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SOME CONSEQUENCES OF THE EQUATION $[x^n, y] = 1$ ON THE STRUCTURE OF A COMPACT GROUP

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Abstract. Given an integer $n \geq 1$ and a compact group G, we find some restrictions for the probability that two randomly picked elements x^n and y of G commute. In the case $n = 1$ this notion was investigated by W. H. Gustafson in 1973 and its influence on the structure of the group has been studied in the researches of several authors in last years.

1. Generalizations of the commutativity degree

P. Erdős and P. Turán introduced in [6] the ratio

$$d(G) = \frac{|\{(x,y) \in G \times G \ : \ [x,y] = 1\}|}{|G|^2}, \tag{1.1}$$

known as the *probability that two randomly chosen elements of a finite group G commute*. It is called the *commutativity degree* of G (see for instance [5, 7, 8, 10, 16]). Several authors have successively investigated whether restrictions on (1.1) originate restrictions on the group structure. The answer is positive and there are many results of classification in the quoted papers in the finite case but few in the infinite case. The short note of W. H. Gustafson [11] shows that it is meaningful to consider the infinite case, when we move in the category of compact groups. However the situation becomes different, since we need of tools which are not common in Group Theory but in Abstract Harmonic Analysis and Topology. There are several works deal with word equations over compact groups which show the influence that a word equation may have on the structure of a group (see [1, 17, 18, 23]). Moreover, [2, 15] are recent advances on the topic, where character theory involved.

From this moment, we will always assume (up to explicit alert) to deal with compact groups satisfying the Hausdorff separability axiom, since this is usual in the context of topological groups.

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The Haar measure ν on a compact group G is a well–known notion and can be found in [11, Section 2] or [13, Chapters 1 and 2]. Then we may follow the idea of W. H. Gustafson, considering $C_2 = \{(x, y) \in G \times G \ : \ [x, y] = 1\}$ and noting that $C_2 = f^{-1}(\{1\})$, where f is the continuous function $(x, y) \in G \times G \mapsto f(x, y) = [x, y] \in G$. This allows us to define

$$d(G) = (\nu \times \nu)(C_2). \tag{1.2}$$

Clearly, if G is finite, then G is a compact group with the discrete topology and so the Haar measure on G is the counting measure. Therefore $d(G) = \frac{|C_2|}{|G|^2}$, which gives exactly (1.1). Further details on (1.2) can be found in [7, 8, 11].

Actually we may ask if two compact groups having the same $d(G)$ can be classified. Recent contributions can be found in [3, 5, 7, 8, 16] and here we will adapt some methods and techniques, where this is possible.

Given an integer $n \geq 1$, we define

$$P_n(G) = (\nu \times \nu)(S_G), \tag{1.3}$$

where

$$S_G = \{(x, y) \in G \times G \ : \ [x^n, y] = 1\}. \tag{1.4}$$

The function $\varphi : (x, y) \in G \times G \mapsto \varphi(x, y) = [x^n, y] \in G$ is continuous and so $S_G = \varphi^{-1}(\{1\})$ is closed and thus a Borél set which is $(\nu \times \nu)$–measurable (see [19]). It turns out that (1.3) expresses the *probability that a randomly chosen pair of element* $(x, y) \in G \times G$ *has the property that* x^n *commute with* y. In particular, $P_1(G) = d(G)$ and this allows us to generalize the results which are already known for (1.1).

2. Structural restrictions and Bell groups

This section is devoted to prove some basic properties of $P_n(G)$. Successively we will give lower and upper bounds for $P_n(G)$ when G is a certain compact group. We will generalized the known result of P. Lescot in [16]. Actually we show $P_n(G) = P_n(H)$ for two isoclinic groups G and H.

The probability $P_n(G)$ helps us to look at some results in [4] from a new point of view. Recall from [4] the following notion.

Definition 2.1. An element x of an abstract group G is called n–Bell, if $[x^n, y] = [x, y^n]$ for all $y \in G$. G is called *n–Bell* if all its elements are n–Bell.

Note that Definition 2.1 can be given for an arbitrary group G without assumptions on its topology. Immediately we can compare (1.1), (1.3) and Definition 2.1 thanks to the following result.

Corollary 2.2. *For a compact group G of exponent n the following conditions are equivalent*:

(i) *G is n–Bell*;
(ii) $P_n(G) = 1$.

