &\leq T(r, F) + m(r, \frac{f^{n+1}(f-1)}{aF}) + S(r, f).\end{aligned}$$

Using above equation and Lemma 2.5, we obtain that

$$(3.3) \quad T(r, F) \geq (n+2)T(r, f) + S(r, f).$$

Similarly, we have

$$(3.4) \quad T(r, G) \geq (n+2)T(r, g) + S(r, g).$$

In addition, from Remark 4, we see

$$T(r, F) \leq (n+2)T(r, f) + S(r, f),$$

and

$$T(r, G) \leq (n+2)T(r, g) + S(r, g).$$

Then we get

$$(3.5) \quad T(r, F) = (n+2)T(r, f) + S(r, f),$$

and

$$(3.6) \quad T(r, G) = (n+2)T(r, g) + S(r, g).$$

Using the second main theorem, we get

$$\begin{aligned}T(r, F) &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F-1}) + S(r, F) \\ &\leq \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f-1}) + \bar{N}(r, \frac{1}{f(z+c)}) + \bar{N}(r, \frac{1}{G-1}) + S(r, f) \\ &\leq 3T(r, f) + (n+2)T(r, g) + S(r, f) + S(r, g)\end{aligned}$$

Combining (3.3) with above equation, we have

$$(3.7) \quad (n-1)T(r, f) \leq (n+2)T(r, g) + S(r, f) + S(r, g).$$

Similarly, we get

$$(3.8) \quad (n-1)T(r, g) \leq (n+2)T(r, f) + S(r, f) + S(r, g).$$

Hence, $S(r, f) = S(r, g)$. The following, we will evaluate the counting functions of F and G .

$$(3.9) \quad N_2(r, \frac{1}{F}) \leq 2\bar{N}(r, \frac{1}{f}) + N(r, \frac{1}{f-1}) + N(r, \frac{1}{f(z+c)}) + S(r, f) \\ \leq 4T(r, f) + S(r, f).$$

And

$$(3.10) \quad \bar{N}(r, \frac{1}{F}) \leq \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f-1}) + \bar{N}(r, \frac{1}{f(z+c)}) + S(r, f) \\ \leq 3T(r, f) + S(r, f).$$

By the same reasoning, we obtain

$$(3.11) \quad N_2(r, \frac{1}{G}) \leq 4T(r, g) + S(r, g),$$

and

$$(3.12) \quad \bar{N}(r, \frac{1}{G}) \leq 3T(r, g) + S(r, g).$$

From the condition of the Theorem 3.1, we know F and G share 1 IM. Let H be given by (3.1). If $H \neq 0$, by Lemma 3.2, we know that (3.2) holds. From (3.5), (3.6) and (3.9)–(3.12), we get

$$(3.13) \quad (n+2)(T(r, f) + T(r, g)) \leq T(r, F) + T(r, G) \\ \leq 17(T(r, f) + T(r, g)) + S(r, f) + S(r, g),$$

which contradicts the assumption that $n \geq 16$. Therefore $H \equiv 0$. Integrating twice, we get from (3.1) that

$$(3.14) \quad \frac{1}{F-1} = \frac{A}{G-1} + B,$$

where $A (\neq 0)$ and B are constants. From (3.14), we have

$$(3.15) \quad F = \frac{(B+1)G + (A-B-1)}{BG + (A-B)}, \quad G = \frac{(B-A)F + (A-B-1)}{BF - (B+1)}.$$

We consider the following three cases.

Case 1. Suppose that $B \neq 0, -1$. From (3.15) we have $\bar{N}(r, \frac{1}{F - \frac{B+1}{B}}) = \bar{N}(r, G)$. From the second fundamental theorem, we have

$$T(r, F) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F - \frac{B+1}{B}}) + S(r, F)$$

$$(3.16) \quad = \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, G) + S(r, F) \leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, F).$$

By (3.3), (3.10) and (3.16), we get that

$$(n+2)T(r, f) \leq 3T(r, f) + S(r, f),$$

which contradicts $n \geq 16$.

Case 2. Suppose that $B = 0$. From (3.15) we have

$$(3.17) \quad F = \frac{G + (A-1)}{A}, \quad G = AF - (A-1).$$

If $A \neq 1$, we get from (3.17) that

$$(3.18) \quad \bar{N}\left(r, \frac{1}{F}\right) = \bar{N}\left(r, \frac{1}{G + (A-1)}\right)$$

Combining the second main theorem with (3.4), (3.7), (3.12) and (3.17), we have

$$\begin{aligned} (n+2)T(r, g) + S(r, g) &\leq T(r, G) \\ &\leq \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{G + (A-1)}\right) + S(r, G) \\ &\leq 3T(r, g) + \bar{N}\left(r, \frac{1}{F}\right) + S(r, g) \\ &\leq \left(3 + \frac{3(n+2)}{n-1}\right)T(r, g) + S(r, g), \end{aligned}$$

which implies $n^2 - 5n + 5 \leq 0$, which contradicts $n \geq 16$. Thus $A = 1$ and $F = G$.

Case 3. Suppose that $B = -1$. From (3.15), we obtain

$$(3.19) \quad F = \frac{A}{-G + (A+1)}, \quad G = \frac{(A+1)F - A}{F}.$$

If $A \neq -1$, then from (3.19), $\bar{N}\left(r, \frac{1}{F - \frac{1}{A+1}}\right) = \bar{N}\left(r, \frac{1}{G}\right)$ follows. By the same reasoning mentioned in Case 1 and Case 2, we get a contradiction. Hence $A = -1$. From (3.15), we have $FG = 1$.

From $F = G$ or $FG = 1$, we get $f(z)^n(f(z)-1)f(z+c)g(z)^n(g(z)-1)g(z+c) \equiv a(z)^2$ or $f^n(f-1)f(z+c) \equiv g^n(g-1)g(z+c)$. The assertion now follows by Case 1 and Case 2 as in [13, p. 407], the conclusion holds. \square

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