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System reliability estimation in multicomponent exponential stress-strength models

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Abstract. A stress-strength model is formulated for a multi-component system consisting of k identical components. The k components of the system with random strengths (X_1, X_2, …, X_k) are subjected to one of the r random stresses (X_{k+1}, X_{k+2}, …, X_{k+r}). The estimation of system reliability based on maximum likelihood estimates (MLEs) and Bayes estimators in k component system are obtained when the system is either parallel or series with the assumption that strengths and stresses follow exponential distribution. A simulation study is conducted to compare MLE and Bayes estimator through the mean squared errors of the estimators.

Key Words: *Exponential Distribution, maximum likelihood estimators, Bayes estimator, stress-strength model, multicomponent system, system reliability.*

1. INTRODUCTION

The problem of estimating reliability P[X>Y] in a stress-strength model has been discussed in the literature extensively when X and Y have some specified distribution. Enis and Geiser(1971), Tong (1974), Kelley, Kelley and Schucany (1976) have considered this problem when X and Y follow independent exponential distributions. Beg and Singh (1979) considered this problem when X and Y follow two parameter exponential distribution. Further, the problem of estimation of $P[X_3>Max ((X_1, X_2)]$ has been considered by Hanagal (1996a) when (X_1, X_2) follow bivariate exponential models and X_3 follow independent exponential distribution. Hanagal (1996b) considered the estimation of $P[X_k<Min(X_1, X_2, …, X_{k-1})]$ when (X_1, X_2, …, X_{k-1}) follow multivariate Pareto distribution. Hanagal (1998) considered the problem of estimating $P[X_{k+1}<Min(X_1, X_2, …, X_k)]$ and $P[X_{k+1}<Max(X_1, X_2, …, X_k)]$ when X_1, X_2, …, X_k are strengths subjected to a common stress X_{k+1}, assuming that X_1, X_2, …, X_{k+1} follow independent two parameter exponential distributions. Hanagal (2003) obtained maximum likelihood estimators for $P[X_{k+1}<Min(X_1, X_2, …, X_k)]$ and $P[X_{k+1}<Max(X_1, X_2, …, X_k)]$ when X_1, X_2, …, X_k are

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strengths subjected to a common stress X_{k+1}, assuming that X_1, X_2, …, X_{k+1} follow independent Pareto or Weibull or Gamma distribution.

Recently few of the papers on Estimation of Reliability in multi-component stress-strength systems are found in the literature. For example, Rao and Kantam (2010) discusses the estimation of reliability when the underlined distribution is Log-Logistic; Rao (2012a, b) discusses the similar problem taking stress and strength distribution as Rayleigh and Generalized Exponential. Further, Gogoi and Borah (2012) considered the estimation of reliability in multi-component standby system.

One can observe that the researchers in this area considered the problem of estimation of reliability for stress-strength models, taking many strengths and maximum of two stresses. However, there are situations where one can think of multiple independent stresses. For example, consider a system of identical Generator sets synchronizing with and running in parallel with the supply utility grid. Each generator with independent strengths such as the voltage magnitude, the frequency of the voltages and the phase angle between the voltages with independent stresses such as power failures, short circuit currents, Earth fault currents and voltage and frequency transients can be considered as a multi-component system with multiple stresses. Again in an optical communication system an optical fiber with independent strengths namely, enormous potential bandwidth, small size and weight, electrical isolation, immunity to interference and cross talk, signal security, low transmission loss, potential low cost etc. are operating on independent stresses such as material absorption, linear scattering losses, filter band losses, dispersion and polarization.

The present paper considers the estimation of reliability of Exponential Stress-Strength model with multi-component system with more than two stresses. Hence, this paper presents a general methodology that can be applied to the situations with many stresses which includes the earlier works as its particular cases.

In this paper we consider the estimation of R_p= P[Max(X_{k+1}, X_{k+2}, …, X_{k+r})< Max (X_1, X_2, …, X_k)] and R_s= P[Max(X_{k+1}, X_{k+2}, …, X_{k+r})<Min (X_1, X_2, …, X_k)] when X_1,X_2,…,X_k are strengths subjected to one of the stresses X_{k+1}, X_{k+2}, …, X_{k+r}, assuming that X_1, X_2, …, X_{k+r} follow independent exponential distributions.