Proof. We claim that (i) implies (ii). [4, Lemma 2] shows that an n–Bell group B has the subgroup $B^n = \langle b^n \mid b \in B \rangle$ which is contained in $C_B(a)$ for all $a \in B$. In particular, this is true for G and so $G^n \subseteq C_G(y)$ for each $y \in G$. Hence

$$G^n \subseteq \bigcap_{y \in G} C_G(y) = Z(G)$$

and all the elements of the form x^n for $x \in G$ are central. We conclude that $P_n(G) = 1$.

Conversely assume that (ii) is true. $P_n(G) = 1$ implies $[x^n, y] = [x, y^n] = 1$ for all $x, y \in G$, then G is n–Bell. □

Remark 2.3. Corollary 2.2 gives information of how G is far from being n–Bell. In particular, (1.3) allows us to have numerical information on this fact.

In the category `Comp` of all compact groups with corresponding morphisms we may consider (1.3) as the following map

$$P_n : \ G \in \texttt{Comp} \longmapsto P_n(G) \in [0, 1].$$

Some classic measure theoretical properties follow. For instance, P_n is multiplicative, that is, its value in a direct product is equal to the product of its values in each direct factor, and *monotone* with respect to suitable subgroups and quotients. These two properties were already noted for $n = 1$ in [11, (i), p. 1033] and in [5, 7, 8, 10, 16]. We will be more general in the successive two statements.

Proposition 2.4. *Let $m \geq 1$ and $G_1, G_2, \ldots, G_m$ be compact groups. Then*

$$P_n(G_1 \times G_2 \times \cdots \times G_m) = P_n(G_1) \cdot P_n(G_2) \cdots P_n(G_m).$$

Proof. It is a well known property of splitting of the probability of distinct events. □

Proposition 2.5. *Let G be a compact group and H a closed subgroup of G. Then*

$$\mu_G(H)^2 P_n(H) \leq P_n(G),$$

where μ_G is the corresponding Haar measure of G. Furthermore, if G is an n–Bell group, then

$$P_n(G) \leq P_n(H).$$

Proof. Assume that μ_H is the Haar measure on H induced by μ_G. It is clear that $\mu_G(H)^2 P_n(H) \leq P_n(G)$, in the case that the index $|G : H| = \infty$. Now, we may assume that $|G : H|$ is finite. Then G is an open subgroup of G and by [7, Lemma 2.1] we have

$$\begin{aligned} P_n(H) &= (\mu_H \times \mu_H)(S_H) \\ &= \int_H \mu_H(C_H(x^n)) d\mu_H(x) \end{aligned}$$

$$\begin{aligned}&\leq |G:H|^2 \int_G \mu_G(C_G(x^n))d\mu_G(x)\\&= |G:H|^2 P_n(G).\end{aligned}$$

Now, let G be an n–Bell group. By [7, Lemma 3.2] and Lebesgue-Fubini's Theorem we have,

$$\begin{aligned}P_n(H) &= \int_H \mu_H(C_H(h^n))d\mu_H(h)\\&\geq \int_H \mu_G(C_G(h^n))d\mu_H(h)\\&= \int_G \mu_H(C_H(g^n))d\mu_G(g)\\&\geq \int_G \mu_G(C_G(g^n))d\mu_G(g) = P_n(G).\end{aligned}$$

□

For a locally compact group G, we denote the space of complex-valued function on G integrable with respect to the Haar measure by $\mathcal{L}^1(G)$. Assume that H is a closed normal subgroup of locally compact group G and $f \in \mathcal{L}^1(G)$. If λ, μ and ν are Haar measures on H, G and G/H respectively, then

$$\int_{\frac{G}{H}} \left(\int_H f(xh)d\lambda(x)\right) d\nu(xH) = \int_G f(x)d\mu(x).$$

This is the so-called *extended Weil formula*, whose proof can be found in [19, Theorem 4.5].

