# On the Invariance of Primitive Ideals via $\phi$-derivations on Banach Algebras

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Abstract. The noncommutative Singer-Wermer conjecture states that every derivation on a Banach algebra (possibly noncommutative) leaves primitive ideals of the algebra invariant. This conjecture is still an open question for more than thirty years. In this note, we approach this question via some sufficient conditions for the separating ideal of $\phi$-derivations to be nilpotent. Moreover, we show that the spectral boundedness of $\phi$-derivations implies that they leave each primitive ideal of Banach algebras invariant.

## 1. Introduction

By a $\phi$-derivation on an algebra $A$, we mean a linear mapping $\Delta : A \to A$ satisfying the identity $\Delta(ab) = \Delta(a)\phi(b) + a\Delta(b)$ for all $a, b \in A$, where $\phi$ is an automorphism of $A$. Of course, $1_A$-derivations (where $1_A$ is the identity mapping on $A$) are ordinary derivations. For example, for any automorphism $\phi$, $\phi - 1_A$ is a $\phi$-derivation, and for each fixed $c \in A$, the mapping $\Delta : x \mapsto c\phi(x) - cx$ $(x \in A)$, is a $\phi$-derivation. For any derivation $D$ of a unital algebra $A$ and an invertible element $c \in A$, the mapping $\Delta : x \mapsto D(x)c$ is a $\phi_c$-derivation, where $\phi_c : x \mapsto c^{-1}xc$ is an inner automorphism. Hence, the notion of a $\phi$-derivation can be considered as a generalization and unification of both the notions of a derivation and an automorphism.

In 1955 Singer and Wermer [11] proved that the range of continuous derivation on a commutative Banach algebra is contained in the Jacobson radical. In the same paper they conjectured that the assumption of continuity is superfluous. In 1969 Johnson [6] proved that the Singer-Wermer conjecture is true when the algebra is semisimple. In 1988 the Singer-Wermer conjecture for a commutative Banach algebra was finally conformed by Thomas [12].

Now the problem concerning derivations on Banach algebras belongs to the noncommutative setting which states that a (possibly discontinuous) derivation

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$D$ on a (possibly noncommutative) Banach algebra $A$ such that the commutator $D(a)a - aD(a)$ belongs to the Jacobson radical of $A$ for all $a \in A$, maps $A$ into its Jacobson radical. Equivalently, every derivation on A leaves primitive ideals of $A$ invariant, which is called the *noncommutative Singer-Wermer conjecture*. But the question whether this is true, is still an open problem. In 1969 Sinclair [9] proved the noncommutative Singer-Wermer conjecture in case the derivation is continuous. There are various partial answers of the noncommutative Singer-Wermer conjecture and these results has been accomplished by a number of authors (for example, see [2, 7]).

The purpose of this note is to present the noncommutative Singer-Wermer conjecture with some conditions via $\phi$-derivations.

Throughout, $A$ will represent an algebra over a complex field $\mathbb{C}$. The *Jacobson radical* (resp. the *prime radical*) of $A$ will be denoted by $rad(A)$ (resp. $prad(A)$). Note that $rad(A)$ (resp. $prad(A)$) is the intersection of all primitive ideals (resp. all prime ideals) of $A$. $A$ is said to be *semisimple* (resp. *semiprime*) if $rad(A) = \{0\}$ (resp. $prad(A) = \{0\}$). We write $[a, b]$ for the commutator $ab - ba$. $N$ will denote the set of all natural numbers, and $\pi_I$ will denote the canonical quotient mapping from $A$ onto $A/I$, where $I$ is any closed two-sided ideal of $A$.

Without loss of generality we assume $A$ to be *unital*. In fact, any Banach algebra $A$ without a unity can be embedded into a unital Banach algebra $A' = A \oplus \mathbb{C}$ as an ideal of codimension one. In particular, we can identify $A$ with the ideal $\{(a, 0) : a \in A\}$ in $A'$ by the isometric isomorphism $a \to (a, 0)$.

