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On G-invariant Minimal Hypersurfaces with Constant Scalar Curvatures in S^5

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Abstract. Let $G = O(2) \times O(2) \times O(2)$. Then a closed G-invariant minimal hypersurface with constant scalar curvature in S^5 is a product of spheres, i.e., the square norm of its second fundamental form, $S = 4$.

1. Introduction

Let M^n be a closed minimally immersed hypersurface in the unit sphere S^{n+1}, and h its second fundamental form. Denote by R and S its scalar curvature and the square norm of h, respectively. It is well known that $S = n(n-1) - R$ from the structure equations of both M^n and S^{n+1}. In particular, S is constant if and only if M has constant scalar curvature. In 1968, J. Simons [6] observed that if $S \le n$ everywhere and S is constant, then $S \in \{0, n\}$. Clearly, M^n is an equatorial sphere if $S = 0$. And when $S = n$, M^n is indeed a product of spheres, due to the works of Chern, do Carmo, and Kobayashi [2] and Lawson [4].

We are concerned about the following conjecture posed by Chern [9].

Chern Conjecture. For any $n \ge 3$, the set R_n of the real numbers each of which can be realized as the constant scalar curvature of a closed minimally immersed hypersurface in S^{n+1} is discrete.

C. K. Peng and C. L. Terng [5] proved

Theorem(Peng and Terng, 1983)**.** *Let M^n be a closed minimally immersed hypersurface with constant scalar curvature in S^{n+1}. If $S > n$, then $S > n + 1/(12n)$.*

S. Chang [1] proved the following theorem by showing that $S = 3$ if $S \ge 3$ and M^3 has multiple principal curvatures at some point.

Theorem(Chang, 1993)**.** *A closed minimally immersed hypersurface with constant*

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scalar curvature in S^4 is either an equatorial 3*-sphere, a product of spheres, or a Cartan's minimal hypersurface. In particular,* $R_3 = \{0, 3, 6\}$.

H. Yang and Q. M. Cheng [8] proved

Theorem(Yang and Cheng, 1998)**.** *Let M^n be a closed minimally immersed hypersurface with constant scalar curvature in S^{n+1}. If $S > n$, then $S \geq n + n/3$.*

Let $G \simeq O(k) \times O(k) \times O(q) \subset O(2k+q)$ and set $2k + q = n + 2$. Then W. Y. Hsiang [3] investigated G-*invariant, minimal hypersurfaces,* M^n in S^{n+1}, by studying their generating curves, M^n/G, in the orbit space S^{n+1}/G. He showed that there exit infinitely many closed minimal hypersurfaces in S^{n+1} for all $n \geq 2$, by proving the following theorem:

Theorem(Hsiang, 1987)**.** *For each dimension $n \geq 2$, there exist infinitely many, mutually noncongruent closed G-invariant minimal hypersurfaces in S^{n+1}, where $G \simeq O(k) \times O(k) \times O(q)$ and $k = 2$ or* 3.

We studied G-invariant minimal hypersurfaces, in stead of minimal ones, with constant scalar curvatures in S^5. In this paper, we shall prove the following classification theorem:

Our Theorem. *A closed G-invariant minimal hypersurface with constant scalar curvature in S^5 is a product of spheres, i.e., $S = 4$, where $G = O(2) \times O(2) \times O(2)$.*

Let M^4 be a closed G-invariant minimal hypersurface with constant scalar curvature in S^5. By virtue of the results of Simons [6], we see that if $S \leq 4$, then $S \in \{0, 4\}$. In Lemma 4.3, we show that if M^4 has 2 distinct principal curvatures at some point, then $S = 4$. Since any equatorial sphere is not $G-$invariant, we see that if $S \leq 4$ then $S = 4$. Moreover, Lemma 4.3 says that if $S > 4$, then M^4 does not have 2 distinct principal curvatures anywhere. Therefore, if $S > 4$ then M^4 must have simple principal curvatures everywhere or 3 distinct principal curvatures at some point. To prove our Theorem, we need only to show that it is impossible. In Lemma 5.1 and Lemma 5.2, we show that if $S > 4$ then M^4 does not have simple principal curvatures everywhere and 3 distinct principal curvatures anywhere, respectively.

2. Preliminary Results

Let M^n be a manifold of dimension n immersed in a Riemannian manifold $\overline{M}^{n+1}$ of dimension $n+1$. Let $\overline{\nabla}$ and $\langle\, ,\, \rangle$ be the connection and metric tensor respectively of $\overline{M}^{n+1}$ and let $\bar{\mathfrak{R}}$ be the curvature tensor with respect to the connection $\overline{\nabla}$ on $\overline{M}^{n+1}$. Choose a local orthonormal frame field $e_1, \ldots, e_{n+1}$ in $\overline{M}^{n+1}$ such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n. Denote the dual coframe by

$\{\omega_A\}$. Here we will always use $i, j, k, \dots,$ for indices running over $\{1, 2, \dots, n\}$ and $A, B, C, \dots,$ over $\{1, 2, \dots, n+1\}$.

As usual, the *second fundamental form* h and the *mean curvature* H of M^n in $\overline{M}^{n+1}$ are respectively defined by

$$h(v, w) = \langle \overline{\nabla}_v w,\, e_{n+1} \rangle \quad \text{and} \quad H = \sum_i h(e_i,\, e_i).$$

M^n is said to be *minimal* if H vanishes identically. And the *scalar curvature* $\bar{R}$ of $\overline{M}^{n+1}$ is defined by

$$\bar{R} = \sum_{A,B} \langle \bar{\mathcal{R}}(e_A,\, e_B) e_B,\, e_A \rangle.$$

Then the structure equations of $\overline{M}^{n+1}$ are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$
$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

where $K_{ABCD} = \langle \bar{\mathcal{R}}(e_A,\, e_B) e_D,\, e_C \rangle$. When $\overline{M}^{n+1}$ is the unit sphere S^{n+1}, we have

$$K_{ABCD} = \delta_{AC}\, \delta_{BD} - \delta_{AD}\, \delta_{BC}.$$

Next, we restrict all tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n. Then

$$\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0.$$

By Cartan's lemma, we can write

$$\omega_{(n+1)i} = -\sum_j h_{ij}\, \omega_j.$$

Here, we see

$$\begin{aligned}(2.1) \qquad h_{ij} &= -\omega_{(n+1)i}(e_j) = -\langle \overline{\nabla}_{e_j} e_{n+1},\, e_i \rangle = \langle \overline{\nabla}_{e_j} e_i,\, e_{n+1} \rangle = \langle \overline{\nabla}_{e_i} e_j,\, e_{n+1} \rangle \\ &= h(e_i, e_j).\end{aligned}$$

Second, from

$$\begin{cases} d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} = \sum_l \omega_{il} \wedge \omega_{lj} - \frac{1}{2} \sum_{l,m} R_{ijlm}\, \omega_l \wedge \omega_m, \end{cases}$$

we find the curvature tensor of M^n is

$$R_{ijlm} = K_{ijlm} + h_{il}\, h_{jm} - h_{im}\, h_{jl}. \tag{2.2}$$

Therefore, if M^n is a piece of minimally immersed hypersurface in the unit sphere S^{n+1} and R is the scalar curvature of M^n, then we have

$$R = n(n-1) - S, \tag{2.3}$$

where $S = \sum_{i,j} h_{ij}^2$ is the *square norm* of h.

Given a symmetric 2-tensor $T = \sum_{i,j} T_{ij}\, \omega_i\, \omega_j$ on M^n, we also define its covariant derivatives, denoted by ∇T, $\nabla^2 T$ and $\nabla^3 T$, etc. with components $T_{ij,k}$, $T_{ij,kl}$ and $T_{ij,klp}$, respectively, as follows:

(2.4)

$$\sum_k T_{ij,k}\, \omega_k = d\, T_{ij} + \sum_s T_{sj}\, \omega_{si} + \sum_s T_{is}\, \omega_{sj},$$

$$\sum_l T_{ij,kl}\, \omega_l = d\, T_{ij,k} + \sum_s T_{sj,k}\, \omega_{si} + \sum_s T_{is,k}\, \omega_{sj} + \sum_s T_{ij,s}\, \omega_{sk},$$

$$\sum_p T_{ij,klp}\, \omega_p = d\, T_{ij,kl} + \sum_s T_{sj,kl}\, \omega_{si} + \sum_s T_{is,kl}\, \omega_{sj} + \sum_s T_{ij,sl}\, \omega_{sk} + \sum_s T_{ij,ks}\, \omega_{sl}.$$

In general, the resulting tensors are no longer symmetric, and the rule to switch sub-index obeys the Ricci formula as follows:

(2.5)

$$T_{ij,kl} - T_{ij,lk} = \sum_s T_{sj}\, R_{sikl} + \sum_s T_{is}\, R_{sjkl},$$

$$T_{ij,klp} - T_{ij,kpl} = \sum_s T_{sj,k}\, R_{silp} + \sum_s T_{is,k}\, R_{sjlp} + \sum_s T_{ij,s}\, R_{sklp},$$

$$T_{ij,klpm} - T_{ij,klmp} = \sum_s T_{sj,kl}\, R_{sipm} + \sum_s T_{is,kl}\, R_{sjpm} + \sum_s T_{ij,sl}\, R_{skpm} + \sum_s T_{ij,ks}\, R_{slpm}.$$

For the sake of simplicity, we always omit the comma (,) between indices in the special case $T = \sum_{i,j} h_{ij}\, \omega_i\, \omega_j$ with $\overline{M}^{n+1} = S^{n+1}$. Since

$$\sum_{C,D} K_{(n+1)iCD}\, \omega_C \wedge \omega_D = 0$$

on M^n when $\overline{M}^{n+1} = S^{n+1}$, we find

$$d\left(\sum_j h_{ij}\, \omega_j\right) = \sum_{j,l} h_{jl}\, \omega_l \wedge \omega_{ji}.$$

Therefore,

$$\sum_{j,l} h_{ijl}\,\omega_l \wedge \omega_j = \sum_j \left(dh_{ij} + \sum_l h_{lj}\,\omega_{li} + \sum_l h_{il}\,\omega_{lj} \right) \wedge \omega_j = 0;$$

i.e., h_{ijl} is symmetric in all indices.