Proposition 2.6. *If H is a closed normal subgroup of a compact group G, then*

$$P_n(G) \leq P_n(G/H).$$

Proof. Assume that λ, μ and ν are corresponding Haar measures on H, G and G/H respectively. Consider FC-center of G, i.e.,

$$FC(G) = \{x \in G \mid |G : C_G(x^n)| \text{ is finite}\}$$

(see [22, Vol. I, Chapter 4]). Thus we have

$$\begin{aligned}P_n(G) &= (\mu \times \mu)(S_G)\\&= \int_G \mu(C_G(x^n))d\mu(x)\\&= \int_{FC(G)} \mu(C_G(x^n))d\mu(x)\\&= \int_{FC(G)} \frac{\mu(C_G(x^n)H)}{|C_G(x^n)H : C_G(x^n)|}d\mu(x)\\&= \int_{FC(G)} \mu(C_G(x^n)H)\lambda(C_H(x^n))d\mu(x)\end{aligned}$$

$$\leq \int_{G} \mu(C_G(x^n)H)\lambda(C_H(x^n))d\mu(x)$$
$$= \int_{\frac{G}{H}} \left(\int_{H} \mu(C_G((xh)^n)H)\lambda(C_H((xh)^n))d\lambda(h) \right) d\nu(xH).$$

By [7, Lemma 3.5] and $\mu(C_G((xh)^n)H) = \nu(C_G((xh)^n)H/H) \leq \nu(C_{G/H}(x^n H))$, we have

$$P_n(G) \leq \int_{\frac{G}{H}} \left(\int_{H} \nu(C_{\frac{G}{H}}(x^n H))\lambda(C_H((xh)^n))d\lambda(h) \right) d\nu(xH)$$
$$= \int_{\frac{G}{H}} \nu(C_{\frac{G}{H}}(x^n H)) \left(\int_{H} \lambda(C_H((xh)^n))d\lambda(h) \right) d\nu(xH).$$
$$\leq \int_{\frac{G}{H}} \nu(C_{\frac{G}{H}}(x^n H))d\nu(xH) = P_n(\frac{G}{H}).$$

□

In order to prove the main results of this section we need of the following lemmas.

Lemma 2.7. *Let G be a compact group with the normalized Haar measure ν, closed subgroup H and $m \geq 1$. If $|G : H| \geq m$, then $\nu(H) \leq \frac{1}{m}$. If $|G : H| \leq m$, then $\nu(H) \geq \frac{1}{m}$. In particular, $|G : H| = m$ if and only if $\nu(H) = \frac{1}{m}$.*

Proof. This is an easy fact whose proof can be found in [7, Lemma 3.1]. □

The following lemma presents the lower and upper bounds for $P_n(G)$ when G is a non-abelian compact group with certain properties. These are the same bounds as we found in [9, Lemma 3.1] when G is finite.

Lemma 2.8. *Let G be a non-abelian compact group, p be a prime and $G/Z(G)$ be a p–elementary abelian group of rank $s \geq 1$. Then $P_n(G) = 1$, if p divides n. Otherwise,*

$$\frac{p^s + p^{s-1} - 1}{p^{2s-1}} \leq P_n(G) \leq \frac{p^s + p - 1}{p^{s+1}}.$$

Proof. Of course $Z(G)$ is a closed subgroup of G so that $G/Z(G)$ is a compact group. Since $G/Z(G) = \{Z(G), a_1 Z(G), \ldots, a_{p^s-1}Z(G)\}$ is a p–elementary abelian group, $(xZ(G))^p = Z(G)$ and therefore $x^p \in Z(G)$ for each $x \notin Z(G)$. $Z_2(G)/Z(G) = Z\big(G/Z(G)\big) = G/Z(G)$ and so G is nilpotent of class 2. First, suppose that p divides n. There exists an integer $t \geq 1$ such that $n = pt$, then for every $(x, y) \in G \times G$ we have

$$[x^n, y] = [x^{pt}, y] = [x^p, y]^{(x^p)^{t-1}}[(x^p)^{t-1}, y] = \cdots = [x^p, y] = 1.$$

Hence $P_n(G) = 1$. Now assume p does not divide n. Then we may write $n = pt + r$ for some $0 < r < p$. Obviously $x^n \notin Z(G)$, whenever $x \notin Z(G)$, because otherwise $x^n Z(G) = x^{pt+r}Z(G) = x^r Z(G) = Z(G)$ and this is a contradiction. On the other hand, we have

$$p^s = |G : Z(G)| = |G : C_G(x^n)||C_G(x^n) : Z(G)| \geq |G : C_G(x^n)|p.$$

From this and Lemma 2.7 we have $\nu(Z(G)) = \frac{1}{p^s}$ and $\frac{1}{p^{s-1}} \leq \nu(C_G(x^n))$. Note that $C_G(x^n)$ is a closed subgroup of G. Separately

$$(\nu \times \nu)(S_G) = \int_G \nu(\{y \in G \ : \ [y, x^n] = 1\}) d\nu(x) = \int_G \nu(C_G(x^n)) d\nu(x).$$