## 2. Nilpotency of the Separating Ideal of $\phi$-derivations and $\Delta$-invariant Primitive Ideals

Let $A$ be a Banach algebra and $\Delta$ a $\phi$-derivation on $A$. Then the separating space of $\Delta$ is defined as

$$\mathcal{S}(\Delta) = \big\{ a \in A : \text{ there exists a sequence } \{a_n\} \to 0 \text{ in } A \text{ with } \Delta(a_n) \to a \big\},$$

which is a closed subspace of of $A$ and $\Delta$ is continuous if and only if $\mathcal{S}(\Delta) = \{0\}$ (see [10]). It was shown in [5, p. 1183] that $\mathcal{S}(\Delta)$ is a separating ideal of $A$, i.e., a separating ideal $J$ of $A$ is a closed two-sided ideal of $A$ with the property that, for each sequence $\{a_n\}$ in $A$, there exists $N \in N$ such that $\overline{Ja_n \dots a_1} = \overline{Ja_N \dots a_1}$ for all $n \geq N$.

The next two lemmas are due to Hejazian and Janfada [5].

**Lemma 2.1.** *Let $A$ be a Banach algebra, let $\phi$ be a continuous automorphism of $A$ and let $\Delta$ be a $\phi$-derivation on $A$. Suppose that $\phi$ and $[\Delta, \phi]$ leave each nilpotent and each primitive ideal of $A$ invariant. If $\mathcal{S}(\Delta)$ is nilpotent, then $\Delta$ leaves each primitive ideal of $A$ invariant.*

**Lemma 2.2.** *Let $A$ be a Banach algebra, let $\phi$ be a continuous automorphism of $A$ and let $\Delta$ be a $\phi$-derivation on $A$ with $[\Delta, \phi] = 0$. If $\mathcal{S}(\Delta) \cap rad(A)$ is nil, then $\mathcal{S}(\Delta)$ is nilpotent.*

We are ready to investigate our main results.

**Theorem 2.3.** *Let $A$ be a Banach algebra, let $\phi$ be an automorphism of $A$ and let $\Delta$ be a $\phi$-derivation on $A$. Then $\mathcal{S}(\Delta)$ is nilpotent if and only if $\bigcap_{n\geq 1}[\mathcal{S}(\Delta)]^n$ is a nil ideal.*

*Proof.* One implication is obvious. Suppose that $\bigcap_{n\geq 1}[\mathcal{S}(\Delta)]^n$ is a nil ideal and $\mathcal{S}(\Delta)$ is not nilpotent. By [4, Theorem 2.5], there exist closed prime ideals $P_1, P_2, \cdots, P_k$ of $A$ that do not contain $\mathcal{S}(\Delta)$ such that

$$\mathcal{S}(\Delta) \cap prad(A) = \mathcal{S}(\Delta) \cap P_1 \cap P_2 \cap \cdots \cap P_k.$$

Since each $P_i$ is closed, we see that $\mathcal{S}(\Delta) \cap prad(A)$ is closed. Let $a$ be an element of $\mathcal{S}(\Delta)$ which is not nilpotent. Since $\mathcal{S}(\Delta)$ is a separating ideal of $A$, it follows from the Mittag-Leffler theorem that $\bigcap_{n\geq 1}\mathcal{S}(\Delta)a^n$ is dense in $\overline{\mathcal{S}(\Delta)a^N}$ for some $N \in \mathbb{N}$. It is clear $\mathcal{S}(\Delta)a^n \subseteq \mathcal{S}(\Delta)a^{n+1}$ for all $n \in \mathbb{N}$. Therefore we have

$$\bigcap_{n\geq 1}\mathcal{S}(\Delta)a^n \subseteq \bigcap_{n\geq 1}[\mathcal{S}(\Delta)]^n \subseteq \mathcal{S}(\Delta) \cap prad(A).$$

Since $\bigcap_{n\geq 1}\mathcal{S}(\Delta)a^n$ is dense in $\overline{\mathcal{S}(\Delta)a^N}$ and $\mathcal{S}(\Delta) \cap prad(A)$ is closed, we get $\overline{\mathcal{S}(\Delta)a^N} \subseteq \mathcal{S}(\Delta) \cap prad(A)$. This implies that $\overline{\mathcal{S}(\Delta)a^N} \subseteq P_i$ for $i = 1, 2, \cdots, k$. But each $P_i$ is a prime ideal, hence $a \in P_i$ for $i = 1, 2, \cdots, k$. Thus we obtain $a \in \mathcal{S}(\Delta) \cap prad(A)$ which tells us that $a$ is nilpotent. This is a contradiction and we have the result. $\square$

