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# Intuitionistic Fuzzy $\delta$-continuous Functions

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**Abstract**

In this paper, we characterize the intuitionistic fuzzy $\delta$-continuous, intuitionistic fuzzy weakly $\delta$-continuous, intuitionistic fuzzy almost continuous, and intuitionistic fuzzy almost strongly $\theta$-continuous functions in terms of intuitionistic fuzzy $\delta$-closure and interior or $\theta$-closure and interior.

**Keywords:** Intuitionistic fuzzy $\delta$-continuous, Weakly $\delta$-continuous, Almost continuous, Almost strongly $\theta$-continuous

## 1. Introduction and Preliminaries

By using the intuitionistic fuzzy sets introduced by Atanassov [1], Çoker and his colleagues [2–4] introduced the intuitionistic fuzzy topological space, which is a generalization of the fuzzy topological space. Moreover, many researchers have studied about this space [5–12].

In the intuitionistic fuzzy topological spaces, Hanafy et al. [13] introduced the concept of intuitionistic fuzzy $\theta$-closure as a generalization of the concept of fuzzy $\theta$-closure by Mukherjee and Sinha [14, 15], and characterized some types of functions. In the previous papers [16, 17], we also introduced and investigated some properties of the concept of intuitionistic fuzzy $\theta$-interior and $\delta$-closure in intuitionistic fuzzy topological spaces.

In this paper, we characterize the intuitionistic fuzzy $\delta$-continuous, intuitionistic fuzzy weakly $\delta$-continuous, intuitionistic fuzzy almost continuous, and intuitionistic fuzzy almost strongly $\theta$-continuous functions in terms of intuitionistic fuzzy $\delta$-closure and interior, or $\theta$-closure and interior.

Let $X$ be a nonempty set and $I$ the unit interval $[0, 1]$. An *intuitionistic fuzzy set* $A$ in $X$ is an object of the form $A = (\mu_A, \gamma_A)$, where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership, respectively, and $\mu_A + \gamma_A \leq 1$. Obviously, every fuzzy set $\mu_A$ in $X$ is an intuitionistic fuzzy set of the form $(\mu_A, 1 - \mu_A)$.

Throughout this paper, $I(X)$ denotes the family of all intuitionistic fuzzy sets in $X$, and "IF" stands for "intuitionistic fuzzy." For the notions which are not mentioned in this paper, refer to [17].

**Theorem 1.1** ( [7]). The following are equivalent:

(1) An IF set $A$ is IF semi-open in $X$.

(2) $A \leq \text{cl}(\text{int}(A))$.

**Corollary 1.2** ( [17]). If $U$ is an IF regular open set, then $U$ is an IF $\delta$-open set.

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**Theorem 1.3** ( [17]). For any IF semi-open set $A$, we have $\mathrm{cl}(A) = \mathrm{cl}_\delta(A)$.

**Lemma 1.4** ( [17]). (1) For any IF set $U$ in an IF topological space $(X, \mathcal{T})$, $\mathrm{int}(\mathrm{cl}(U))$ is an IF regular open set.

(2) For any IF open set $U$ in an IF topological space $(X, \mathcal{T})$ such that $x_{(\alpha,\beta)}qU$, $\mathrm{int}(\mathrm{cl}(U))$ is an IF regular open $q$-neighborhood of $x_{(\alpha,\beta)}$.

**Theorem 1.5** ( [12]). Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and $U = (\mu_U, \gamma_U)$ an IF set in $X$. Then $x_{(\alpha,\beta)} \in \mathrm{cl}(U)$ if and only if $UqN$, for any IF $q$-neighborhood $N$ of $x_{(\alpha,\beta)}$.

## 2. Intuitionistic Fuzzy $\delta$-continuous and Weakly $\delta$-continuous Functions

Recall that a fuzzy set $N$ in $(X, \mathcal{T})$ is said to be a *fuzzy $\delta$-neighborhood* of a fuzzy point $x_\alpha$ if there exists a fuzzy regular open $q$-neighborhood $V$ of $x_\alpha$ such that $V\tilde{q}N^c$, or equivalently $V \leq N$ (See [14]). Now, we define a similar definition in the intuitionistic fuzzy topological spaces.

**Definition 2.1.** An intuitionistic fuzzy set $N$ in $(X, \mathcal{T})$ is said to be an *intuitionistic fuzzy $\delta$-neighborhood* of an intuitionistic fuzzy point $x_{(\alpha,\beta)}$ if there exists an intuitionistic fuzzy regular open $q$-neighborhood $V$ of $x_{(\alpha,\beta)}$ such that

$$V \leq N.$$

**Lemma 2.2.** An IF set $A$ is an IF $\delta$-open set in $(X, \mathcal{T})$ if and only if for any IF point $x_{(\alpha,\beta)}$ with $x_{(\alpha,\beta)}qA$, $A$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$.