In section 2, we derive the expression for system reliability of series and parallel systems for an exponential stress-strength model. The MLEs for the parameters and reliability functions with their asymptotic distributions are derived in section 3. In Section 4, the Bayes estimators are derived. Section 5 deals with evaluating the performance of the MLEs and Bayes estimators of reliability functions by estimating the mean squared errors (MSEs) through simulations. Some remarks and conclusions are given in section 6.

2. SYSTEM RELIABILITY

Consider a multi-component system with k identical components. Here, we assume that strengths of k components are subjected to one of the r stresses. Let X_1, X_2, …, X_k be strengths having exponential distribution with parameter θ_1, subjected to one of the stresses X_{k+1}, X_{k+2}, …, X_{k+r}, that follow exponential distribution with parameter θ_2 .

The p.d.f. of X_i is given by

$$f_i(x) = \theta_1 e^{-\theta_1 x} \quad , \; x > 0 \; , \; \theta_1 > 0 \; , \; i = 1,2,...,k$$

and

$$f_i(x) = \theta_2 e^{-\theta_2 x} \quad , \; x > 0 \; , \; \theta_2 > 0 \; , \; i = k+1, k+2,..., k+r$$

Then the distribution function of $U = Max(X_1, X_2,..., X_k)$ is given by

$$G_1(u) = P[U < u] = \left[1 - e^{-\theta_1 u}\right]^k$$

and the distribution function of $V = Max(X_{k+1}, X_{k+2},..., X_{k+r})$ is given by

$$G_2(v) = P[V < v] = \left[1 - e^{-\theta_2 v}\right]^r$$

Now in parallel system, the system reliability is

$$\begin{aligned} R_p &= P[V < U] \\ &= \int_0^\infty \overline{G}_1(v) dG_2(v) \\ &= k\theta_1 \int_0^\infty (1 - e^{-\theta_1 v})^{k-1} (1 - e^{-\theta_2 v})^r e^{-\theta_1 v} dv \\ &= k\theta_1 \sum_{l=0}^{k-1} \sum_{m=0}^{r} \left[\frac{\binom{k-1}{l}\binom{r}{m}(-1)^{l+m}}{\left((l+1)\theta_1 + m\theta_2\right)} \right]. \end{aligned}$$

On the similar lines, the distribution function of $W = Min(X_1, X_2,..., X_K)$ is given by

$$G_3(w) = 1 - e^{-\theta_1 wk}$$

Then the system reliability for a series system is obtained as

$$\begin{aligned} R_s &= P\left[V < W\right] \\ &= r\theta_2 \int_0^\infty e^{-(k\theta_1 + \theta_2) w} \left(1 - e^{-\theta_2 w}\right)^{r-1} dw \\ &= r\theta_2 \sum_{m=0}^{r-1} \left[\frac{\binom{r-1}{m}(-1)^m}{\left((m+1)\theta_2 + k\theta_1\right)} \right] \end{aligned}$$

As the reliability function of both series and parallel systems involve θ_1 and θ_2, first we consider the estimation of θ_1 and θ_2, using method of maximum likelihood and then the system reliability estimates are obtained in the next section.

3. MAXIMUM LIKELIHOOD ESTIMATORS FOR PARAMETERS θ_1, θ_2 AND RELIABILITY

Consider a k-component system, in which components are subjected to r stresses. Let $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$ (i =1,2,…,n) be a random sample of strengths of n systems, that are exponentially distributed with parameter θ_1 and $X_{i,k+1}, X_{i,k+2}, \ldots, X_{i,k+r}$ (i =1,2,…,n) be a random sample of stresses corresponding to n systems, that are exponentially distributed with parameter θ_2.

The MLEs of R_p and R_s based on $\underline{\theta} = (\theta_1, \theta_2)$ are given by

$$\hat{R}_p = R_p(\hat{\underline{\theta}}) \ , \ \hat{R}_s = R_s(\hat{\underline{\theta}})$$

where $\hat{\underline{\theta}} = (\hat{\theta}_1, \hat{\theta}_2)$.