**Theorem 2.4.** *Let $A$ be a Banach algebra, let $\phi$ be a continuous automorphism of $A$ and let $\Delta$ be a $\phi$-derivation on $A$. Suppose that $\phi$ leaves each nilpotent and each primitive ideal of $A$ invariant and $[\Delta, \phi] = 0$. If the Jacobson radical $rad(A)$ of $A$ is finite dimensional, then $\Delta$ leaves each primitive ideal of $A$ invariant.*

*Proof.* Let $a \in \mathcal{S}(\Delta) \cap rad(A)$. Then there exists a sequence $\{a_n\}$ in $A$ with $\{a_n\} \to 0$ such that $\Delta(a_n) \to a$. Then $aa_n \to 0$ in $rad(A)$ and $\Delta(aa_n) = \Delta(a)\phi(a_n) + a\Delta(a_n)$. Thus we have $\Delta(aa_n) \to a^2$. Since $\Delta$ is continuous on $rad(A)$, we get $a^2 = 0$. This implies that $\mathcal{S}(\Delta) \cap rad(A)$ is nilpotent and so $\mathcal{S}(\Delta)$ is nilpotent by Lemma 2.2. Now Lemma 2.1 gives the conclusion. $\square$

**Remark 2.5([8]).** Let $l^2 = \left\{a = \{\lambda_n\} : \sum_{n=1}^{\infty}|\lambda_n|^2 < \infty\right\}$ Then $l^2$ is a Banach space with norm $\|a\| = (\sum_{n=1}^{\infty}|\lambda_n|^2)^{\frac{1}{2}} < \infty$. It is well-known that $l^2$ is a Banach algebra under the pointwise multiplication. Let $l_0^2$ be the dense subalgebra of $l^2$ consisting of elements which vanish outside a finite set. Let $A_0 = l_0^2 \oplus \mathbb{C}$ be the linear space direct sum. Define a multiplication and a norm in $A_0$ by $(a, \alpha)(b, \beta) = (ab, 0)$ and $\|(a, \alpha)\| = \max(\|a\|, |\alpha - \sum_{n=1}^{\infty}\lambda_n|)$. Let $A$ be the completion of $A_0$ with respect to this norm. Then the Jacobson radical $rad(A)$ of the Banach algebra $A$ is one-dimensional since $rad(A) = \{0\} \oplus \mathbb{C}$.

**Theorem 2.6.** *Let $A$ be a Banach algebra, let $\phi$ be an inner automorphism of $A$*

and let $\Delta$ be a $\phi$-derivation on $A$. Suppose that $[\Delta, \phi]$ leave each nilpotent and each primitive ideal of $A$ invariant. If $\mathcal{S}(\Delta)$ is finite dimensional and semisimple, then $\Delta$ leaves each primitive ideal of $A$ invariant.

*Proof.* It is sufficient to show that $\mathcal{S}(\Delta)$ is nilpotent. Suppose on the contrary that $\mathcal{S}(\Delta)$ is not nilpotent, that is, $\mathcal{S}(\Delta)^n \neq \{0\}$ for all $n \in \mathbb{N}$. Since $\mathcal{S}(\Delta)$ is finite dimensional and semisimple, it follows from the Wedderburn theorem that $\mathcal{S}(\Delta)$ has an identity $e$ which is a central idempotent in $A$. Then we have

$$\Delta(e) = \Delta(e^2) = \Delta(e)\phi(e) + e\Delta(e) = \Delta(e)e + e\Delta(e) = 2e\Delta(e)$$

which yields that $e\Delta(e) = 2e^2\Delta(e) = 2e\Delta(e)$. Hence $e\Delta(e) = 0$ and $\Delta(e) = 2e\Delta(e) = 0$. On the other hand, for all $a \in A$

$$\Delta(ea) = \Delta(e)\phi(a) + e\Delta(a) = e\Delta(a) \in eA;$$
$$\Delta(a - ea) = \Delta(a) - \Delta(e)\phi(a) - e\Delta(a) = (1 - e)\Delta(a) \in (1 - e)A.$$

