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Some New Application of Power Increasing Sequences

Waad Sulaiman
Department of Computer Engineering, College of Engineering, University of Mosul, Mosul, Iraq
e-mail : waadsulaiman@hotmail.com

Abstract. A general theorem for absolute summability using quasi $f-$power increasing sequence is obtained. This result generalized and improved the result of Bor and Özarslan [2].

1. Introduction

Let $\sum a_n$ be an infinitive series with the sequence of partial sums (s_n). By (C, α) we mean the Cesàro matrix of order α, and σ_n^α denotes the n-th term of the $(C, \alpha)-$transform of (s_n), that is

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v. \tag{1.1}$$

A series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k < \infty, \tag{1.2}$$

and it is said to be summable $\varphi - |C, \alpha|_k$, $k \geq 1$, $\alpha > -1$ if (see Balcı [1])

$$\sum_{n=1}^{\infty} \left|\varphi_n \left(\sigma_n^\alpha - \sigma_{n-1}^\alpha\right)\right|^k < \infty, \tag{1.3}$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-k} \left|\varphi_n t_n^\alpha\right|^k < \infty, \tag{1.4}$$

where

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v. \tag{1.5}$$

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In the special case when $\varphi_n = n^{1-1/k}, n^{\delta+1-1/k}$ $(n \in N,\ k \geq 1)$, $\varphi-\left|C,\alpha\right|_k$ −summability reduces to $\left|C,\alpha\right|_k$ −summability, $\left|C,\alpha,\delta\right|_k$ −summability respectively. Let (p_n) be a sequence of positive real numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \text{ as } n \to \infty,\ P_{-1} = p_{-1} = 0. \tag{1.6}$$

A series $\sum a_n$ is said to be summable $\left|N,p_n\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|u_n - u_{n-1}\right|^k < \infty, \tag{1.7}$$

where

$$u_n = \frac{1}{P_n}\sum_{n=1}^{\infty} p_{n-v}s_v. \tag{1.8}$$

For $k = 1$, $\left|N,p_n\right|_k$ −summability reduces to $\left|N,p_n\right|$ −summability, and for $p_n = A_n^{\alpha}$, $\left|N,p_n\right|_k$ −summability reduces to $\left|C,\alpha\right|_k$ −summability. By M we denote the set of sequences $p = (p_n)$ satisfying

$$\frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, \quad p_n > 0, \quad n = 0, 1, \ldots. \tag{1.9}$$

It is known that (see Das [4]) for $p \in M$, $k = 1$, (1.7) holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n}\left|\sum_{v=1}^{n} p_{n-v}va_v\right| < \infty. \tag{1.10}$$

For $p \in M$, the series $\sum a_n$ is summable $\left|N,p_n\right|_k$, $k \geq 1$, if (see Sulaiman [7])

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k}\left|\sum_{v=1}^{n} p_{n-v}va_v\right|^k < \infty. \tag{1.11}$$

Here we give the following definition: For $p \in M$, the series $\sum a_n$ is summable $\varphi - \left|N,p_n\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{-k}\left|\varphi_n \Phi_n\right|^k < \infty, \tag{1.12}$$

where

$$\Phi_n = \frac{1}{P_n}\sum_{v=1}^{n} p_{n-v}va_v. \tag{1.13}$$

A positive sequence $a = (a_n)$ is said to be quasi $\beta-$power increasing sequence if there exists a constant $K = K(\beta, a) \geq 1$ such that (see Leindler [5])

$$Kn^{\beta}a_n \geq m^{\beta}a_m \qquad (n \geq m \geq 1,\ n, m \in N).$$

A positive sequence (a_n) is said to be quasi $f-$power increasing sequence if there exists a constant $K = K(f, a) \geq 1$ such that (see Sulaiman [8])

$$Kf_n a_n \geq f_m a_m \qquad (n \geq m \geq 1,\ n, m \in N).$$

where $f = (f_n)$, $f = n^{\beta}(\log n)^{\gamma}$, $0 < \beta < 1$, $\gamma \geq 0$. Recently, Bor and Özarslan [2] presented the following theorem