In the case that M^n is minimal, by differentiating $\sum_l h_{ll} = 0$ we have

$$0 = e_j\, e_i \left(\sum_l h_{ll} \right) = \sum_l e_j(h_{lli}) = \sum_l h_{llij} \tag{2.6}$$

and so,

(2.7)

$$\begin{aligned}
\sum_l h_{ijll} &= \sum_l h_{lijl} = \sum_l \left\{ h_{lilj} + \sum_m (h_{mi}R_{mljl} + h_{lm}R_{mijl}) \right\} \\
&= (n-1)h_{ij} + \sum_{l,m} \{ -h_{mi}h_{ml}h_{lj} + h_{lm}(\delta_{mj}\delta_{il} - \delta_{ml}\delta_{ij} + h_{mj}h_{il} - h_{ml}h_{ij}) \} \\
&= nh_{ij} - \sum_{l,m} h_{lm}h_{ml}h_{ij} = (n-S)h_{ij}.
\end{aligned}$$

It follows that

$$\frac{1}{2}\,\Delta S = (n-S)S + \sum_{i,j,l} h_{ijl}^2. \tag{2.8}$$

In the case that S is constant, by differentiating $S = \sum_{i,j} h_{ij}^2$ twice , we have

$$0 = \sum_{i,j} h_{ij}h_{ijkl} + \sum_{i,j} h_{ijk}\,h_{ijl}. \tag{2.9}$$

3. G-invariant Hypersurface in S^{n+1}

For $G \simeq O(k)\times O(k)\times O(q)$, R^{n+2} splits into the orthogonal direct sum of irreducible invariant subspaces, namely

$$R^{n+2} \simeq R^k \oplus R^k \oplus R^q = \{(X, Y, Z)\}$$

where X and Y are generic k-vectors and Z is a generic q-vector. Here if we set $x = |X|, y = |Y|$ and $z = |Z|$, then the orbit space R^{n+2}/G can be parametrized by $(x,\, y,\, z)$; $x,\, y,\, z \in R_+$ and the orbital distance metric is given by $ds^2 = dx^2 +$

$dy^2 + dz^2$. By restricting the above G-action to the unit sphere $S^{n+1} \subset R^{n+2}$, it is easy to see that

$$S^{n+1}/G \simeq \{(x, y, z) : x^2 + y^2 + z^2 = 1;\ x,\ y,\ z \geq 0\}$$

which is isometric to a spherical triangle of $S^2(1)$ with $\pi/2$ as its three angles. The orbit labeled by (x, y, z) is exactly $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$.

In this section, M^n is a closed G- invariant hypersurface in S^{n+1}. ∇ and $\overline{\nabla}$ are the Riemannian connections of M^n and S^{n+1}, respectively. To investigate those G-invariant minimal hypersurfaces, we study their generating curves, $\gamma(s) = (x(s), y(s), z(s)) = M^n/G$, in the orbit space S^{n+1}/G.

Let us start with the following two lemmas which play very important roles in proving our Theorem.

Lemma 3.1. *Let M^n be a G-invariant hypersurface in S^{n+1}. Then there is a local orthonormal frame field $e_1, \ldots, e_{n+1}$ on S^{n+1} such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n and $h_{ij} = 0$ if $i \neq j$.*

Proof. Let $(X_0, Y_0, Z_0) \in M^n \subset S^{n+1}$ with $x = |X_0|$, $y = |Y_0|$ and $z = |Z_0|$ and choose a local orthonormal frame field on a neighborhood of (X_0, Y_0, Z_0) as follows.

First, we choose vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}$, $\widetilde{v}_1, \ldots, \widetilde{v}_{k-1}$, $\widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ on a neighborhood U of (X_0, Y_0, Z_0) in the orbit $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ such that:

(1) $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}$ are lifts of orthonormal tangent vector fields $u_1, \ldots, u_{k-1}$ on a neighborhood of X_0 in $S^{k-1}(x)$ to $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ respectively,

(2) $\widetilde{v}_1, \ldots, \widetilde{v}_{k-1}$ are lifts of orthonormal tangent vector fields $v_1, \ldots, v_{k-1}$ on a neighborhood of Y_0 in $S^{k-1}(y)$ to $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ respectively,

(3) $\widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ are lifts of orthonormal tangent vector fields $w_1, \ldots, w_{q-1}$ on a neighborhood of Z_0 in $S^{q-1}(z)$ to $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ respectively.

Second, let $N(s) = (n_1(s), n_2(s), n_3(s))$ be a local unit normal vector field on γ in S^{n+1}/G. For each $p = (X, Y, Z) \in U$, let $\widetilde{\gamma}(p, s)$ be the lift curve of $\gamma(s)$ in S^{n+1} through p. and let $\widetilde{N}(p, s)$ be the lift vector field of $N(s)$ on $\widetilde{\gamma}(p, s)$. Then we know

$$\widetilde{\gamma}(p, s) = (X(s), Y(s), Z(s)) = \left(x(s)\frac{X}{x},\ y(s)\frac{Y}{y},\ z(s)\frac{Z}{z} \right) \tag{3.1}$$

and so,

$$\widetilde{\gamma}'(p, s) = \left(x'(s)\frac{X}{x},\ y'(s)\frac{Y}{y},\ z'(s)\frac{Z}{z} \right) \tag{3.2}$$

and

$$\widetilde{N}(p,s) = \left(n_1(s)\frac{X(s)}{x(s)},\, n_2(s)\frac{Y(s)}{y(s)},\, n_3(s)\frac{Z(s)}{z(s)} \right). \tag{3.3}$$

The two orthonormal vector fields $\widetilde{\gamma}'$ and $\widetilde{N}$ are defined on a neighborhood in M^n.

Third, let us extend $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ over a neighborhood in M as follows:

Let $\bar{\alpha}_i(u) = (\alpha_i(u), Y, Z)$ be a curve in $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ through $p = (X, Y, Z)$ such that $\bar{\alpha}_i(0) = p$ and $\bar{\alpha}_i'(0) = (\alpha_i'(0), 0, 0) = \widetilde{u}_i(p)$. From (2.1),

$$\bar{\bar{\alpha}}_i(u) = \left(x(s)\frac{\alpha_i(u)}{x},\, y(s)\frac{Y}{y},\, z(s)\frac{Z}{z} \right)$$

is a a curve in the orbit $S^{k-1}(x(s)) \times S^{k-1}(y(s)) \times S^{q-1}(z(s))$ through $\widetilde{\gamma}(p,s)$ and

$$\bar{\bar{\alpha}}_i'(0) = \frac{x(s)}{x}(\alpha_i'(0),\, 0,\, 0) \quad (\text{: parallel to } \widetilde{u}_i(p) \text{ in the Euclidean space})$$

is tangent to the orbit $S^{k-1}(x(s)) \times S^{k-1}(y(s)) \times S^{q-1}(z(s))$ and so, to M^n. It says that the vector field obtained by Euclidean parallel translation of $\widetilde{u}_i$ along $\widetilde{\gamma}$ is tangent to M^n. Hence,

($*$) extend $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ over a neighborhood in M by Euclidean parallel translation along $\widetilde{\gamma}$.

Then these vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}', \widetilde{N}$ is a local orthonormal frame field on M^n and $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}'$ are tangent to M^n.

Last, let us extend $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}', \widetilde{N}$ over a neighborhood in S^{n+1} as follows:

From (2.1), we have

$$h_{ij} = \langle\, \overline{\nabla}_{\widetilde{u}_i}\widetilde{u}_j,\, \widetilde{N}\, \rangle = -\left\langle\, \widetilde{u}_j,\, \overline{\nabla}_{\widetilde{u}_i}\widetilde{N}\, \right\rangle. \tag{3.4}$$

Here, $\overline{\nabla}_{\widetilde{u}_i}\widetilde{N}$ depends only the values of $\widetilde{N}$ along any smooth curve $\bar{\alpha}_i$ such that $\bar{\alpha}_i' = \widetilde{u}_i$. Since $\widetilde{N}$ is already defined on a neighborhood in M^n and $\widetilde{u}_i$ is a tangent vector field on the neighborhood in M^n, $\overline{\nabla}_{\widetilde{u}_i}\widetilde{N}$ does not depend on the choice of extending $\widetilde{N}$. Hence,

($**$) extend all vector fields over a neighborhood in S^{n+1} properly.

The extended vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}', \widetilde{N}$ is a local orthonormal frame field on S^{n+1}. After restriction these vector fields to M^n,

$\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}'$ are tangent to M^n. For convenience, we write them as $e_1, \ldots, e_{n+1}$ in order.