Since $eA$ is contained in $\mathcal{S}(\Delta)$, $eA$ is finite dimensional. Then the derivation $\Delta$ induces a derivation $\bar{\Delta}$ on the Banach algebra $A/(1 - e)A$, defined by $\bar{\Delta}(a + (1 - e)A) = \Delta(a) + (1 - e)A$ for all $a \in A$. Since $A = eA \oplus (1 - e)A$, $\bar{\Delta}$ is continuous on $A/(1 - e)A$. This means that $\mathcal{S}(\Delta) \subseteq (1 - e)A$ by [10, Lemma 1.4]. Thus we get $e \in \mathcal{S}(\Delta) \subseteq (1 - e)A$ but $e \notin (1 - e)A$. This is a contradiction. Hence $\mathcal{S}(\Delta)$ is nilpotent and Lemma 2.1 completes the proof.                                    □

**Lemma 2.7.** *Let $A$ be a Banach algebra, let $\phi$ be a continuous automorphism of $A$ and let $\Delta$ be a $\phi$-derivation on $A$ with $[\Delta, \phi] = 0$. If $\bigcap_{n \geq 1}[rad(A)]^n = \{0\}$, then $\mathcal{S}(\Delta)$ is nilpotent.*

*Proof.* Let $a \in \mathcal{S}(\Delta) \cap rad(A)$. Since $\mathcal{S}(\Delta)$ is a separating ideal, there is $N \in \mathbb{N}$ such that $\overline{\mathcal{S}(\Delta)a^N} = \overline{\mathcal{S}(\Delta)a^n}$ for all $n \geq N$. Hence we have

$$\overline{\mathcal{S}(\Delta)a^N} = \bigcap_{n \geq N} \overline{\mathcal{S}(\Delta)a^n} = \bigcap_{n \geq 1} \overline{\mathcal{S}(\Delta)a^n}.$$

Applying Mittag-Leffler theorem, we obtain

$$\overline{\mathcal{S}(\Delta)a^N} = \bigcap_{n \geq 1} \overline{\mathcal{S}(\Delta)a^n} = \overline{\bigcap_{n \geq 1} \mathcal{S}(\Delta)a^n} \subseteq \bigcap_{n \geq 1}[rad(A)]^n = \{0\}.$$

Thus we get $a^{N+1} = 0$ which implies that $\mathcal{S}(\Delta) \cap rad(A)$ is nilpotent and so is $\mathcal{S}(\Delta)$ by Lemma 2.2.                                    □

**Lemma 2.8.** *Let $A$ be an algebra and let $P$ be a prime ideal of $A$. If $\phi$ is an inner automorphism of $A$ and $\Delta$ is a $\phi$-derivation on $A$ with $[\Delta, \phi] = 0$, then $Q = \{a \in P : \Delta^n(a) \in P \text{ for all } n \in \mathbb{N}\}$ is a prime ideal of $A$.*

*Proof.* For convenience, take $\Delta^0(a) = a \, (a \in A)$. From ([3, Theorem 2.1]), we know

that, for any nonnegative integer $n$,

$$(2.1) \qquad \Delta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \Delta^{n-i}(a)\phi^{n-i}(\Delta^i(b)).$$

It is clear that $Q$ is an ideal of $A$. We show that $Q$ is prime as follows. Consider any $a, b \in A - Q$. Choose nonnegative integers $r$ and $s$ as small as possible so that $\Delta^r(a)$ and $\Delta^s(b)$ are not in $P$, and then choose $c \in A$ such that $\Delta^r(a)\phi^r(c)\phi^r(\Delta^s(b)) \notin P$. Now use the relation (2.1) to expand $\Delta^{r+s}(acb)$, as follows:

$$(2.2) \quad \Delta^{r+s}(acb) = \sum_{i=0}^{r+s} \binom{r+s}{i} \Delta^{r+s-i}(a)\phi^{r+s-i}(\Delta^i(cb))$$

$$= \sum_{i=0}^{r+s} \sum_{j=0}^{i} \binom{r+s}{i}\binom{i}{j} \Delta^{r+s-i}(a)\phi^{r+s-i}(\Delta^{i-j}(c))\phi^{r+s-j}(\Delta^j(b)).$$

Since we have $\Delta^{r+s-i}(a) \in P$ if $i > s$ and $\Delta^j(b) \in P$ if $j < s$, all of the terms in the last summation of (2.2) are in $P$ except for $\binom{r+s}{s}\binom{s}{s}\Delta^r(a)\phi^r(c)\phi^r(\Delta^s(b))$, which is not in $P$ since $\Delta^r(a)\phi^r(c)\phi^r(\Delta^s(b))$ is not in $P$. Hence $\Delta^{r+s}(acb) \notin P$ and so $acb \notin Q$, which shows that $Q$ is prime. $\qquad\qquad\square$