Then we have

$$\begin{aligned}
P_n(G) &= (\nu \times \nu)(S_G) \\
&= \int_G \nu(C_G(x^n)) d\nu(x) \\
&= \int_{Z(G)} \nu(C_G(x^n)) d\nu(x) + \int_{G-Z(G)} \nu(C_G(x^n)) d\nu(x) \\
&= \nu(Z(G)) + \int_{G-Z(G)} \nu(C_G(x^n)) d\nu(x) \\
&\geq \nu(Z(G)) + \frac{1}{p^{s-1}}(1 - \nu(Z(G))) \\
&= \frac{1}{p^s} + \frac{1}{p^{s-1}} - \frac{1}{p^{s-1}} \nu(Z(G)) \\
&= \frac{p^s + p^{s-1} - 1}{p^{2s-1}}.
\end{aligned}$$

The first inequality follows.

We note from Lemma 2.7 that

$$\nu(C_G(x^n)) \leq \frac{1}{p} \ \text{ for all } x \notin Z(G).$$

Then

$$\begin{aligned}
P_n(G) &= \int_{Z(G)} \nu(C_G(x^n)) d\nu(x) + \sum_{i=1}^{p^s-1} \Big(\int_{a_i Z(G)} \nu(C_G(x^n)) d\nu(x) \Big) \\
&= \nu(Z(G)) + \sum_{i=1}^{p^s-1} \Big(\int_{a_i Z(G)} \nu(C_G(x^n)) d\nu(x) \Big) \\
&\leq \nu(Z(G)) + \frac{1}{p} \sum_{i=1}^{p^s-1} \nu(a_i Z(G)) \\
&= \frac{1}{p^s} + \frac{1}{p}(p^s - 1) \frac{1}{p^s} \\
&= \frac{p^s + p - 1}{p^{s+1}}.
\end{aligned}$$

So the result follows. □

Remark 2.9. If $s = 2$ in Lemma 2.8 and p does not divide n, then the lower and upper bound coincides. This means that $P_n(G) = \frac{p^2+p-1}{p^3}$. Furthermore, if $p = 2$, i.e., $G/Z(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, then

$$P_n(G) = \begin{cases} 5/8, & \text{if} \quad n \text{ odd} \\ 1, & \text{if} \quad n \text{ even.} \end{cases}$$

As already noted, we may have further concrete example, considering direct products of the form $D = G \times \mathbb{T}$, in which $\mathbb{T}$ is the group of unit circle. Proposition 2.4 implies $P_n(D) = P_n(G) \cdot P_n(\mathbb{T}) = P_n(G)$.

Example 2.10. Consider the finite metacyclic group

$$G = \langle a, b | a^9 = b^3, bab^{-1} = a^4 \rangle.$$

We can check that $G/Z(G) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ and $P_n(G) = 11/27$ for all n not divisible by 3. It is also easy to check that $P_5(G) = P_{10}(G) = 11/27$ and $P_6(G) = P_9(G) = 1$. Note that GAP allows us to calculations of this kind. Thanks to Proposition 2.4, an infinite example can be easily done by considering $D = G \times \mathbb{T}$ and we get $P_n(D) = P_n(G)$.

Now we recall the notion of isoclinism between two groups, which was introduced by P. Hall in [12].

Definition 2.11. For two groups G and H; a pair (φ, ψ) is called an *isoclinism* from G to H if
(i) φ is an isomorphism from $G/Z(G)$ to $H/Z(H)$;
(ii) ψ is an isomorphism from G' to H';
(iii) the following diagram is commutative:

$$\begin{array}{ccc} \dfrac{G}{Z(G)} \times \dfrac{G}{Z(G)} & \overset{\varphi \times \varphi}{\rightarrow} & \dfrac{H}{Z(H)} \times \dfrac{H}{Z(H)} \\ \downarrow & & \downarrow \\ G' & \overset{\psi}{\rightarrow} & H' \end{array}$$

where, $a_G(g_1 Z(G), g_2 Z(G)) = [g_1, g_2]$ and $a_H(h_1 Z(H), h_2 Z(H)) = [h_1, h_2]$.