Now, let us compute $h_{ij}(p)$. From (3.2) and (3.3), we have

$$\begin{cases} \widetilde{\gamma}'(\bar{\alpha}_i(u), 0) = \left(x'(0)\dfrac{\alpha_i(u)}{x},\, y'(0)\dfrac{Y}{y},\, z'(0)\dfrac{Z}{z} \right), \\ \widetilde{N}(\bar{\alpha}_i(u), 0) = \left(n_1(0)\dfrac{\alpha_i(u)}{x},\, n_2(0)\dfrac{Y}{y},\, n_3(0)\dfrac{Z}{z} \right). \end{cases} \tag{3.5}$$

If ∇^* is the Riemannian connection of R^{n+2}, then $\overline{\nabla} = \nabla^{*\top}$. Hence, (3.5) implies

$$\begin{cases} \overline{\nabla}_{\widetilde{u}_i(p)}\widetilde{\gamma}' = \left\{ \dfrac{x'(0)}{x}\,(\alpha_i'(0),\, 0,\, 0) \right\}^{\top} = \left\{ \dfrac{x'(0)}{x}\,\widetilde{u}_i(p) \right\}^{\top} = \dfrac{x'(0)}{x}\,\widetilde{u}_i(p), \\ \overline{\nabla}_{\widetilde{u}_i(p)}\widetilde{N} = \left\{ \dfrac{n_1(0)}{x}\,(\alpha_i'(0),\, 0,\, 0) \right\}^{\top} = \left\{ \dfrac{n_1(0)}{x}\,\widetilde{u}_i(p) \right\}^{\top} = \dfrac{n_1(0)}{x}\,\widetilde{u}_i(p). \end{cases} \tag{3.6}$$

Thus, from (3.4) and (3.6) we have at p

$$h_{ij} = -\langle\, \widetilde{u}_j(p),\, \overline{\nabla}_{\widetilde{u}_i(p)}\widetilde{N}\, \rangle = -\left\langle\, \widetilde{u}_j(p),\, \frac{n_1(0)}{x}\widetilde{u}_i(p)\, \right\rangle = -\frac{n_1(0)}{x}\delta_{ij}. \tag{3.7}$$

Similarly, we have at p

$$\begin{cases} h_{(k-1+i)(k-1+j)} = \langle\, \overline{\nabla}_{\widetilde{v}_i(p)}\widetilde{v}_j,\, \widetilde{N}\, \rangle = -\dfrac{n_2(0)}{y}\delta_{ij}, \\ h_{(2k-2+i)(2k-2+j)} = \langle\, \overline{\nabla}_{\widetilde{w}_i(p)}\widetilde{w}_j,\, \widetilde{N}\, \rangle = -\dfrac{n_3(0)}{z}\delta_{ij}. \end{cases} \tag{3.8}$$

And since $\nabla_{\gamma'}\gamma' = (x''(0), y''(0), z''(0))^{\top}$ on S^{n+1}/G, we have at p

$$\begin{aligned} h_{nn} &= \langle\, \overline{\nabla}_{\widetilde{\gamma}'}\widetilde{\gamma}',\, \widetilde{N}\, \rangle \\ &= \langle\, (x''(0)\frac{X}{x},\, y''(0)\frac{Y}{y},\, z''(0)\frac{Z}{z})^{\top},\, (n_1(0)\frac{X}{x},\, n_2(0)\frac{Y}{y},\, n_3(0)\frac{Z}{z})\, \rangle \\ &= x''(0)\, n_1(0) + y''(0)\, n_2(0) + z''(0)\, n_3(0) \\ &= \langle\, (x''(0), y''(0), z''(0)),\, N\, \rangle \\ &= \langle\, \nabla_{\gamma'}\gamma',\, N\, \rangle = \kappa_g(\gamma), \end{aligned} \tag{3.9}$$

where $\kappa_g(\gamma)$ is the geodesic curvature of γ at (x, y, z). Recall that

$$\gamma(s) = (\sin r(s)\cos\theta(s), \sin r(s)\sin\theta(s), \cos r(s)) = (x(s), y(s), z(s)). \tag{3.10}$$

Let $(x, y, z) = \gamma(0) = (\sin r\cos\theta, \sin r\sin\theta, \cos r)$. Then

$$\gamma'(0) = \frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta},$$

where $\partial/\partial r = (\cos r\cos\theta, \cos r\sin\theta, -\sin r)$ and $\partial/\partial\theta = \sin r(-\sin\theta, \cos\theta, 0)$.

Now, let $U = (\partial/\partial r)\times 1/\sin r\,(\partial/\partial\theta)$ be a unit normal vector field on a neighborhood of (x, y, z) in S^{n+1}/G. Then we have

(3.11)

$$\begin{aligned} N(0) &= (n_1(0), n_2(0), n_3(0)) \\ &= U\times T = U\times\gamma'(0) = U\times\left(\frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta}\right) \\ &= \frac{1}{\sin r}\frac{dr}{ds}\frac{\partial}{\partial\theta} - \sin r\frac{d\theta}{ds}\frac{\partial}{\partial r} \\ &= -\sin r\frac{d\theta}{ds}(\cos r\cos\theta, -\sin r\cos r\sin\theta, -\sin r) + \frac{dr}{ds}(-\sin\theta, \cos\theta, 0). \end{aligned}$$

Therefore, from (3.7), (3.8), (3.9), (3.10) and (3.11) we obtain

$$(3.12)\qquad \begin{cases} h_{11} = \cdots = h_{(k-1)(k-1)} = -\dfrac{n_1(0)}{x} = \cos r\dfrac{d\theta}{ds} + \dfrac{\tan\theta}{\sin r}\dfrac{dr}{ds}, \\ h_{kk} = \cdots = h_{(2k-2)(2k-2)} = -\dfrac{n_2(0)}{y} = \cos r\dfrac{d\theta}{ds} - \dfrac{\cot\theta}{\sin r}\dfrac{dr}{ds}, \\ h_{(2k-1)(2k-1)} = \cdots = h_{(n-1)(n-1)} = -\dfrac{n_3(0)}{z} = -\dfrac{\sin^2 r}{\cos r}\dfrac{d\theta}{ds}, \\ h_{nn} = \kappa_g(\gamma), \\ h_{ij} = 0 \quad \text{if } i\neq j, \end{cases}$$

which completes the proof of Lemma 3.1. □

Note. In Lemma 3.1, those all h_{ii}'s are called the *principal curvatures* of M^n. All principal curvatures h_{ii}'s are constant on each orbit from (3.12) and the vector fields $e_1, \cdots, e_{n-1}$ are tangent to each orbit from $(*)$ of Lemma 3.1. Hence we have

$$(3.13)\qquad e_j(h_{11}) = \cdots = e_j(h_{nn}) = 0, \qquad \text{for all } j = 1, \cdots, n-1.$$

From now on throughout this paper, $\{e_A\}$ is a local orthonormal frame field on S^{n+1} such as the frame field in Lemma 3.1.

Lemma 3.2. *Let M^n be a G-invariant hypersurface in S^{n+1}. Then,*
(1) all $h_{ijl} = 0$ except when $\{i, j, l\}$ is a permutation of $\{i, i, n\}$,
(2) all $h_{ijlm} = 0$ except when $\{i, j, l, m\}$ is a permutation of $\{i, i, j, j\}$.

Proof. (1) Since h_{ijl} is symmetric in all indices, it suffices to show that $h_{ijl} = 0$ if $i \le j \le l$ and $\{i, j, l\} \neq \{i, i, n\}$.

(1.a) *Case* 1. $j \neq i$: (2.4) together with Lemma 3.1 gives

$$(3.14)\qquad h_{ijl} = e_l(h_{ij}) + \sum_s h_{sj}\,\omega_{si}(e_l) + \sum_s h_{is}\,\omega_{sj}(e_l) = (h_{jj} - h_{ii})\,\omega_{ji}(e_l).$$

If $i, j \leq k-1$, then from (3.12) $h_{ii} = h_{jj}$. Hence, (3.14) implies $h_{ijl} = 0$ for all l.

If $k \leq i, j \leq 2k-2$ or $2k-1 \leq i, j \leq n-1$, then also $h_{ijl} = 0$ for all l.

And, if $i \leq k-1$ and $k \leq j < n$, then for all l $(i \lneq j \leq l)$ we have

$$h_{ijl} = h_{lij} = e_j(h_{li}) + (h_{ii} - h_{ll})\, \omega_{il}(e_j) = (h_{ii} - h_{ll}) \langle \nabla_{e_j} e_i,\, e_l \rangle = 0, \tag{3.15}$$

since $\nabla_{e_j} e_i = 0$ by the Koszul formula. In the similar cases, we also have $h_{ijl} = 0$.

Now, from (2.4) and Lemma 3.1, we have

$$h_{mml} = e_l(h_{mm}) + \sum_s h_{sm}\, \omega_{sm}(e_l) + \sum_s h_{ms}\, \omega_{sm}(e_l) = e_l(h_{mm}). \tag{3.16}$$

Hence, if $j = l = n$, then $h_{inn} = h_{nni} = e_i(h_{nn}) = 0$ from (3.13) since $i < j (= n)$.

(1.b) *Case* 2. $j = i$ and $l \neq n$: $h_{ijl} = h_{iil} = e_l(h_{ii}) = 0$ from (3.13).

Therefore, (1.a) and (1.b) imply that (1) holds.

(2) (2.a) *Case* 1. i, j, l, m are distinct : Without loss of generality, it suffices to show that $h_{ijln} = h_{ijnl} = 0$ and $h_{ijlm} = 0$ for all i, j, l, m such that i, j, l, $m < n$.

By using (1) of this Lemma, we easily see that

$$h_{ijln} = e_n(h_{ijl}) + \sum_s h_{sjl}\, \omega_{si}(e_n) + \sum_s h_{isl}\, \omega_{sj}(e_n) + \sum_s h_{ijs}\, \omega_{sl}(e_n) = 0, \tag{3.17}$$

since i, j, $l < n$ and i, j, l are distinct. And, from (2.5) and Lemma 3.1 we have

$$h_{ijnl} = h_{ijln} + \sum_s h_{sj}\, R_{sinl} + \sum_s h_{is}\, R_{sjnl} = h_{jj}\, R_{jinl} + h_{ii}\, R_{ijnl} = 0. \tag{3.18}$$

If i, j, l, $m < n$, then from (1) of this Lemma we can easily see

$$h_{ijlm} = e_m(h_{ijl}) + \sum_s \{h_{sjl}\, \omega_{sj}(e_m) + h_{isl}\, \omega_{sj}(e_m) + h_{ijs}\, \omega_{sl}(e_m)\} = 0. \tag{3.19}$$

From (3.17), (3.18) and (3.19), we complete the proof of (2.a)

(2.b) *Case* 2. $j \neq l$: Let us show that $h_{iijl} = h_{jlii} = h_{jjjl} = h_{ljjj} = 0$.