*Proof.* Let $A$ be an IF $\delta$-open set in $(X, \mathcal{T})$ such that $x_{(\alpha,\beta)}qA$. Then $x_{(\alpha,\beta)} \not\leq A^c$. Since $A^c$ is an IF $\delta$-closed set, we have $x_{(\alpha,\beta)} \notin A^c = \mathrm{cl}_\delta(A^c)$. Then there exists an IF regular open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $U\tilde{q}A^c$. Thus $U \leq A$. Hence $A$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$.

Conversely, to show that $A^c$ is an IF $\delta$-closed set, take any $x_{(\alpha,\beta)} \notin A^c$. Then we have $x_{(\alpha,\beta)}qA$. Thus $A$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$. Therefore there exists an IF regular open $q$-neighborhood $V$ of $x_{(\alpha,\beta)}$ such that $V \leq A^c$, i.e. $x_{(\alpha,\beta)} \notin \mathrm{cl}_\delta(A^c)$. Since $\mathrm{cl}_\delta(A^c) \leq A^c$, we have $A^c$ is an IF $\delta$-closed set. Hence $A$ is an IF $\delta$-open set.

Recall that a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be a *fuzzy $\delta$-continuous* function if for each fuzzy point $x_\alpha$ in $X$ and for any fuzzy regular open $q$-neighborhood $V$ of $f(x_\alpha)$, there exists an fuzzy regular open $q$-neighborhood $U$ of $x_\alpha$ such that $f(U) \leq V$ (See [18]). We define a similar definition in the intuitionistic fuzzy topological spaces as follows.

**Definition 2.3.** A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be *intuitionistic fuzzy $\delta$-continuous* if for each intuitionistic fuzzy point $x_{(\alpha,\beta)}$ in $X$ and for any intuitionistic fuzzy regular open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$, there exists an intuitionistic fuzzy regular open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that

$$f(U) \leq V.$$

Now, we characterize the intuitionistic fuzzy $\delta$-continuous function in terms of IF $\delta$-closure and IF $\delta$-interior.

**Theorem 2.4.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent:

(1) $f$ is an IF $\delta$-continuous function.

(2) $f(\mathrm{cl}_\delta(U)) \leq \mathrm{cl}_\delta(f(U))$ for each IF set $U$ in $X$.

(3) $\mathrm{cl}_\delta(f^{-1}(V)) \leq f^{-1}(\mathrm{cl}_\delta(V))$ for each IF set $V$ in $Y$.

(4) $f^{-1}(\mathrm{int}_\delta(V)) \leq \mathrm{int}_\delta(f^{-1}(V))$ for each IF set $V$ in $Y$.

*Proof.* (1) $\Rightarrow$ (2). Let $x_{(\alpha,\beta)} \in \mathrm{cl}_\delta(U)$, and let $B$ be an IF regular open $q$-neighborhood of $f(x_{(\alpha,\beta)})$ in $Y$. By (1), there exists an IF regular open $q$-neighborhood $A$ of $x_{(\alpha,\beta)}$ such that $f(A) \leq B$. Since $x_{(\alpha,\beta)} \in \mathrm{cl}_\delta(U)$ and $A$ is an IF regular open $q$-neighborhood of $x_{(\alpha,\beta)}$, $AqU$. So $f(A)qf(U)$. Since $f(A) \leq B$, $Bqf(U)$. Then $f(x_{(\alpha,\beta)}) \in \mathrm{cl}_\delta(f(U))$. Hence $f(\mathrm{cl}_\delta(U))) \leq \mathrm{cl}_\delta(f(U))$.

(2) $\Rightarrow$ (3). Let $V$ be an IF set in $Y$. Then $f^{-1}(V)$ is an IF set in $X$. By (2), $f(\mathrm{cl}_\delta(f^{-1}(V))) \leq \mathrm{cl}_\delta(f(f^{-1}(V))) \leq \mathrm{cl}_\delta(V)$. Thus $\mathrm{cl}_\delta(f^{-1}(V)) \leq f^{-1}(\mathrm{cl}_\delta(V))$.

(3) $\Rightarrow$ (1). Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $V$ be an IF regular open $q$-neighborhood of $f(x_{(\alpha,\beta)})$ in $Y$. Since $V^c$ is an IF regular closed set, $V^c$ is an IF semi-open set. By Theorem 1.3, $\mathrm{cl}(V^c) = \mathrm{cl}_\delta(V^c)$. Since $f(x_{(\alpha,\beta)})qV$, $f(x_{(\alpha,\beta)}) \notin V^c = \mathrm{cl}(V^c) = \mathrm{cl}_\delta(V^c)$. Therefore $x_{(\alpha,\beta)} \notin f^{-1}(\mathrm{cl}_\delta(V^c))$. By (3), $x_{(\alpha,\beta)} \notin \mathrm{cl}_\delta(f^{-1}(V^c))$. Then there exists an IF regular open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $U\tilde{q}f^{-1}(V^c) = (f^{-1}(V))^c$. So $U \leq f^{-1}(V)$, i.e.$f(U) \leq V$. Hence $f$ is an IF $\delta$-continuous function.