In Lemma 2.8 the lower bound is achieved if $\nu(C_G(x^n)) = \frac{1}{p^{s-1}}$ for all $x \notin Z(G)$. The upper bound is achieved if G is isoclinic to an extra special p–group of order p^{s+1}.

In the following theorem we are going to generalize a corresponding situation, known for commutativity degree (see [7, Theorem 3.8] and [20, 21]).

Theorem 2.12. *Let G and H be two isoclinic compact groups. Then $P_n(G) = P_n(H)$.*

Proof. Let (φ, ψ) be an isoclinism from G to H and μ_X, λ_X and ν_X are Haar measures on X, $Z(X)$ and $X/Z(X)$ respectively, where $X \in \{G, H\}$. Since $Z(G)$ is a closed subgroup of compact group G, then extended Weil formula implies

$$P_n(G) = (\mu_G \times \mu_G)(S_G)$$

$$
\begin{aligned}
&= \int_G \int_G \chi_{S_G}(x^n, y) d\mu_G(x) d\mu_G(y) \\
&= \int_G \left(\int_{\frac{G}{Z(G)}} \int_{Z(G)} \chi_{S_G}((xz)^n, y) d\lambda_G(z) d\nu_G(\bar{x}) \right) d\mu_G(y) \\
&= \int_G \left(\int_{\frac{G}{Z(G)}} \int_{Z(G)} \chi_{S_G}(x^n z^n, y) d\lambda_G(z) d\nu_G(\bar{x}) \right) d\mu_G(y) \\
&= \int_G \left(\int_{\frac{G}{Z(G)}} \int_{Z(G)} \chi_{S_G}(x^n, y) d\lambda_G(z) d\nu_G(\bar{x}) \right) d\mu_G(y) \\
&= \int_G \left(\int_{\frac{G}{Z(G)}} \chi_{S_G}(x^n, y) \int_{Z(G)} d\lambda_G(z) d\nu_G(\bar{x}) \right) d\mu_G(y) \\
&= \int_G \int_{\frac{G}{Z(G)}} \chi_{S_G}(x^n, y) d\nu_G(\bar{x}) d\mu_G(y).
\end{aligned}
$$

By Lebesgue-Fubini's Theorem we have

$$
\begin{aligned}
P_n(G) &= \int_{\frac{G}{Z(G)}} \int_G \chi_{S_G}(x^n, y) d\mu_G(y) d\nu_G(\bar{x}) \\
&= \int_{\frac{G}{Z(G)}} \left(\int_{\frac{G}{Z(G)}} \int_{Z(G)} \chi_{S_G}(x^n, yz) d\lambda_G(z) d\nu_G(\bar{y}) \right) d\nu_G(\bar{x}) \\
&= \int_{\frac{G}{Z(G)}} \int_{\frac{G}{Z(G)}} \chi_{S_G}(x^n, y) d\nu_G(\bar{x}) d\nu_G(\bar{y}) \\
&= \int_{\frac{G}{Z(G)}} \int_{\frac{G}{Z(G)}} \chi_{S_G}(a_G(\bar{x}^n, \bar{y})) d\nu_G(\bar{x}) d\nu_G(\bar{y}) \\
&= \int_{\frac{G}{Z(G)}} \int_{\frac{G}{Z(G)}} \chi_{S_G}(\psi(a_G(\bar{x}^n, \bar{y}))) d\nu_G(\bar{x}) d\nu_G(\bar{y}) \\
&= \int_{\frac{G}{Z(G)}} \int_{\frac{G}{Z(G)}} \chi_{S_G}(a_H(\varphi(\bar{x}^n), \varphi(\bar{y}))) d\nu_G(\bar{x}) d\nu_G(\bar{y}).
\end{aligned}
$$

Since φ is an isomorphism of compact groups, [7, Corollary 2.5] allows us to write

$$
\begin{aligned}
P_n(G) &= \int_{\frac{H}{Z(H)}} \int_{\frac{H}{Z(H)}} \chi_{S_H}(a_H(\bar{a}^n, \bar{b})) d\nu_H(\bar{a}) d\nu_H(\bar{b}) \\
&= \int_{\frac{H}{Z(H)}} \int_{\frac{H}{Z(H)}} \chi_{S_H}(a^n, b) d\nu_H(\bar{a}) d\nu_H(\bar{b}) \\
&= \int_H \int_H \chi_{S_H}(a^n, b) d\mu_H(a) d\mu_H(b) \\
&= (\mu_H \times \mu_H)(S_H) = P_n(H)
\end{aligned}
$$

and the proof of the theorem is completed. □

Theorem 2.13. *Let G be a non-abelian compact group and $G/Z(G)$ be a p-group, where p is a prime. Then G is isoclinic to an extra special p-group of order p^3, if and only if $P_n(G) = (p^2 + p - 1)/p^3$, whenever p does not divide n.*

Proof. Assume that G is isoclinic to an extra special p-group of order p^3. Then $G/Z(G)$ is a p–elementary abelian group of rank 2 and the result follows by Theorem 2.12 and Remark 2.9.