First, we show that $h_{iijl} = h_{jlii} = 0$. Since $j \neq l$, one of $\{j,\, l\}$ is not n. And

$$h_{iijl} = h_{iilj} + \sum_s h_{si}\, R_{sijl} + \sum_s h_{is}\, R_{sijl} = h_{iilj} + 2h_{ii} R_{iijl} = h_{iilj}. \tag{3.20}$$

Hence, we may assume $l \neq n$. So, $e_l(h_{iij}) = 0$. Because $h_{iij} = e_j(h_{ii})$ is also constant on each orbit since h_{ii} is constant on each orbit. Therefore, we have

$$\text{(i)} \qquad \begin{aligned} h_{iijl} &= e_l(h_{iij}) + \sum_s h_{sij}\,\omega_{si}(e_l) + \sum_s h_{isj}\,\omega_{si}(e_l) + \sum_s h_{iis}\,\omega_{sj}(e_l) \\ &= 2h_{jij}\,\omega_{ji}(e_l) - h_{iin}\,\omega_{nj}(e_l) = 0, \end{aligned}$$

since $h_{jij} = 0$ if $i \neq n$ and $\omega_{nj}(e_l) = \langle \nabla_{e_l} e_n, e_j \rangle = 0$ from the first of (3.6).

And since $j \neq l$, from (2.5), Lemma 3.1 and (i) we also have

$$\text{(ii)} \qquad \begin{aligned} h_{jlii} = h_{ijli} &= h_{ijil} + \sum_s h_{sj}\,R_{sili} + \sum_s h_{is}\,R_{sjli} \\ &= h_{iijl} + h_{jj}\,R_{jili} + h_{ii}\,R_{ijli} = 0. \end{aligned}$$

Second, we show that $h_{jjjl} = h_{ljjj} = 0$. From (2.4), we have

$$h_{jjjl} = e_l(h_{jjj}) + \sum_s h_{sjj}\,\omega_{sj}(e_l) + \sum_s h_{jsj}\,\omega_{sj}(e_l) + \sum_s h_{jjs}\,\omega_{sj}(e_l). \tag{3.21}$$

Hence, (3.21) and (1) of this Lemma give

$$h_{jjjl} = \begin{cases} 3h_{jjn}\,\omega_{nj}(e_l) & \text{if } j \neq n, \\ e_l(h_{nnn}) & \text{if } j = n. \end{cases} \tag{3.22}$$

Here,

$$\begin{cases} \omega_{nj}(e_l) = \begin{cases} \langle \nabla_{e_l} e_n, e_j \rangle = 0 \quad \text{from (3.6)} & \text{if } l \neq n, \\ -\langle e_n, \nabla_{e_n} e_j \rangle = 0 \quad \text{from (*) in Lemma 3.1} & \text{if } l = n, \end{cases} \\ e_l(h_{nnn}) = 0 \quad \text{since } h_{nnn} \text{ is also constant on each orbit } \; (l \neq j = n). \end{cases} \tag{3.23}$$

From (3.21), (3.22) and (3.23), we have

$$\text{(iii)} \qquad h_{jjjl} = 0$$

and

$$\text{(iiii)} \quad h_{ljjj} = h_{jjlj} = h_{jjjl} + \sum_s h_{sj}\,R_{sjlj} + \sum_s h_{js}\,R_{sjlj} = h_{jjjl} + 2h_{jj}\,R_{jjlj} = 0.$$

From (i), (ii), (iii) and (iiii), we complete the proof of (2.b) and Lemma 3.2. □

4. G-invariant Minimal Hypersurface in S^5.

From now on, we assume that $G \simeq O(2) \times O(2) \times O(2)$ and M^4 is a closed

G-invariant minimal hypersurface with constant scalar curvature in S^5. Then by differentiating $\sum_i h_{ii} = 0$ and $\sum_i h_{ii}^2 = S$ with respect to e_4 respectively, we have

$$\begin{cases} h_{114} + h_{224} + h_{334} + h_{444} = 0, \\ h_{11}h_{114} + h_{22}h_{224} + h_{33}h_{334} + h_{44}h_{444} = 0. \end{cases} \tag{4.1}$$

By differentiating (4.1) with respect to e_4 respectively, we have

$$\begin{cases} h_{1144} + h_{2244} + h_{3344} + h_{4444} = 0, \\ \sum_i h_{ii}h_{ii44} + \sum_i h_{ii4}^2 = 0. \end{cases} \tag{4.2}$$

Since $e_4(h_{ii44}) = h_{ii444}$ from (2.4), by differentiating (4.2) with respect to e_4 respectively, we also have

$$\begin{cases} h_{11444} + h_{22444} + h_{33444} + h_{44444} = 0, \\ \sum_i h_{ii}h_{ii444} + 3\sum_i h_{ii4}h_{ii44} = 0. \end{cases} \tag{4.3}$$

From (2.7), we have

$$h_{ii11} + h_{ii22} + h_{ii33} + h_{ii44} = (4 - S)h_{ii}. \tag{4.4}$$

Since S is constant, (2.8) and Lemma 3.2 give

$$3\sum_{i\neq 4} h_{ii4}^2 + h_{444}^2 = S(S-4). \tag{4.5}$$

Now, by differentiating it once and twice with respect to e_4 respectively, we have

$$\begin{cases} 3\sum_{i\neq 4} h_{ii4}\, h_{ii44} + h_{444}\, h_{4444} = 0, \\ 3\sum_{i\neq 4} h_{ii4}\, h_{ii444} + h_{444}\, h_{44444} + 3\sum_{i\neq 4} h_{ii44}^2 + h_{4444}^2 = 0. \end{cases} \tag{4.6}$$

Moreover, if $i \neq 4$, from (2.4) we know

$$\begin{cases} h_{ii4} &= h_{i4i} = (h_{44} - h_{ii})\, \omega_{4i}(e_i), \\ h_{iiii} &= 3h_{ii4}\, \omega_{4i}(e_i), \\ h_{44ii} &= (h_{444} - 2h_{ii4})\omega_{4i}(e_i). \end{cases} \tag{4.7}$$

And, if i, $j \neq 4$ and $i \neq j$, then

(4.8)
$$h_{iijj} = e_j(h_{iij}) + \sum_s \{h_{sij}\omega_{si}(e_j) + h_{isj}\omega_{si}(e_j) + h_{iis}\omega_{sj}(e_j)\} = h_{ii4}\, \omega_{4j}(e_j).$$

The following (4.9), (4.10) and (4.11) are needed to prove Lemma 4.1.

If $i \neq 4$, then (2.4) and Lemma 3.2 give

$$
(4.9)\quad \begin{cases} e_4(h_{44ii}) & = h_{44ii4} - \sum_s \{h_{s4ii}\omega_{s4}(e_4) + h_{4sii}\omega_{s4}(e_4) \\ & \qquad\qquad + h_{44si}\omega_{si}(e_4) + h_{44is}\omega_{si}(e_4)\} = h_{44ii4}, \\ h_{444ii} & = e_i(h_{444i}) + \sum_s \{h_{s44i}\omega_{s4}(e_i) + h_{4s4i}\omega_{s4}(e_i) \\ & \qquad + h_{44si}\omega_{s4}(e_i) + h_{444s}\omega_{si}(e_i)\} = (h_{4444} - 3h_{44ii})\omega_{4i}(e_i). \end{cases}
$$

Furthermore, if $i \neq 4$, then (2.2) and (2.5), (4.7) give

$$
(4.10)\quad \begin{cases} R_{i4i4} = K_{i4i4} + h_{ii}h_{44} = 1 + h_{ii}h_{44} = -R_{4ii4}, \\ (h_{44ii} - h_{ii44})\,\omega_{4i}(e_i) = (h_{44} - h_{ii})(1 + h_{44}h_{ii})\,\omega_{4i}(e_i) \\ \qquad\qquad\qquad\qquad\qquad = h_{ii4}(1 + h_{44}h_{ii}), \end{cases}
$$

respectively. Here $h_{44i4} = h_{444i} = 0$ by Lemma 3.2. And so (2.5) and (4.10) give

(4.11)

$$
\begin{aligned}
h_{44i4i} &= e_i(h_{44i4}) + h_{i4i4}\,\omega_{i4}(e_i) + h_{4ii4}\,\omega_{i4}(e_i) + h_{4444}\,\omega_{4i}(e_i) + h_{44ii}\,\omega_{i4}(e_i) \\
&= e_i(h_{444i}) - h_{ii44}\,\omega_{4i}(e_i) - h_{ii44}\,\omega_{4i}(e_i) + h_{4444}\,\omega_{4i}(e_i) + h_{44ii}\,\omega_{i4}(e_i) \\
&= h_{444ii} - h_{i44i}\,\omega_{i4}(e_i) - h_{4i4i}\,\omega_{i4}(e_i) - h_{44ii}\,\omega_{i4}(e_i) - h_{4444}\,\omega_{4i}(e_i) \\
&\qquad - h_{ii44}\,\omega_{4i}(e_i) - h_{ii44}\,\omega_{4i}(e_i) + h_{4444}\,\omega_{4i}(e_i) + h_{44ii}\,\omega_{i4}(e_i) \\
&= h_{444ii} + 2(h_{44ii} - h_{ii44})\,\omega_{4i}(e_i) \\
&= h_{444ii} + 2h_{ii4}(1 + h_{44}h_{ii}).
\end{aligned}
$$

Hence, we have the following lemma that is needed to prove our Theorem.