**Theorem 2.9.** *Let $A$ be a Banach algebra. Suppose that $\phi$ is an inner automorphism of $A$ and $\Delta$ is a $\phi$-derivation on $A$ with $[\Delta, \phi] = 0$. Then, for each primitive ideal $P$ of $A$, $\Delta$ leaves $P$ invariant if and only if $J = \{a \in P : \Delta^n(a) \in P \text{ for all } n \in \mathbb{N}\}$ is closed in $A$.*

*Proof.* If $\Delta(P) \subseteq P$, then $J = P$ is closed.

Assume that $J$ is closed. According to Lemma 2.8, it follows that $J$ is a prime ideal of $A$ because any primitive ideal is prime. Since we have $\phi(J) \subseteq J$ and $\Delta(J) \subseteq J$, $\phi$ drops to an automorphism $\bar{\phi}$ of the prime Banach algebra $A/J$ and so $\Delta$ induces the $\bar{\phi}$-derivation $\bar{\Delta}$ on $A/J$ defined by $\bar{\Delta}(x + J) = \Delta(x) + J$ for all $x \in A$, respectively. Also the hypothesis $[\Delta, \phi] = 0$ on $A$ induces $[\bar{\Delta}, \bar{\phi}] = 0$ on $A/J$. Since $J \subseteq P$, we see that $J \subseteq rad(A)$ and

$$(2.3) \qquad rad(A/J) = rad(A)/J \subseteq P/J.$$

Now we show that

$$\bigcap_{n \geq 1} (P/J)^n = \{0\}.$$

Let $x \in \bigcap_{n \geq 1}(P/J)^n$. Then for each $n \in \mathbb{N}$, there exist elements $a_n \in P^n$ such that $x = a_n + J$. Since $a_1 - a_{n+1} \in J$ for all $n \in \mathbb{N}$, we have $\Delta^n(a_1 - a_{n+1}) \in P$.

On the other hand, we have $\Delta^n(a_{n+1}) \in P$ so we see that $\Delta^n(a_1) \in P$. Since $n$ is arbitrary, then we obtain $a_1 \in J$ and $x = 0$. But $x$ is arbitrary, hence we get

$$\bigcap_{n \geq 1} (P/J)^n = \{0\}.$$

Then the relation (2.3) gives

$$\bigcap_{n \geq 1} [rad(A/J)]^n = \{0\}$$

and so $\mathcal{S}(\bar{\Delta})$ is nilpotent by Lemma 2.7.

Since $A/J$ is prime and so has no non-zero nilpotent ideal, we obtain $\mathcal{S}(\bar{\Delta}) = \{0\}$, that is, $\bar{\Delta}$ is continuous on $A/J$. This means that we also define a mapping

$$\Phi \circ \bar{\Delta}^n \circ \pi_J : A \to A/J \to A/J \to A/P$$

by $(\Phi \circ \bar{\Delta}^n \circ \pi_J)(x) = (\pi_J \circ \Delta^n)(x)$ for all $x \in A$, where $\Phi$ is the canonical inclusion mapping from $A/J$ onto $A/P$ (which exists since $J \subseteq P$). We therefore conclude that $\|\pi_P \circ \Delta^n\| \leq \|\bar{\Delta}\|^n$ for all $n \in \mathbb{N}$, since the other mappings are norm depressing. Now, taking into account [13, Lemma 1.1], we can proceed analogously to the proof of [3, Theorem 3.2] to obtain that $\pi_P(\Delta(P))$ consists of quasinipotent elements, that is, elements $x$ such that $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0$. As in the proof of [3, Corollary 3.3], this gives $\Delta(P) \subseteq P$. □

**Corollary 2.10.** *Let $A$ be a Banach algebra. Suppose that $\phi$ is an inner automorphism of $A$ and that $\Delta$ is a $\phi$-derivation on $A$ with $[\Delta, \phi] = 0$. If every prime ideal of $A$ is closed, then $\Delta$ leaves each primitive ideal of $A$ invariant.*

*Proof.* Take $J = \{a \in P : \Delta^n(a) \in P \text{ for all } n \in \mathbb{N}\}$, where $P$ is a primitive ideal of $A$. Since $J$ is a prime ideal of $A$, it follows that $J$ is closed. By Theorem 2.9, we have the result. □