Conversely assume that $P_n(G) = (p^2 + p - 1)/p^3$. If x is a non-central element, then $\nu(C_G(x^n)) \leq 1/p$ by Lemma 2.7. We may use an argument as in the proof of Lemma 2.8. In fact we have

$$\frac{p^2+p-1}{p^3} = P_n(G) = \int_{Z(G)} \nu(C_G(x^n))d\nu(x) + \int_{G-Z(G)} \nu(C_G(x^n))d\nu(x)$$
$$\leq \frac{(p-1)}{p}\nu(Z(G)) + \frac{1}{p},$$

and so

$$\frac{1}{p^2} \leq \nu(Z(G)).$$

Again by Lemma 2.7 we conclude $|G : Z(G)| \leq p^2$. If $|G : Z(G)| \in \{1, p\}$, then G is abelian, which is a contradiction. Hence $|G : Z(G)| = p^2$ and so $G/Z(G)$ is an elementary abelian group of rank 2. Assume that $G/Z(G) = \langle\overline{x}\rangle \times \langle\overline{y}\rangle$ where $x, y \in G$ and H is an extra-special p-group of order p^3. Then $H/Z(H) \cong \mathbb{Z}_p \times \mathbb{Z}_p = \langle\overline{a}\rangle \times \langle\overline{b}\rangle$ where $a, b \in H$ and $H' = Z(H) \cong \mathbb{Z}_p$.

Define $\varphi : G/Z(G) \to H/Z(H)$ by the rule $\varphi(\overline{x}) = \overline{a}$ and $\varphi(\overline{y}) = \overline{b}$, which is an isomorphism. It is known that if $|G : Z(G)| = p^n$, then G' is a p-group and $|G'| \leq p^{\frac{n(n-1)}{2}}$ (see [24]). By this fact, we have $|G'| = \mathbb{Z}_p$. One can check that $G' = \langle[x, y]\rangle$ and $H' = \langle[a, b]\rangle$, therefore $\psi : G' \to H'$ by the rule $\psi([x, y]) = [a, b]$ is an isomorphism. It is now enough to check the commutativity of diagram of Definition 2.11. Let $\overline{x}^t\overline{y}^s, \overline{x}^u\overline{y}^v \in G/Z(G)$. Then

$$a_H\varphi^2(\overline{x}^t\overline{y}^s, \overline{x}^u\overline{y}^v) = a_H(\overline{a}^t\overline{b}^s, \overline{a}^u\overline{b}^v) = [a^tb^s, a^ub^v] = [a, b]^{tv-su}$$
$$= \psi([x, y]^{tv-su}) = \psi([x^ty^s, x^uy^v]) = \psi a_G(\overline{x}^t\overline{y}^s, \overline{x}^u\overline{y}^v).$$

Therefore G and H are isoclinic by Definition 2.11 and the proof of the theorem is completed. □

Here are some examples which confirm the above results and one example which shows the limits of (1.3) in the compact case.

Example 2.14. Consider the finite group

$$G = \langle a, b, c : a^3 = b^3 = c^3 = 1, bac = ab, ca = ac, cb = bc\rangle.$$

One can easily see that $|G| = 27$, $|G'| = |Z(G)| = 3$ and $G/Z(G)$ is a 3–elementary abelian of rank 2. G is an extra special 3–group of order 27. Now, by using GAP, we can compute $P_n(G)$ for some values of n. For instance, if

$n = 1, 2, 4, 5, 7, 8, 10$, then $P_n(G) = 11/27$ and for $n = 3, 6, 9$ we have $P_n(G) = 1$ which is an evidence for Theorem 2.13. For the infinite case, we may consider $B = G \times A$, where A is the direct product of finitely many copies of $\mathbb{T}$. From Proposition 2.4, $P_n(B) = P_n(G)$.