Lemma 4.1. *If $i \neq 4$, then*

$$
(4.12)\qquad h_{ii444} = h_{444ii} + (5 + 6h_{ii}h_{44} - h_{44}^2)h_{ii4} - (2 + 3h_{ii}h_{44} - h_{ii}^2)h_{444}.
$$

Proof. By using (4.9), (4.10) and (4.11), we have

$$
\begin{aligned}
h_{ii444} &= e_4(h_{ii44}) + \sum_s \{h_{si44}\omega_{si}(e_4) + h_{is44}\omega_{si}(e_4) + h_{iis4}\omega_{s4}(e_4) + h_{ii4s}\omega_{s4}(e_4)\} \\
&= e_4(h_{ii44}) \\
&= e_4\{h_{44ii} + (h_{ii} - h_{44})(1 + h_{ii}h_{44})\} \\
&= h_{44ii4} + (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\
&= h_{44i4i} + h_{i4i}R_{i4i4} + h_{4ii}R_{i4i4} + h_{444}R_{4ii4} \\
&\qquad + (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\
&= h_{444ii} + 2h_{ii4}(1 + h_{44}h_{ii}) + (2h_{ii4} - h_{444})R_{i4i4} \\
&\qquad + (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\
&= h_{444ii} + (5h_{ii4} - 2h_{444})(1 + h_{ii}h_{44}) + (h_{ii}h_{44} - h_{44}^2)h_{ii4} + (h_{ii}^2 - h_{ii}h_{44})h_{444} \\
&= h_{444ii} + (5 + 6h_{ii}h_{44} - h_{44}^2)h_{ii4} - (2 + 3h_{ii}h_{44} - h_{ii}^2)h_{444}
\end{aligned}
$$

and it completes the proof of Lemma 4.1. □

For the sake of simplicity, we sometimes let $h_{ii} = \lambda_i$ from now on throughout this paper. To prove our Theorem we need another lemmas.

Lemma 4.2. *Suppose $h_{ii} = h_{44} = \lambda$ at some point p for $i = 1, 2$ or 3. Then,*

$$S = \frac{12\lambda^4 + 4\lambda^2}{5\lambda^2 - 1}. \tag{4.13}$$

Proof. Without loss of generality, we can assume $h_{33} = h_{44} = \lambda$ at some point p. Then (4.7) implies $h_{334}(p) = 0$. Together with (4.7) and (4.8), it implies

$$h_{3311} = h_{3322} = h_{3333} = 0, \quad \text{at } p. \tag{4.14}$$

Hence, (4.4) and (4.14) imply

$$h_{3344} = (4 - S)h_{33}, \quad \text{at } p \tag{4.15}$$

and (2.5) implies

$$h_{4433} = h_{3344} + (h_{44} - h_{33})(1 + h_{44}h_{33}) = h_{3344}, \quad \text{at } p. \tag{4.16}$$

In the equation (2.9), $\sum_{i,j} h_{ij3}^2 = 0$ at p. Hence, we have

$$h_{11}h_{1133} + h_{22}\, h_{2233} + h_{33}\, h_{3333} + h_{44}\, h_{4433} = 0, \quad \text{at } p. \tag{4.17}$$

By using (2.5) and (4.14) we know, at p

$$\begin{cases} h_{1133} & = h_{3311} + (h_{11} - \lambda)(1 + h_{11}\,\lambda) = (\lambda_1 - \lambda)(1 + \lambda_1\,\lambda), \\ h_{2233} & = h_{3322} + (h_{22} - \lambda)(1 + h_{22}\,\lambda) = (\lambda_2 - \lambda)(1 + \lambda_2\,\lambda). \end{cases} \tag{4.18}$$

Hence, (4.17) and (4.18) imply

$$\lambda_1\,(\lambda_1 - \lambda)(1 + \lambda_1\,\lambda) + \lambda_2\,(\lambda_2 - \lambda)(1 + \lambda_2\,\lambda) + \lambda\,(4 - S)\lambda = 0. \tag{4.19}$$

Here, since

$$\begin{cases} \lambda_1 + \lambda_2 + 2\lambda = 0, \;\; \lambda_1^2 + \lambda_2^2 + 2\lambda^2 = S, \;\; \lambda_1\,\lambda_2 = 3\lambda^2 - \frac{S}{2}, \\ \lambda_1^3 + \lambda_2^3 = (\lambda_1^2 + \lambda_2^2 - \lambda_1\,\lambda_2)(\lambda_1 + \lambda_2) = 10\lambda^3 - 3S\lambda, \end{cases} \tag{4.20}$$

(4.19) becomes

$$S + 4\lambda^2 + 12\lambda^4 - 5S\lambda^2 = 0,$$

and so,

$$S = \frac{12\lambda^4 + 4\lambda^2}{5\lambda^2 - 1}.$$

It completes the proof of Lemma 4.2. □

The following Lemma 4.3 and Lemma 5.1 are proved in the same methods as in our early paper [7].

Lemma 4.3. *If M^4 has 2 distinct principal curvatures at some point, then $S = 4$.*

Proof. Suppose M^4 has 2 distinct principal curvatures at some point, say, p. Without loss of generality, we can assume either one of the following three cases for some $\lambda \neq 0$:

Case 1. Suppose $h_{22} = h_{33} = h_{44} = \lambda$ and $h_{11} = -3\lambda$ at p . Then

$$S = h_{11}^2 + h_{22}^2 + h_{33}^2 + h_{44}^2 = 12\lambda^2. \tag{4.21}$$

Hence, (4.13) and (4.21) imply $S = 4$, i.e., $M^4 = S^1(\sqrt{1/4}\,) \times S^3(\sqrt{3/4}\,)$.

Case 2. Suppose $h_{11} = h_{22} = -\lambda$, $h_{33} = h_{44} = \lambda$ at p. Then

$$S = h_{11}^2 + h_{22}^2 + h_{33}^2 + h_{44}^2 = 4\lambda^2. \tag{4.22}$$

Hence, (4.13) and (4.22) imply $S = 4$, i.e., $M^4 = S^2(\sqrt{1/2}\,) \times S^2(\sqrt{1/2}\,)$. But, it is not G-invariant.

Case 3. Suppose $h_{11} = h_{22} = h_{33} = \lambda$ and $h_{44} = -3\lambda$ at p. Then from (3.12), we have at p

$$\cos r\,\frac{d\theta}{ds} + \frac{\tan\theta}{\sin r}\frac{dr}{ds} = \cos r\,\frac{d\theta}{ds} - \frac{\cot\theta}{\sin r}\frac{dr}{ds} = -\frac{\sin^2 r}{\cos r}\frac{d\theta}{ds}. \tag{4.23}$$

From (4.23), we have

$$\frac{dr}{ds} = 0 \quad \text{and} \quad \frac{d\theta}{ds} = 0, \tag{4.24}$$

which means that $h_{11} = h_{22} = h_{33} = h_{44} = \lambda = 0$ at p. It is contrary to the hypothesis and completes the proof of Lemma 4.3. □

Lemma 4.4. *If $S > 4$ and $i = 1, 2, 3$, then*

(1) *for each i, there exists a point q_i in M such that $h_{ii}(q_i) = 0$ and*

(2) *for all i, $h_{44} \neq h_{ii}$ anywhere.*

Proof. (1) Suppose that the conclusion is not valid. Without loss of generality, we can assume that $h_{33} > 0$ everywhere. Consider a point p_0, such that

$$h_{33}(p_0) = \min_{M^4} h_{33} > 0. \tag{4.25}$$

Then, due to the maximal principle, we have

$$e_4(h_{33})(p_0) = h_{334}(p_0) = 0 \quad \text{and} \quad Hess.\, h_{33}(e_4,\, e_4)(p_0) \geq 0. \tag{4.26}$$

Now, we have

$$\text{(4.27)}\quad Hess.\, h_{33}(e_4,\, e_4) = (e_4e_4 - \nabla_{e_4}e_4)(h_{33}) = h_{3344} - \sum_s \omega_{4s}(e_4)h_{33s} = h_{3344}.$$

Here, since $h_{334}(p_0) = 0$, by using (4.7) and (4.8) we have at p_0

$$h_{3311} = h_{3322} = h_{3333} = 0$$

and so,

$$\text{(4.28)}\qquad h_{3344} = (4 - S)h_{33}.$$

From (4.26), (4.27) and (4.28), we have

$$h_{3344} = (4 - S)h_{33}(p_0) \geq 0,$$

which is contrary to the hypotheses that $S > 4$ and $h_{33}(p_0) > 0$.