The MLEs of θ_1 and θ_2 are obtained as

$$\hat{\theta}_1 = \frac{nk}{\sum_{i=1}^{n}\sum_{j=1}^{k} x_{i,j}} \quad , \quad \hat{\theta}_2 = \frac{nr}{\sum_{i=1}^{n}\sum_{j=k+1}^{k+r} x_{i,j}}$$

The asymptotic variances of the MLEs of $\hat{\theta}_1$ and $\hat{\theta}_2$ are given by

$$V(\hat{\theta}_1) = \frac{\theta_1^2}{nk} \ , \ V(\hat{\theta}_2) = \frac{\theta_2^2}{nr}.$$

The MLEs $\hat{\underline{\theta}}$ are consistent asymptotically normal with mean $\underline{\theta}$ and variance-covariance matrix $diag\left(\frac{\theta_1^2}{nk}, \frac{\theta_2^2}{nr}\right)$. Since $\hat{R}_p$ and $\hat{R}_s$ are functions of $\hat{\underline{\theta}}$, asymptotic distribution of $\hat{R}_p$ and $\hat{R}_s$ are given as below.

The distribution of $\hat{R}_p$ is AN $\left(R_p, B_p' \Lambda B_p\right)$ and that of $\hat{R}_s$ is AN $\left(R_s, B_s' \Lambda B_s\right)$ where $B_p' = \left(\frac{\partial R_p}{\partial \theta_1}, \frac{\partial R_p}{\partial \theta_2}\right)$, $B_s' = \left(\frac{\partial R_s}{\partial \theta_1}, \frac{\partial R_s}{\partial \theta_2}\right)$ and $\Lambda = \frac{1}{n} diag\left(\frac{\theta_1^2}{k}, \frac{\theta_2^2}{r}\right)$.

Here

$$\frac{\partial R_p}{\partial \theta_1} = k \sum_{l=0}^{k-1}\sum_{m=0}^{r} \left(\frac{\binom{k-1}{l}\binom{r}{m}(-1)^{l+m}}{\left((l+1)\theta_1 + m\theta_2\right)} \right) - k(l+1)\theta_1 \sum_{l=0}^{k-1}\sum_{m=0}^{r} \left(\frac{\binom{k-1}{l}\binom{r}{m}(-1)^{l+m}}{\left((l+1)\theta_1 + m\theta_2\right)^2} \right),$$

$$\frac{\partial R_p}{\partial \theta_2} = -km\theta_1 \sum_{l=0}^{k-1}\sum_{m=0}^{r} \left(\frac{\binom{k-1}{l}\binom{r}{m}(-1)^{l+m}}{\left((l+1)\theta_1 + m\theta_2\right)^2} \right),$$

$$\frac{\partial R_s}{\partial \theta_1} = -rk\theta_2 \sum_{m=0}^{r-1} \left(\frac{\binom{r-1}{m}(-1)^{l+m}}{((m+1)\theta_2 + k\theta_1)^2} \right),$$

$$\frac{\partial R_s}{\partial \theta_2} = r\sum_{m=0}^{r-1} \left(\frac{\binom{r-1}{m}(-1)^{l+m}}{((m+1)\theta_2 + k\theta_1)} \right) - r\theta_2(m+1) \sum_{m=0}^{r-1} \left(\frac{\binom{r-1}{m}(-1)^{l+m}}{((m+1)\theta_2 + k\theta_1)^2} \right).$$

4. BAYES ESTIMATION OF RELIABILITY FUNCTION

In this section Bayes estimator of R_p and R_s are derived by considering the prior distribution of the parameters θ_1 and θ_2 as,

$$g(\theta_1) = \frac{1}{\Gamma(\alpha)} e^{-\theta_1} \theta_1^{\alpha-1} \quad \theta_1 > 0,\ \alpha \ge 0$$

$$g(\theta_2) = \frac{1}{\Gamma(\beta)} e^{-\theta_2} \theta_2^{\beta-1} \quad \theta_2 > 0,\ \beta \ge 0 \cdot$$