Theorem 1.1. *Let (X_n) be a quasi-$\beta-$power increasing sequence for some $\beta\,(0 < \beta < 1)$. Suppose also that there exist sequences (β_n) and $(\lambda_n) \in BV_0$ such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{1.14}$$

$$\beta_n \to \infty,\ \ (n \to \infty), \tag{1.15}$$

$$\sum_{n=1}^{\infty} n\,|\Delta\beta_n|\,X_n < \infty, \tag{1.16}$$

and

$$|\lambda_n|\,X_n = O(1). \quad (n \to \infty) \tag{1.17}$$

If there exists an $\epsilon > 0$ such that the sequence $\left(n^{\epsilon-k}\,|\varphi_n|^k\right)$ is non-increasing and if the sequence (ω_n^{α}) is defined by

$$\omega_n^{\alpha} = \begin{cases} |t_n^{\alpha}|, & \alpha = 1 \\ \max\limits_{1\leq v\leq n}\{|t_v^{\alpha}|\}, & 0 < \alpha < 1 \end{cases} \tag{1.18}$$

satisfying the following condition:

$$\sum_{n=1}^{m} n^{-k}\left(|\varphi_n|\,\omega_n^{\alpha}\right)^k = O(X_m), \quad (m \to \infty), \tag{1.19}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - |C, \alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$, $k\alpha + \epsilon > 1$.

2. Lemmas

In this section, the following lemmas are needed for our aim.

Lemma 2.1. *If* $0 \leq m \leq n$, p_{n-v}/p_{m-v} *is decreasing with respect to* v, *then*

$$\left|\sum_{v=0}^{m} p_{n-v}s_v\right| \leq \max_{0\leq\mu\leq m}\left|\sum_{v=0}^{\mu} p_{\mu-v}s_v\right|. \tag{2.1}$$

Proof. The proof is similar to that given in [3]. □

Lemma 2.2. *Condition* (1.14) – (1.16) *implies*

$$n\beta_n X_n = O(1),\ n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{2.3}$$

Proof. The proof is similar to that given in [8]. □

3. Results

The following is our main result.

Theorem 3.1. *Let* (X_n) *be a quasi-*$f-$*power increasing sequence,* $f = (f_n)$, $f_n = n^{\beta}(\log n)^{\gamma}, 0 < \beta < 1$, $\gamma \geq 0$, *such that the sequences* (X_n) *and* (λ_n) *satisfying conditions* (1.14) – (1.17), *and*

$$\sum_{n=1}^{m} \frac{1}{n^k X_n^{k-1}} \left(|\varphi_n|\,\omega_n\right)^k = O(X_m), \quad (m\to\infty), \tag{3.1}$$

$$\sum_{n=v+1}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k} = O\left(\frac{|\varphi_v|^k}{v^{k-1}P_v^k}\right) \tag{3.2}$$

where

$$\omega_n = \begin{cases} \Phi_n, & v = n \\ \max\{\Phi_v\}, & 1 \leq v \leq n. \end{cases} \tag{3.3}$$

Then the series $\sum a_n\lambda_n$ *is summable* $\varphi - |N, p_n|_k$, $k \geq 1$.

Proof. Let T_n be n-th $(N, p_n)-$mean of the sequence $(na_n\lambda_n)$. Then, we have

$$T_n = \frac{1}{P_n}\sum_{v=1}^{n} v a_v p_{n-v}\lambda_v$$

We have, via Abel's transformation

$$\begin{aligned}
|T_n| &= \frac{1}{P_n}\left|\sum_{v=1}^{n} v a_v p_{n-v}\lambda_v\right| \\
&= \frac{1}{P_n}\left|\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v} r p_{n-r} a_r\right)\Delta\lambda_v + \lambda_n \sum_{v=1}^{n} v p_{n-v} a_v\right| \\
&\leq \frac{1}{P_n}\sum_{v=0}^{n-1} P_v\left|\omega_v\right|\left|\Delta\lambda_v\right| + \left|\lambda_n\right|\left|\omega_n\right| \\
&= T_{n1} + T_{n2}.
\end{aligned}$$