(2) Suppose the conclusion is not valid. Without loss of generality, we can assume that $h_{33} = h_{44} = \lambda$ at some point p. Then since $S > 4$, it follows h_{11}, h_{22}, λ are distinct at p by Lemma 4.3 and $\lambda \neq 0$ by Lemma 4.2. From now on, all computations are performed at p. (4.7) gives $h_{334} = 0$. From (4.2), we have

$$\text{(4.29)}\qquad \begin{cases} h_{1144} + h_{2244} + h_{3344} + h_{4444} = 0, \\ \lambda_1\, h_{1144} + \lambda_2\, h_{2244} + \lambda\, h_{3344} + \lambda\, h_{4444} = -h_{114}^2 - h_{224}^2 - h_{444}^2. \end{cases}$$

It follows that

$$\text{(4.30)}\qquad (\lambda - \lambda_1)\, h_{1144} + (\lambda - \lambda_2)\, h_{2244} = h_{114}^2 + h_{224}^2 + h_{444}^2.$$

Here, from (2.5) and (4.7) we have

$$\text{(4.31)}\qquad \begin{cases} h_{1144} & = h_{4411} + (h_{11} - h_{44})(1 + h_{11}h_{44}) \\ & = (h_{444} - 2h_{114})\omega_{41}(e_1) + (\lambda_1 - \lambda)(1 + \lambda_1\lambda) \\ & = (h_{444} - 2h_{114})h_{114}/(\lambda - \lambda_1) + (\lambda_1 - \lambda)(1 + \lambda_1\lambda), \\ h_{2244} & = h_{4422} + (h_{22} - h_{44})(1 + h_{22}h_{44}) \\ & = (h_{444} - 2h_{224})\omega_{42}(e_2) + (\lambda_2 - \lambda)(1 + \lambda_2\lambda) \\ & = (h_{444} - 2h_{224})h_{224}/(\lambda - \lambda_2) + (\lambda_2 - \lambda)(1 + \lambda_2\lambda). \end{cases}$$

Hence, by using (4.1) and (4.31) we have

$$\begin{aligned} &\text{LHS of }\ (4.30) = (\lambda - \lambda_1)\, h_{1144} + (\lambda - \lambda_2)\, h_{2244} \\ &= h_{444}(h_{114} + h_{224}) - 2h_{114}^2 - 2h_{224}^2 - \{(\lambda_1 - \lambda)^2(1 + \lambda_1\lambda) + (\lambda_2 - \lambda)^2(1 + \lambda_2\lambda)\} \\ &= -h_{444}^2 - 2h_{114}^2 - 2h_{224}^2 - \{(\lambda_1 - \lambda)^2(1 + \lambda_1\lambda) + (\lambda_2 - \lambda)^2(1 + \lambda_2\lambda)\} \\ &= -h_{444}^2 - 2h_{114}^2 - 2h_{224}^2, \end{aligned}$$

since

$$\begin{aligned}&(\lambda_1-\lambda)^2(1+\lambda_1\lambda)+(\lambda_2-\lambda)^2(1+\lambda_2\lambda)\\ &\quad=\lambda_1^2+\lambda_2^2+2\lambda^2-2(\lambda_1+\lambda_2)\lambda+(\lambda_1^3+\lambda_2^3)\lambda-2(\lambda_1^2+\lambda_2^2)\lambda^2+(\lambda_1+\lambda_2)\lambda^3\\ &\quad=S+4\lambda^2+12\lambda^4-5S\lambda^2=0\end{aligned}$$

by using (4.20) and Lemma 4.2. Hence, from (4.30) and (4.5) we obtain

$$0=3h_{114}^2+3h_{224}^2+2h_{444}^2=S(S-4)+h_{444}^2.$$

It contradicts to the hypothesis that $S>4$ and completes the proof. □

5. Proof of Our Theorem

From Lemma 4.3, we know that if $S\le 4$, then $S=4$. Moreover, Lemma 4.3 says that if $S>4$, then M^4 does not have 2 distinct principal curvatures anywhere. Therefore, if $S>4$, then M^4 must have simple principal curvatures everywhere or 3 distinct principal curvatures at some point. To prove our Theorem, it suffices to show that if $S>4$, then M^4 does not have simple principal curvatures everywhere and 3 distinct principal curvatures anywhere.

Lemma 5.1. *If $S>4$, then M^4 does not have simple principal curvatures everywhere.*

Proof. Suppose that M^4 has only simple principal curvatures everywhere. Then since all principal curvatures h_{ii}'s are constant on each orbit, without loss of generality we can assume everywhere either one of the following three cases:

$$\begin{aligned}&(1)\quad h_{11}<h_{22}<h_{33}<h_{44},\\ &(2)\quad h_{11}<h_{22}<h_{44}<h_{33},\\ &(3)\quad h_{44}<h_{11}<h_{22}<h_{33}.\end{aligned}$$

Now, from (1) of Lemma 4.4 we know there exist points q_1 and q_3 in M^4 such that $h_{11}(q_1)=0$ and $h_{33}(q_3)=0$ respectively. Hence the above each case is contrary to the fact that

$$\begin{aligned}&h_{11}(q_1)+h_{22}(q_1)+h_{33}(q_1)+h_{44}(q_1)=0\ \text{ or}\\ &h_{11}(q_3)+h_{22}(q_3)+h_{33}(q_3)+h_{44}(q_3)=0.\end{aligned}$$

Therefore, M^4 does not have simple principal curvatures everywhere. □

Lemma 5.2. *If $S>4$, then M^4 does not have* 3 *distinct principal curvatures anywhere.*

Proof. Suppose that M^4 has 3 distinct principal curvatures at some point p. Then by (2) of Lemma 4.4, without loss of generality we may assume that $\lambda_1 = \lambda_2 = \lambda$ and $\lambda, \lambda_3, \lambda_4$ are distinct at p. All computations are performed at p. From (4.1), we have

$$\begin{cases} h_{114} + h_{224} + h_{334} + h_{444} = 0, \\ \lambda\, h_{114} + \lambda\, h_{224} + \lambda_3\, h_{334} + \lambda_4\, h_{444} = 0. \end{cases} \tag{5.1}$$

Let $h_{114} = b\, h_{224}$ for some real number b. Then, (5.1) becomes

$$\begin{cases} (1+b)\, h_{224} + h_{334} + h_{444} = 0, \\ (1+b)\lambda\, h_{224} + \lambda_3\, h_{334} + \lambda_4\, h_{444} = 0. \end{cases} \tag{5.2}$$

It follows that

$$\begin{cases} h_{114} &= (\lambda_4 - \lambda_3)\, a\, b, \qquad h_{224} = (\lambda_4 - \lambda_3)\, a, \\ h_{334} &= (\lambda - \lambda_4)\, a\, (1+b), \quad h_{444} = (\lambda_3 - \lambda) a (1+b) \end{cases} \tag{5.3}$$

for some real number a. Here since $S > 4$, $a \neq 0$ from (4.5).

Now (2.5) implies

$$h_{3311} - h_{1133} = (\lambda_3 - \lambda)(1 + \lambda_3\, \lambda) = h_{3322} - h_{2233}. \tag{5.4}$$

And, (4.8), (4.7) and (5.3) imply

$$\begin{cases} h_{3311} - h_{1133} &= h_{334}\, \omega_{41}(e_1) - h_{114}\, \omega_{43}(e_3) \\ &= h_{334} h_{114}/(\lambda_4 - \lambda) - h_{114}\, h_{334}/(\lambda_4 - \lambda_3) \\ &= (\lambda_3 - \lambda) a^2 b (1+b), \\ h_{3322} - h_{2233} &= h_{334}\, \omega_{42}(e_2) - h_{224}\, \omega_{43}(e_3) = (\lambda_3 - \lambda) a^2 (1+b). \end{cases} \tag{5.5}$$

Hence, from (5.4) and (5.5) we get

$$(\lambda_3 - \lambda) a^2 b (1+b) = (\lambda_3 - \lambda)(1 + \lambda_3\, \lambda) = (\lambda_3 - \lambda) a^2 (1+b) \tag{5.6}$$

and so,

$$b = -1 \quad \text{or} \quad b = 1. \tag{5.7}$$

To prove our Lemma 5.2, it therefore suffices to show that $b \neq -1$ and $b \neq 1$.
Case 1. In the case $b = -1$: (5.6) implies $(\lambda_3 - \lambda)(1 + \lambda_3\, \lambda) = 0$, i.e.,

$$\lambda \neq 0, \quad \lambda_3 = \frac{-1}{\lambda} \text{ and } \lambda_4 = \frac{1}{\lambda} - 2\lambda. \tag{5.8}$$

Hence,

$$S = 2\lambda^2 + \lambda_3^2 + \lambda_4^2 = 6\lambda^2 + \frac{2}{\lambda^2} - 4. \tag{5.9}$$

From (5.3) and (4.7), we have

$$h_{114} = -h_{224}, \quad h_{334} = h_{444} = 0, \quad \omega_{41}(e_1) = -\omega_{42}(e_2), \quad \omega_{43}(e_3) = 0. \tag{5.10}$$

Hence, from (4.5) and (5.10) we have

$$6h_{114}^2 = S(S-4). \tag{5.11}$$

Let $h_{114}\omega_{41}(e_1) = c$. Then, by using (4.7) and (5.8) we have

$$c(\lambda_4 - \lambda) = h_{114}^2 \quad \text{and so} \quad c = \frac{h_{114}^2}{\lambda_4 - \lambda} = \frac{h_{114}^2\,\lambda}{1 - 3\lambda^2}. \tag{5.12}$$

Moreover, by using (4.7), (4.8), (4.4) and (5.10) we also have

$$\begin{cases} h_{1111} &= 3c, \quad h_{1122} = -c, \quad h_{1133} = 0, \quad h_{1144} = (4-S)\lambda - 2c, \\ h_{2211} &= -c, \quad h_{2222} = 3c, \quad h_{2233} = 0, \quad h_{2244} = (4-S)\lambda - 2c, \\ h_{3311} &= 0, \quad h_{3322} = 0, \quad h_{3333} = 0, \quad h_{3344} = (4-S)\lambda_3, \\ h_{4411} &= -2c, \quad h_{4422} = -2c, \quad h_{4433} = 0, \quad h_{4444} = (4-S)\lambda_4 + 4c. \end{cases} \tag{5.13}$$

Now, we can not draw anymore here and have to appeal to covariant derivatives of h up to the third order.

We compute $6h_{114}h_{11444}$ in *Step* 1 and *Step* 2 respectively by using different ways, and show that in *Step* 3 they are not equal mutually to prove $b \neq -1$.