The Bayes estimator of Reliability function $\hat{R}_{pB}$ is obtained as the posterior expectation of $\hat{R}_p$ is given by

$$\hat{R}_{pB} = \int_0^\infty \int_0^\infty R_p \ f\left(\theta_1, \theta_2 \left| x_{i,j}, y_{i,j} \right.\right) d\theta_1 d\theta$$

where

$$f\left(\theta_1, \theta_2 \left| x_{i,j}, y_{i,j} \right.\right) = \frac{1}{\Gamma(n_1 k + \alpha)\Gamma(n_2 r + \beta)} A_1 \ A_2 \ A_3 \ \theta_1^{n_1 k + \alpha - 1} \ \theta_2^{n_2 r + \beta - 1}$$

$$A_1 = \left(\sum_{i=1}^{k} \sum_{j=1}^{n_1} x_{i,j} \right)^{n_1 k + \alpha}$$

$$A_2 = \left(\sum_{i=1}^{r} \sum_{j=1}^{n_2} y_{i,j} \right)^{n_2 r + \beta}$$

$$A_3 = \exp\left(-\theta_1 \sum_{i=1}^{k} \sum_{j=1}^{n_1} x_{i,j} - \theta_2 \sum_{i=1}^{r} \sum_{j=1}^{n_2} y_{i,j} \right)$$

Therefore,

$$\hat{R}_{pB} = \frac{A_1 A_2\, k}{\Gamma(n_1 k+\alpha)\Gamma(n_2 r+\beta)} \sum_{l=0}^{k-1}\sum_{m=0}^{r} \binom{k-1}{l}\binom{r}{m}(-1)^{l+m} \int_0^\infty\int_0^\infty \frac{A_3\, \theta_1^{n_1 k+\alpha}\, \theta_2^{n_2 r+\beta-1}}{[(l+1)\theta_1 + m\theta_2]}\, d\theta_1 d\theta_2$$

Similarly, the Bayes estimator of Reliability function $\hat{R}_{sB}$ is obtained as the posterior expectation of $\hat{R}_s$ is given by

$$\hat{R}_{sB} = \frac{A_1 A_2\, r}{\Gamma(n_1 k+\alpha)\Gamma(n_2 r+\beta)} \sum_{m=0}^{r-1} \binom{r-1}{m}(-1)^{m} \int_0^\infty\int_0^\infty \frac{A_3}{[k\theta_1 + (m+1)\theta_2]}\, \theta_1^{n_1 k+\alpha-1}\, \theta_2^{n_2 r+\beta}\, d\theta_1 d\theta_2$$

5. SIMULATION STUDY

A simulation study is conducted to evaluate MSEs of reliabilities of series and parallel systems with different strengths and stresses. A simulation study of 100,000 samples of size n=5, 6, 8, 10 are generated for different values of k, the number of strengths, r, the number of stresses and the parameters (θ_1, θ_2) as specified in tables.
Based on the simulation study for the parameters considered, the values of MLE and Bayes estimators for R_p and R_s with their MSEs are presented in Table 1, 2, 3 and 4 for different combinations of (θ_1, θ_2, k, r, α, β). The actual values of R_p and R_s are also given which can be compared with estimates.

Table 1. MLEs, Bayes estimators and MSE for estimates of R_p and R_s
$(\theta_1 = 1 \quad \theta_2 = 2.5 \quad k=3 \quad r=2 \quad \alpha=1.8 \quad \beta=2.1 \quad R_p = 0.879329 \quad R_s = 0.284090)$

$n_1 = n_2$	$\hat{R}_p$	$\hat{R}_s$	$\hat{R}_{pB}$	$\hat{R}_{sB}$	$MSE(\hat{R}_p)$	$MSE(\hat{R}_s)$	$MSE(\hat{R}_{pB})$	$MSE(\hat{R}_{sB})$
5	0.846445	0.258412	0.845167	0.281921	0.004304	0.005035	0.004025	0.004273
6	0.909326	0.351222	0.905904	0.369566	0.004146	0.011215	0.003698	0.013761
8	0.821581	0.238086	0.821807	0.253442	0.008443	0.007697	0.008001	0.006477
10	0.876202	0.294144	0.874218	0.306486	0.002544	0.005098	0.002441	0.005391

Table 2. MLEs, Bayes estimators and MSE for estimates of R_p and R_s
$(\theta_1 = 1 \quad \theta_2 = 2.5 \quad k=3 \quad r=3 \quad \alpha=1.8 \quad \beta=2.1 \quad R_p = 0.838702 \quad R_s = 0.202922)$

$n_1 = n_2$	$\hat{R}_p$	$\hat{R}_s$	$\hat{R}_{pB}$	$\hat{R}_{sB}$	$MSE(\hat{R}_p)$	$MSE(\hat{R}_s)$	$MSE(\hat{R}_{pB})$	$MSE(\hat{R}_{sB})$
5	0.779619	0.182047	0.771831	0.194386	0.018542	0.006661	0.018262	0.006275
6	0.842329	0.235178	0.834170	0.245044	0.009429	0.008576	0.009057	0.009090
8	0.862519	0.236461	0.855189	0.244547	0.002554	0.003903	0.002267	0.004419
10	0.835093	0.222416	0.829934	0.228865	0.007064	0.006987	0.007002	0.007137