Example 2.15. Let L be a nontrivial compact Lie group and let $\mathbb{Z}$ act automorphically on $P = L^{\mathbb{Z}}$ by the shift. Set $G = P \rtimes \mathbb{Z}$. Then $N = P \times \{1\}$ is a compact normal subgroup, and G/N is discrete, and thus a Lie group. G is not compact but it is locally compact. If P is abelian, then G is metabelian. If L is abelian, then G is pro–solvable. See [13, 14] for details. G is an example of a group which is locally compact, center–free and pro-solvable but it is neither compact nor pro-Lie nor totally disconnected. On the other hand, G has a unique maximal compact subgroup $C(G) = P$ and we can consider $P_n(P)$ but not $P_n(G)$. This fact shows that our techniques can be further extended to wider contexts.

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References

[1] M. Abert, *On the probability of satisfying a word in a group*, Cornell University Library, preprint, 2005, available online at http://arxiv.org/abs/math/0504312.

[2] A. M. Alghamdi and F. G. Russo, *A generalization of the probability that the commutator of two group elements is equal to a given element*, Bull. Iranian Math. Soc., in press.

[3] N. M. Ali and N. Sarmin, *On some problems in group theory of probabilistic nature*, Technical Report, Universiti Teknologi Malaysia, Johor Bahru, Malaysia, 2009.

[4] R. Brandl and L.-C. Kappe, *On n–Bell groups*, Comm. Algebra **17** (1989), no. 4, 787–807.

[5] K. Chiti, M. R. R. Moghaddam, and A. R. Salemkar, *n–isoclinism classes and n–nilpotency degree of finite groups*, Algebra Colloq. **12** (2005), no. 2, 255–261.

[6] P. Erdős and P. Turán, *On some problems of a statistical group-theory. IV*, Acta Math. Acad. Sci. Hungar **19** (1968), 413–435.

[7] A. Erfanian and R. Rezaei, *On the commutativity degree of compact groups*, Arch. Math. (Basel) **93** (2009), no. 4, 201–212.

[8] A. Erfanian and F. G. Russo, *Probability of mutually commuting n–tuples in some classes of compact groups*, Bull. Iranian Math. Soc. **34** (2008), no. 2, 27–37.

[9] A. Erfanian, B. Tolue, and N. Sarmin, *Some consideration on the n–th commutativity degrees of finite groups*, Ars. Comb. (2010), in press.

[10] P. X. Gallagher, *The number of conjugacy classes in a finite group*, Math. Z. **118** (1970), 175–179.

[11] W. H. Gustafson, *What is the probability that two groups elements commute?*, Amer. Math. Monthly **80** (1973), 1031–1034.

[12] P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130–141.

[13] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups*, de Gruyter, Berlin, 2006.

[14] K. H. Hofmann, S. Morris, and M. Stroppel, *Locally compact groups, residual Lie groups, and varieties generated by Lie groups*, Topology Appl. **71** (1996), no. 1, 63–91.

[15] K. H. Hofmann and F. G. Russo, *The probability that x and y commute in a compact group*, Cornelly University Library, 2010, available online at http://arxiv.org/abs/1001.4856.
[16] P. Lescot, *Isoclinism classes and commutativity degrees of finite groups*, J. Algebra **177** (1995), no. 3, 847–869.
[17] M. Levy, *On the probability of satisfying a word in nilpotent groups of class 2*, Cornell University Library, preprint, 2011, available online at http://arxiv.org/abs/1101.4286.
[18] Y. Medvedev, *On compact Engel groups*, Israel J. Math. **135** (2003), 147–156.
[19] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford, Clarendon Press, 1968.
[20] R. Rezaei and F. G. Russo, *Bounds for the relative n-th nilpotency degree in compact groups*, Asian-Eur. J. Math. **4** (2011), no. 3, 495–506.
[21] ———, *n-th relative nilpotency degree and relative n-isoclinism classes*, Carpathian J. Math. **27** (2011), no. 1, 123–130.
[22] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Springer-Verlag, Berlin, 1972.
[23] D. Segal, *Words: Notes on verbal width in groups*, LMS Lecture Notes Serie **361**, Cambridge University Press, Cambridge, 2009.
[24] J. Wiegold, *Multiplicators and groups with finite central factor-groups*, Math. Z. **89** (1965), 345–347.

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