**Remark 2.11([8]).** Let $A = \left\{a = \sum_{n=0}^{\infty} a_n x^n : \|a\| = \sum_{n=0}^{\infty} |a_n| w^n < \infty\right\}$ in one indeterminant $x$ with complex coefficients where $\{w_n : n = 0, 1, 2, \cdots\}$ is a sequence in $(0, \infty)$ such that $w_0 = 1, w_{n+m} \leq w_n w_m$ and $\lim_{n\to\infty}(w_n)^{\frac{1}{n}} = 0$. Then $A$ is a Banach algebra of power series. Furthermore, $A$ has a unique maximal ideal $M = \left\{\sum_{n=0}^{\infty} a_n x^n : a_0 = 0\right\}$. If $\{w_n\}$ is chosen properly, then the only prime ideals of $A$ are $\{0\}$ and $A$. Hence every prime ideal of $A$ is closed.

The following is the Sinclair's version [9] of $\phi$-derivations.

**Corollary 2.12.** *Let $A$ be a Banach algebra. Suppose that $\phi$ is an inner automorphism of $A$ and $\Delta$ is a $\phi$-derivation on $A$ with $[\Delta, \phi] = 0$. If $\Delta$ is continuous on $A$, then $\Delta$ leaves each primitive ideal of $A$ invariant.*

*Proof.* Let $P$ be a primitive ideal of $A$. Since $\Delta$ is continuous, it is easy to see that $J = \{a \in P : \Delta^n(a) \in P \text{ for all } n \in \mathbb{N}\}$ is closed. Hence Theorem 2.9 gives the conclusion. □

**Theorem 2.13.** *Let $A$ be a Banach algebra in which every closed prime ideal has a finite codimension. If $\phi$ is an inner automorphism of $A$ and $\Delta$ is a $\phi$-derivation on $A$ with $[\Delta, \phi] = 0$, then $\Delta$ leaves each primitive ideal of $A$ invariant.*

*Proof.* We claim that $\mathcal{S}(\Delta)$ is nilpotent. Suppose that $\mathcal{S}(\Delta)$ is not nilpotent. Then

it follows from [4, Theorem 2.5] that there exists a minimal prime ideal $P$ such that $P$ is closed and $\mathcal{S}(\Delta) \not\subseteq P$. By Lemma 2.6, we see that $Q \subseteq P$ and we also obtain $P \subseteq Q$ by the minimality of $P$, therefore we have $P = Q$, i.e., $\Delta(P) \subseteq P$. Since $\phi$ induces an inner automorphism $\bar{\phi}$ of $A/P$, we can define a $\bar{\phi}$-derivation $\bar{\Delta} : A/P \to A/P$ by $\bar{\Delta}(a+P) = \Delta(a)+P$ for all $a \in A$. By the hypothesis, we have $\dim(A/P) < \infty$ and so $\bar{\Delta}$ is continuous on $A/P$. Hence [10, Lemma 1.4] yields that $\mathcal{S}(\Delta) \subseteq P$. This is a contradiction and we have the result on account of Lemma 2.1. □

**Remark 2.14([8]).** $A = C^n[0,1]$ is a Banach algebra of $n$ times continuously differential complex valued functions defined on the unit interval $[0,1]$ with the norm

$$\|f\|_n = \max_{t \in [0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}$$

for $f \in C^n[0,1]$. Then every closed prime ideal of $A$ has a finite codimension because the only closed prime ideals are the maximal ideals.

## 3. Spectral Boundedness of $\phi$-derivations and $\Delta$-invariant Primitive Ideals

Let $A$ and $B$ be Banach algebras. A linear mapping $T : A \to B$ is called *spectrally bounded* if there is $M > 0$ such that $r(T(a)) \leq Mr(a)$ for all $a \in A$. If $r(T(a)) = r(a)$ for all $a \in A$, we say that $T$ is a *spectrally isometry*. If $r(a) = 0$, then $a$ is called *quasinilpotent*. (Herein, $r(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}$ denotes the spectral radius of the element $a$).

Observe that the canonical epimorphism $\sigma := \pi_{rad(A)} : A \to A/rad(A)$ is spectrally isometry.