Table 3. MLEs, Bayes estimators and MSE for estimates of R_p and R_s
$(\theta_1 = 1 \quad \theta_2 = 2.5 \quad k = 4 \quad r = 3 \quad \alpha = 1.8 \quad \beta = 2.1 \quad R_p = 0.893476 \quad R_s = 0.139353)$

$n_1 = n_2$	$\hat{R}_p$	$\hat{R}_s$	$\hat{R}_{pB}$	$\hat{R}_{sB}$	$MSE(\hat{R}_p)$	$MSE(\hat{R}_s)$	$MSE(\hat{R}_{pB})$	$MSE(\hat{R}_{sB})$
5	0.893960	0.167165	0.887890	0.173073	0.004528	0.006304	0.004397	0.007542
6	0.862057	0.125870	0.857030	0.139121	0.004387	0.002008	0.004539	0.001933
8	0.879792	0.152703	0.875876	0.162987	0.007493	0.004027	0.007205	0.004536
10	0.892604	0.155051	0.890223	0.163623	0.002453	0.003915	0.002610	0.004272

Table 4. MLEs, Bayes estimators and MSE for estimates of R_p and R_s
$(\theta_1 = 1 \quad \theta_2 = 2.5 \quad k = 5 \quad r = 3 \quad \alpha = 1.8 \quad \beta = 2.1 \quad R_p = 0.925672 \quad R_s = 0.10)$

$n_1 = n_2$	$\hat{R}_p$	$\hat{R}_s$	$\hat{R}_{pB}$	$\hat{R}_{sB}$	$MSE(\hat{R}_p)$	$MSE(\hat{R}_s)$	$MSE(\hat{R}_{pB})$	$MSE(\hat{R}_{sB})$
5	0.907061	0.126030	0.904532	0.142190	0.011715	0.004897	0.009803	0.006505
6	0.917737	0.110199	0.913780	0.123952	0.002885	0.001886	0.002804	0.002556
8	0.909369	0.097404	0.904837	0.107475	0.002121	0.001227	0.002483	0.00139
10	0.904645	0.095929	0.902394	0.103825	0.002888	0.001353	0.003367	0.001451

Table 5. MLEs, Bayes estimators and MSE for estimates of R_p and R_s
$(\theta_1 = 1 \quad \theta_2 = 2.5 \quad k = 5 \quad r = 5 \quad \alpha = 1.8 \quad \beta = 2.1 \quad R_p = 0.889284 \quad R_s = 0.047619)$

$n_1 = n_2$	$\hat{R}_p$	$\hat{R}_s$	$\hat{R}_{pB}$	$\hat{R}_{sB}$	$MSE(\hat{R}_p)$	$MSE(\hat{R}_s)$	$MSE(\hat{R}_{pB})$	$MSE(\hat{R}_{sB})$
5	0.863880	0.053039	0.853786	0.060150	0.005661	0.001486	0.006369	0.001741
6	0.865678	0.046744	0.85605	0.052741	0.003848	0.000607	0.004387	0.000688
8	0.878542	0.049224	0.871384	0.053955	0.002533	0.000384	0.002686	0.0004523
10	0.902895	0.055263	0.903949	0.0599	0.000546	0.000156	0.000524	0.000251

6. SOME REMARKS AND CONCLUSIONS

1. In this paper, we have considered exponential stress-strength model for estimating the parameters and reliability functions.
2. In exponential stress-strength models, the parameters are estimated using the method of maximum likelihood (ML) and then the estimators for reliability function for parallel and series system, when the system has k independent strength and r independent stresses.

3. The Bayes estimators are also derived for the reliability of series and parallel systems with respect to conjugate priors and their MSEs are evaluated using simulation. It can be observed that MLEs are slightly better than Bayes estimators.
4. The simulation study reveals that the estimators for parameters and reliability are very close to the actual value having very small MSE. Hence, the need for deriving uniformly minimum variance estimators are very limited as the MLEs obtained here are very efficient.

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