Since $|T_{n1}+T_{n2}|^k \leq 2^k\left(|T_{n1}|^k + |T_{n2}|^k\right)$, then in order to complete the proof, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k}\left|\varphi_n T_{nj}\right|^k < \infty, \qquad j = 1, 2.$$

Applying Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-k}\left|\varphi_n T_{n1}\right|^k &= \sum_{n=2}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k}\left|\sum_{v=1}^{n-1} P_v\left|\omega_v\right|\left|\Delta\lambda_v\right|\right|^k \\
&\leq \sum_{n=2}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k}\sum_{v=1}^{n-1} P_v^k\left|\omega_v\right|^k \beta_v X_v^{1-k}\left(\sum_{v=1}^{n-1} X_v\beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k}\sum_{v=1}^{n-1} P_v^k\left|\omega_v\right|^k \beta_v X_v^{1-k} \\
&= O(1)\sum_{v=1}^{m} P_v^k\left|\omega_v\right|^k \beta_v X_v^{1-k}\sum_{n=v+1}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k} \\
&= O(1)\sum_{v=1}^{m} \frac{v}{v^k X_v^{k-1}}\left|\varphi_v\right|^k\left|\omega_v\right|^k \beta_v \\
&= O(1)\sum_{v=1}^{m}\left(\sum_{r=1}^{v} \frac{|\varphi_r|^k\left|\omega_r\right|^k}{r^k X_r^{k-1}}\right)\Delta\left(v\beta_v\right) \\
&\quad + O(1) m\beta_m \sum_{v=1}^{m} \frac{1}{v^k X_v^{k-1}}\left|\varphi_v\right|^k\left|\omega_v\right|^k \\
&= O(1)\sum_{v=1}^{m} \beta_v X_v + O(1)\sum_{v=1}^{m} v\left|\Delta\beta_v\right| X_v + O(1) m\beta_m X_m \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{m} n^{-k}\left|\varphi_n T_{n2}\right|^k &= \sum_{n=1}^{m} \frac{|\varphi_n|^k}{n^k}|\lambda_n|^k|\omega_n|^k \\
&= \sum_{n=1}^{m} \frac{|\varphi_n|^k}{n^k X_n^{k-1}}|\lambda_n||\omega_n|^k\left(|\lambda_n| X_n\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m} \frac{|\varphi_n|^k|\omega_n|^k}{n^k X_n^{k-1}}|\lambda_n| \\
&= O(1)\sum_{n=1}^{m}\left(\sum_{v=1}^{n} \frac{|\varphi_v|^k|\omega_v|^k}{v^k X_v^{k-1}}\right)\Delta|\lambda_n| + O(1)|\lambda_m|\sum_{n=1}^{m} \frac{|\varphi_n|^k|\omega_n|^k}{n^k X_n^{k-1}} \\
&= O(1)\sum_{n=1}^{m}|\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1)\sum_{n=1}^{m}\beta_n X_n + O(1)|\lambda_m| X_m \\
&= O(1). \qquad \square
\end{aligned}$$

Remark 1. (1) It may be mentioned that an improvement to Theorem 1.1 follows from Theorem 3.1 by putting $p_n = A_n^{\alpha-1}$, provided from some $\epsilon > 0$, $\left(n^{\epsilon-k}|\varphi_n|^k\right)$ is non-increasing, and $k\alpha + \epsilon > 1$.
(2) One advantage of condition (3.1) is that we have no loss of power of $|\lambda_n|$ through estimation such as by having $|\lambda_n|^{k-1} = O(1)$.

Remark 2. Condition (3.1) does not imply condition (1.19).

Proof.

$$\begin{aligned}
\sum_{n=1}^{m} n^{-k}\left(|\varphi_n|\omega_n^\alpha\right)^k &= \sum_{n=1}^{m} \frac{1}{n^k X_n^{k-1}}\left(|\varphi_n|\omega_n^\alpha\right)^k X_n^{k-1} \\
&= O(X_m^{k-1})\sum_{n=1}^{m} \frac{1}{n^k X_n^{k-1}}\left(|\varphi_n|\omega_n^\alpha\right)^k \\
&= O(X_m^k) \\
&\neq O(X_m), \text{ for } k > 1. \qquad \square
\end{aligned}$$

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