Brešar and Mathieu [2, Lemma 2.1] showed that if $\delta$ is a derivation on a unital Banach algebra $A$, then the spectral boundedness of $\delta$ implies that $\delta$ leaves each primitive ideal of $A$ invariant.

We now have the following results concerning $\phi$-derivations by modifying the above Brešar and Mathieu's result from derivations.

**Theorem 3.1.** *Let $A$ be a Banach algebra, let $\phi$ be an inner automorphism of $A$ and let $\Delta$ be a $\phi$-derivation on $A$. If $\Delta$ is spectrally bounded, then $\Delta$ leaves each primitive ideal of $A$ invariant.*

*Proof.* Suppose that $r(\Delta(a)) \leq Mr(a)$ for some $M > 0$ and all $a \in A$. Then we see

that

$$r(a\Delta(b)) = r(\Delta(ab) - \Delta(a)\phi(b))$$
$$= r(\sigma(\Delta(ab) - \Delta(a)\phi(b)))$$
$$= r(\sigma(\Delta(ab)) - \sigma(\Delta(a)\phi(b)))$$
$$= r(\sigma(\Delta(ab)))$$
$$= r(\Delta(ab)) \leq Mr(ab) = 0$$

for all $b \in rad(A)$ and all $a \in A$. Hence we have $\Delta(rad(A)) \subseteq rad(A)$ by [1, p.126, Prop. 1(ii)]. Since $\phi$ induces an inner automorphism $\bar{\phi}$ of $A/rad(A)$, we can define a $\bar{\phi}$-derivation $\bar{\Delta} : A/rad(A) \to A/rad(A)$ by $\bar{\Delta}(a + rad(A)) = \Delta(a) + rad(A)$ for all $a \in A$, i.e., $\bar{\Delta}$ is a $\bar{\phi}$-derivation on the semisimple Banach algebra $A/rad(A)$, and hence is continuous by [3, Corollary 4.3]. It follows from [9] that $\bar{d}$ leaves each primitive ideal $\bar{P}$ of $A/rad(A)$ invariant. If $P$ is a primitive ideal of $A$, then $\bar{P} = P/rad(A)$ is a primitive ideal of $A/rad(A)$ whence $\bar{d}(\bar{P}) \subseteq \bar{P}$ implies that $d(P) \subseteq P$. Thus we conclude that $d$ leaves each primitive ideal of $A$ invariant. □

**Theorem 3.2.** *Let $A$ be a Banach algebra, let $\phi$ be an inner automorphism of $A$ and let $\Delta$ be a $\phi$-derivation on $A$. If $\sup\{r(x^{-1}\Delta(a)) : a \in A$ invertible$\} < \infty$, then $\Delta$ leaves each primitive ideal of $A$ invariant.*

*Proof.* Assume that $s = \sup\{r(z^{-1}\Delta(c)) | c \in A$ invertible$\} < \infty$. Given $b \in rad(A)$, we have $(1+b)^{-1} = 1 - b(1+b)^{-1} \in 1 + rad(A)$ and hence

$$r((1+b)^{-1}\Delta(1+b)) = r((1 - b(1+b)^{-1})\Delta(b))$$
$$= r(\Delta(b) - b(1+b)^{-1}\Delta(b))$$
$$= r(\sigma(\Delta(b) - b(1+b)^{-1}\Delta(b)))$$
$$= r(\sigma(\Delta(b)) - \sigma(b(1+b)^{-1}\Delta(b)))$$
$$= r(\sigma(\Delta(b))) = r(\Delta(b)).$$

By the assumption, it follows that $r(\Delta(b)) \leq s < \infty$ for all $b \in rad(A)$, whence $r(\Delta(b)) = 0$ for all $b \in rad(A)$. Then we see that

$$r(a\Delta(b)) = r(\Delta(ab) - \Delta(a)\phi(b))$$
$$= r(\sigma(\Delta(ab) - \Delta(a)\phi(b)))$$
$$= r(\sigma(\Delta(ab)) - \sigma(\Delta(a)\phi(b)))$$
$$= r(\sigma(\Delta(ab)))$$
$$= r(\Delta(ab)) = 0$$

for all $b \in rad(A)$ and all $a \in A$. Hence $\Delta(rad(A)) \subseteq rad(A)$ as before. The remainder follows the same fashion as in the proof of Theorem 3.1. Hence $\Delta$ leaves each primitive ideal of $A$ invariant. We complete the proof. □

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