Step 1. First we compute $6h_{114}h_{11444}$ by using one way. From (4.9), (4.12) and (5.10), we have

$$h_{44433} = 0, \quad h_{33444} = h_{44433}, \quad \text{and so,} \quad h_{33444} = 0. \tag{5.14}$$

Since $h_{1144} = h_{2244}$ from (5.13), by using (4.3), (5.10) and (5.14) we have

$$\begin{cases} h_{11444} + h_{22444} + h_{44444} = 0, \\ \lambda h_{11444} + \lambda h_{22444} + \lambda_4 h_{44444} = 0. \end{cases} \tag{5.15}$$

If follows that

$$h_{11444} = -h_{22444} \quad \text{and} \quad h_{44444} = 0. \tag{5.16}$$

Hence, from (4.6), (5.10) and (5.16) we obtain

$$6h_{114}h_{11444} = -6h_{1144}^2 - 3h_{3344}^2 - h_{4444}^2. \tag{5.17}$$

Step 2. Second we compute $6h_{114}h_{11444}$ in another way. From (4.12), (4.9) and (5.10), we also have

$$(5.18)\qquad \begin{aligned} 6h_{114}h_{11444} &= 6h_{114}h_{44411} + 6(5 + 6h_{11}h_{44} - h_{44}^2)h_{114}^2 \\ &= 6(h_{4444} - 3h_{4411})c + 6(5 + 6\lambda\,\lambda_4 - \lambda_4^2)h_{114}^2. \end{aligned}$$

Step 3. We must show that (5.17) $\neq$ (5.18). Suppose (5.17) = (5.18). Then

$$(5.19)\quad 6h_{1144}^2 + 3h_{3344}^2 + h_{4444}^2 + 6(h_{4444} - 3h_{4411})c + 6(5 + 6\lambda\,\lambda_4 - \lambda_4^2)h_{114}^2 = 0.$$

By using (5.11), (5.13) and the fact that $S - 4 \neq 0$, (5.19) becomes

(5.20)
$$(S-4)(6\lambda^2 + 3\lambda_3^2 + \lambda_4^2) + (24\lambda - 14\lambda_4)c + S(5 + 6\lambda\,\lambda_4 - \lambda_4^2) + \frac{100c^2}{S-4} = 0.$$

Let $\lambda^2 = t$. Then, by using (5.8), (5.9), (5.11) and (5.12) we have

$$(5.21)\quad \begin{cases} S = 6t + \dfrac{2}{t} - 4, \quad (S-4)t = 2(3t-1)(t-1), \\ c = \dfrac{h_{114}^2}{\lambda_4 - \lambda} = \dfrac{S(S-4)\lambda}{6(1-3t)}, \qquad \lambda c = \dfrac{S(S-4)t}{6(1-3t)}, \\ 6\lambda^2 + 3\lambda_3^2 + \lambda_4^2 = 6\lambda^2 + 3\dfrac{1}{\lambda^2} + \left(\dfrac{1}{\lambda} - 2\lambda\right)^2 = 2S - 2t + 4, \\ (24\lambda - 14\lambda_4)c = -14(\lambda_4 - \lambda)c + 10\lambda c = -\dfrac{7}{3}S(S-4) + \dfrac{5S(S-4)t}{3(1-3t)}, \\ 5 + 6\lambda\,\lambda_4 - \lambda_4^2 = -(3\lambda^2 + \dfrac{1}{\lambda^2} - 2) - 13\lambda^2 + 13 = -\dfrac{S}{2} - 13t + 13. \end{cases}$$

Substituting (5.21) to (5.20), we have

$$(5.22)\qquad (55t - 85)S^2 - (990t^2 - 1500t + 390)S + 432t^2 - 1008t + 288 = 0.$$

By eliminating S from the above two equations (5.21) and (5.22), we have

$$(5.23)\qquad 990t^5 - 1923t^4 + 1262t^3 - 142t^2 - 200t + 85 = 0.$$

Here, since $S = 6t + 2/t - 4 > 4$, we have $0 < t < 1/3$ or $t > 1$.
For all t such that $0 < t < 1/3$,

$$\begin{aligned} \text{LHS of (5.23)} &= 990t^5 - 1923t^4 + 1262t^3 - 142t^2 - 200t + 85 \\ &= 110(1-3t)^2t^3 + 421(1-3t)t^3 + 16(1-3t)(1+3t) + 67(1-3t) \\ &\quad + 731t^3 + 2t^2 + t + 2 > 0. \end{aligned}$$

Moreover, for all t such that $t > 1$

$$\begin{aligned}\text{LHS of } (5.23) &= 990t^5 - 1923t^4 + 1262t^3 - 142t^2 - 200t + 85\\ &= 962(t-1)^2t^3 + 100(t-1)^2 + 242(t-1)t^2 + 15(t^3-1)\\ &\quad + 28t^5 + t^4 + 43t^3 > 0.\end{aligned}$$

Hence, there is no a root of the equation (5.23). It follows that $b \neq -1$.

Case 2. In the case $b = 1$: From (5.3) and (4.7), we have

$$(5.24)\quad \begin{cases} h_{114} = h_{224} = (\lambda_4 - \lambda_3)\, a, \quad h_{334} = 2(\lambda - \lambda_4)a, \quad h_{444} = 2(\lambda_3 - \lambda)a,\\ \omega_{41}(e_1) = \omega_{42}(e_2) = h_{114}/(\lambda_4 - \lambda), \quad \omega_{43}(e_3) = h_{334}/(\lambda_4 - \lambda_3)\end{cases}$$

and from (4.5) and (5.24), we also have

$$(5.25)\quad \begin{aligned} S(S-4) &= 3h_{114}^2 + 3h_{224}^2 + 3h_{334}^2 + h_{444}^2\\ &= \{6(\lambda_4-\lambda_3)^2 + 12(\lambda-\lambda_4)^2 + 4(\lambda_3-\lambda)^2\}\, a^2.\end{aligned}$$

We compute h_{1144} in *Step* 1 and *Step* 2 respectively by using different ways, and show that in *Step* 3 they are not equal mutually to prove $b \neq 1$.

Step 1. First we compute h_{1144} in one way. Now, (4.4), (4.7) and (5.24) give

$$(5.26)\quad \begin{aligned} h_{1144} &= (4-S)\lambda - h_{1111} - h_{1122} - h_{1133}\\ &= (4-S)\lambda - h_{114}\{3\omega_{41}(e_1) + \omega_{42}(e_2) + \omega_{43}(e_3)\}\\ &= (4-S)\lambda - \frac{4(\lambda_4-\lambda_3)^2}{\lambda_4-\lambda}a^2 + 2(\lambda_4-\lambda)a^2.\end{aligned}$$

Step 2. Second we compute h_{1144} by using another way. Here,

$$h_{2244} = (4-S)\lambda - h_{224}\{\omega_{41}(e_1) + 3\omega_{42}(e_2) + \omega_{43}(e_3)\} = h_{1144}.$$

Hence, (4.2) and (4.6) imply a system of equations:

$$(5.27)\quad \begin{cases} 2h_{1144} + h_{3344} + h_{4444} = 0,\\ 2\lambda\, h_{1144} + \lambda_3\, h_{3344} + \lambda_4\, h_{4444} = -8Sa^2,\\ 6h_{114}\, h_{1144} + 3h_{334}\, h_{3344} + h_{444}\, h_{4444} = 0,\end{cases}$$

since

$$\begin{aligned} 2h_{114}^2 + h_{334}^2 + h_{444}^2 &= \{2(\lambda_4-\lambda_3)^2 + 4(\lambda-\lambda_4)^2 + 4(\lambda_3-\lambda)^2\}\, a^2\\ &= \{8\lambda^2 + 8\lambda_3^2 + 8\lambda_4^2 - 2(\lambda_3^2+\lambda_4^2+2\lambda_3\lambda_4) - 8\lambda(\lambda_3+\lambda_4)\}a^2\\ &= 8(2\lambda^2 + \lambda_3^2 + \lambda_4^2)a^2 = 8S\, a^2.\end{aligned}$$

By using (5.24) and (5.25), from the system (5.27) of equations we also compute

$$h_{1144} = \frac{8(h_{444} - 3h_{334})Sa^2}{6h_{114}(\lambda_4 - \lambda_3) + 3h_{334}(2\lambda - 2\lambda_4) + h_{444}(2\lambda_3 - 2\lambda)} = \frac{8(h_{444} - 3h_{334})Sa^3}{6h_{114}^2 + 3h_{334}^2 + h_{444}^2} = \frac{32(\lambda_4 - 3\lambda)}{S - 4}\, a^4. \tag{5.28}$$

Step 3. We want to show that (5.26) $\neq$ (5.28). From (5.6), we have

$$1 + \lambda_3\,\lambda = 2a^2. \tag{5.29}$$

Case $2-1$. Suppose that $\lambda = 0$. Then, it follows from (5.29) that

$$a^2 = \frac{1}{2}, \quad \lambda_4 = -\lambda_3 \neq 0 \quad \text{and} \quad S = 2\lambda_4^2. \tag{5.30}$$

Hence, (5.30) and (5.25) imply

$$\begin{cases} (5.26) = (4-S)\lambda - \dfrac{4(\lambda_4 - \lambda_3)^2}{\lambda_4 - \lambda} a^2 + 2(\lambda_4 - \lambda)a^2 = -7\lambda_4 \\ (5.28) = \dfrac{32(\lambda_4 - 3\lambda)}{S - 4}\, a^4 = \dfrac{32(\lambda_4 - 3\lambda)S}{S(S-4)}\, a^4 = \dfrac{64\lambda_4^3}{40\lambda_4^2}\, a^2 = \dfrac{4}{5}\lambda_4. \end{cases}$$

Hence, (5.26) $\neq$ (5.28), and so $b \neq 1$.

Case $2-2$. Suppose $\lambda \neq 0$ and (5.26) = (5.28). Then, we have

$$(4-S)\lambda - \frac{4(\lambda_4 - \lambda_3)^2}{\lambda_4 - \lambda} a^2 + 2(\lambda_4 - \lambda)a^2 = \frac{32(\lambda_4 - 3\lambda)}{S - 4}\, a^4. \tag{5.31}$$

Let $\lambda^2 = t$ and $2a^2 - 1 = u$. Then, from (5.29) we have

$$\lambda_3 = \frac{u}{\lambda}, \quad \lambda_4 = \frac{-u}{\lambda} - 2\lambda, \quad S = 2\lambda^2 + \lambda_3^2 + \lambda_4^2 = 6t + \frac{2u^2}{t} + 4u. \tag{5.32}$$

Substituting (5.32) to (5.25) and (5.31), respectively, we obtain

$$\begin{cases} u^4 - tu^3 - (4t^2 + 7t)u^2 - (5t^3 + 18t^2)u + (9t^4 - 23t^3) = 0, \\ 5u^5 + (14t + 7)u^4 + (28t^2 + 26t)u^3 + (4t^3 + 124t^2 - 10t)u^2 \\ \quad -(93t^4 - 222t^3 - 4t^2)u - (54t^5 - 69t^4 - 38t^3) = 0. \end{cases} \tag{5.33}$$

To find such pairs of numbers t, u that satisfy the above system (5.33) of equations, let us eliminate u. First, by eliminating u^5 and u^4 from (5.33), we have

$$(67t + 68)u^3 + (105t^2 + 375t + 39)u^2 + (-43t^3 + 714t^2 + 130t)u - (225t^4 - 443t^3 - 199t^2) = 0. \tag{5.34}$$

$\{(5.34) \times u\}$ and (5.33) imply

$$(5.35)\qquad (172t^2 + 443t + 39)u^3 + (225t^3 + 1455t^2 + 606t)u^2 + (110t^4 + 1989t^3 + 1423t^2)u - (603t^5 - 929t^4 - 1564t^3) = 0.$$

$\{(5.34) \times (172t^2 + 443t + 39) - (5.35) \times (67t + 68)\} \div 3$ becomes

$$(5.36)\qquad (995t^4 - 590t^3 + 12462t^2 - 3102t + 507)u^2 = (4922t^5 + 12328t^4 - 35464t^3 + 3776t^2 - 1690t)u - 567t^6 + 14906t^5 - 17914t^4 + 306t^3 - 2587t^2.$$

Second, $(5.34) \times (995t^4 - 590t^3 + 12462t^2 - 3102t + 507)$ and (5.36) give

$$(5.37)\qquad \begin{aligned}&(67t + 68)u\{(4922t^5 + 12328t^4 - 35464t^3 + 3776t^2 - 1690t)u - 567t^6 + 14906t^5 - 17914t^4 + 306t^3 - 2587t^2\}\\ &+ (105t^2 + 375t + 39)\{(4922t^5 + 12328t^4 - 35464t^3 + 3776t^2 - 1690t)u - 567t^6 + 14906t^5 - 17914t^4 + 306t^3 - 2587t^2\}\\ &+ (-43t^3 + 714t^2 + 130t)(995t^4 - 590t^3 + 12462t^2 - 3102t + 507)u\\ &- (225t^4 - 443t^3 - 199t^2)(995t^4 - 590t^3 + 12462t^2 - 3102t + 507) = 0.\end{aligned}$$

Here, $(5.37) \div 2t(67t + 68)$ becomes

$$(5.38)\qquad (2461t^4 + 6164t^3 - 17732t^2 + 1888t - 845)u^2 + (3254t^5 + 32788t^4 - 32704t^3 - 1620t^2 - 5174t)u + (-2115t^6 + 16520t^5 - 10652t^4 + 10788t^3 - 9933t^2) = 0.$$

Third, $(5.38) \times (995t^4 - 590t^3 + 12462t^2 - 3102t + 507)$ and (5.36) give

$$\begin{aligned}&(2461t^4 + 6164t^3 - 17732t^2 + 1888t - 845)\{(4922t^5 + 12328t^4 - 35464t^3 + 3776t^2 - 1690t)u - 567t^6 + 14906t^5 - 17914t^4 + 306t^3 - 2587t^2\}\\ &+ (3254t^5 + 32788t^4 - 32704t^3 - 1620t^2 - 5174t)(995t^4 + \cdots + 507)u\\ &+ (-2115t^6 + 16520t^5 - 10652t^4 + 10788t^3 - 9933t^2)(995t^4 + \cdots + 507) = 0.\end{aligned}$$

And dividing the above equation by $4t(67t + 68)$ we obtain

$$(5.39)\qquad (57279t^7 + 282846t^6 - 697135t^5 + 698506t^4 - 129559t^3 - 69294t^2 + 36855t - 4394)u = (13059t^7 - 203082t^6 + 164525t^5 + 376306t^4 - 906107t^3 + 494522t^2 - 124805t + 10478)t.$$

In the same way as above, (5.36) $\times$ $(57279t^7 + \cdots - 4394)$ and (5.39) imply an equation. And dividing the equation by $(995t^4 + \cdots + 507)$ we also obtain

$$(5.40)\qquad \begin{aligned}&(13059t^7 - 203082t^6 + 164525t^5 + 376306t^4 - 906107t^3 + 494522t^2\\ &\quad - 124805t + 10478)u = (31959t^7 - 126930t^6 + 959993t^5\\ &\quad - 2470086t^4 + 2650385t^3 - 1084542t^2 + 226831t - 12506)t.\end{aligned}$$

Last, using (5.39) and (5.40) we obtain an equation in which u is eliminated and dividing both sides of the equation by $32(995t^4 + \cdots + 507)$ we obtain

$$(5.41)\qquad \begin{aligned}&52137t^{10} + 253062t^9 - 2033508t^8 + 5141910t^7 - 7134618t^6\\ &\quad + 6230014t^5 - 3591608t^4 + 1378538t^3 - 343231t^2 + 50684t - 3380\\ &\quad = (t-1)^2(3t-1)^2(5793t^6 + 43566t^5 - 123930t^4 + 139498t^3\\ &\qquad\qquad - 79719t^2 + 23644t - 3380) = 0.\end{aligned}$$

From (5.39), (5.40) and (5.32), we see that if $t = 1$ or $\frac{1}{3}$, then $u = -1$ and $S = 4$. But since $S > 4$, we know $t \neq 1$ and $t \neq \frac{1}{3}$. Hence, from (5.41) we have an equation

$$(5.42)\qquad 5793t^6 + 43566t^5 - 123930t^4 + 139498t^3 - 79719t^2 + 23644t - 3380 = 0.$$

Let

$$f(t) = 5793t^6 + 43566t^5 - 123930t^4 + 139498t^3 - 79719t^2 + 23644t - 3380.$$

Then, we have

$$\begin{aligned}f'(t) &= 34758t^5 + 217830t^4 - 495720t^3 + 418494t^2 - 159438t + 23644,\\ f''(t) &= 6(28965t^4 + 145220t^3 - 247860t^2 + 139498t - 26573),\\ f'''(t) &= 6(115860t^3 + 435660t^2 - 495720t + 139498)\\ &= 6(28965t + 131172)(2t-1)^2 + 6(26832t^2 + 3t + 8326) > 0.\end{aligned}$$

Since $f'''(t) > 0$ for all $t > 0$, f'' is increasing. And since $f''(0) < 0$, there is only one real number α $(5/12 < \alpha < 1/2)$ such that $f''(\alpha) = 0$. That is, f' has only one local minimum at α. For the α,

$$\begin{aligned}f'(\alpha) &= 34758\alpha^5 + 217830\alpha^4 - 495720\alpha^3 + 418494\alpha^2 - 159438\alpha + 23644\\ &= \left(\frac{6\alpha}{5} - 1\right)(28965\alpha^4 + 145220\alpha^3 - 247860\alpha^2 + 139498\alpha - 26573)\\ &\qquad + 72531\alpha^4 - 53068\alpha^3 + 3236\alpha^2 + 11947\alpha - 2929 + \frac{2}{5}\alpha^2 + \frac{3}{5}\alpha\\ &= 72531\alpha^4 - 53068\alpha^3 + 3236\alpha^2 + 11947\alpha - 2929 + \frac{2}{5}\alpha^2 + \frac{3}{5}\alpha\\ &> (8059\alpha^2 - 524\alpha - 886)(3\alpha-1)^2 + (2\alpha+11)(\alpha-1)^2 + 7175\alpha - 2054 > 0,\end{aligned}$$

since $8059\alpha^2 - 524\alpha - 886 > 0$ and $7175\alpha - 2054 > 0$. Hence $f'(t) > 0$ for all $t > 0$, and so f is increasing. It implies that the equation (5.42) has only one root β (≈ 0.654) between $3/5$ and $2/3$, since $f(3/5) < 0$ and $f(2/3) > 0$. Since $S = 6t + 2u^2/t + 4u > 4$, we have

$$u^2 + 2tu + 3t^2 - 2t > 0$$

and for the root $t = \beta$ we also have

$$u^2 + 2\beta u + 3\beta^2 - 2\beta > 0.$$

Hence, we have

$$u < -\beta - \sqrt{2\beta(1-\beta)} \quad \text{and} \quad u > -\beta + \sqrt{2\beta(1-\beta)}. \tag{5.43}$$

In fact, since $3/5 < \beta < 2/3$ we have

$$-\beta - \sqrt{2\beta(1-\beta)} < -1 \quad \text{and} \quad -\beta + \sqrt{2\beta(1-\beta)} > 0. \tag{5.44}$$

Since $u = 2a^2 - 1 > -1$, from (5.43) and (5.44) we need at least that $u > 0$. But from (5.39) and (5.40) we can compute that $u \approx -1.12 < 0$. Therefore there is no a pair t, u satisfying (5.33) such that $t > 0, t \neq \frac{1}{3}, t \neq 1$ and $u > 0$. That is, it follows that $b \neq 1$, which completes the proof of Lemma 5.2. □

We completes the proof of our Theorem by Lemma 5.1 and Lemma 5.2.

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