(3) $\Rightarrow$ (4). Let $V$ be an IF set in $Y$. By (3), $\mathrm{cl}_\delta(f^{-1}(V^c)) \leq f^{-1}(\mathrm{cl}_\delta(V^c))$. Thus

$$\begin{aligned} f^{-1}(\mathrm{int}_\delta(V)) &= f^{-1}((\mathrm{cl}_\delta(V^c))^c) = (f^{-1}(\mathrm{cl}_\delta((V^c))))^c \\ &\leq (\mathrm{cl}_\delta(f^{-1}(V^c)))^c = (\mathrm{cl}_\delta((f^{-1}(V))^c))^c \\ &= \mathrm{int}_\delta(f^{-1}(V)). \end{aligned}$$

$(4) \Rightarrow (3)$. Let $V$ be an IF set in $Y$. Then $V^c$ is an IF set in $Y$. By the hypothesis, $f^{-1}(\text{int}_\delta(V^c)) \leq \text{int}_\delta(f^{-1}(V^c))$. Thus

$$\begin{aligned}\text{cl}_\delta(f^{-1}(V)) &= (\text{int}_\delta((f^{-1}(V))^c))^c = (\text{int}_\delta(f^{-1}(V^c)))^c \\ &\leq (f^{-1}(\text{int}_\delta(V^c)))^c = f^{-1}((\text{int}_\delta(V^c))^c) \\ &= f^{-1}(\text{cl}_\delta(V)).\end{aligned}$$

Hence $\text{cl}_\delta(f^{-1}(V)) \leq f^{-1}(\text{cl}_\delta(V))$.

The intuitionistic fuzzy $\delta$-continuous function is also characterized in terms of IF $\delta$-open and IF $\delta$-closed sets.

**Theorem 2.5.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent:

(1) $f$ is an IF $\delta$-continuous function.

(2) $f^{-1}(A)$ is an IF $\delta$-closed set for each IF $\delta$-closed set $A$ in $X$.

(3) $f^{-1}(A)$ is an IF $\delta$-open set for each IF $\delta$-open set $A$ in $X$.

*Proof.* $(1) \Rightarrow (2)$. Let $A$ be an IF $\delta$-closed set in $X$. Then $A = \text{cl}_\delta(A)$. By Theorem 2.4, $\text{cl}_\delta(f^{-1}(A)) \leq f^{-1}(\text{cl}_\delta(A)) = f^{-1}(A)$. Hence $f^{-1}(A) = \text{cl}_\delta(f^{-1}(A))$. Therefore, $f^{-1}(A)$ is an IF $\delta$-closed set.

$(2) \Rightarrow (3)$. Trivial.

$(3) \Rightarrow (1)$. Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $V$ be an IF regular open $q$-neighborhood of $f(x_{(\alpha,\beta)})$. By Corollary 1.2, $V$ is an IF $\delta$-open set. By the hypothesis, $f^{-1}(V)$ is an IF $\delta$-open set. Since $x_{(\alpha,\beta)} q f^{-1}(V)$, by Lemma 2.2, we have that $f^{-1}(V)$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$. Therefore, there exists an IF regular open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $U \leq f^{-1}(V)$. Hence $f(U) \leq V$.

The intuitionistic fuzzy $\delta$-continuous function is also characterized in terms of IF $\delta$-neighborhoods.

**Theorem 2.6.** A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is IF $\delta$-continuous if and only if for each IF point $x_{(\alpha,\beta)}$ of $X$ and each IF $\delta$-neighborhood $N$ of $f(x_{(\alpha,\beta)})$, the IF set $f^{-1}(N)$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$.

*Proof.* Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $N$ be an IF $\delta$-neighborhood of $f(x_{(\alpha,\beta)})$. Then there exists an IF regular open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$ such that $V \leq N$. Since $f$ is an an IF $\delta$-continuous function, there exists an IF regular open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $f(U) \leq V$. Thus, $U \leq f^{-1}(V) \leq N$. Hence $f^{-1}(N)$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$.

Conversely, let $x_{(\alpha,\beta)}$ be an IF point in $X$, and $V$ an IF regular open $q$-neighborhood of $f(x_{(\alpha,\beta)})$. Then $V$ is an IF $\delta$-neighborhood of $f(x_{(\alpha,\beta)})$. By the hypothesis, $f^{-1}(V)$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$. By the definition of IF $\delta$-neighborhood, there exists an IF regular open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $U \leq f^{-1}(V)$. Thus$f(U) \leq V$. Hence $f$ is an IF $\delta$-continuous function.

**Theorem 2.7.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a bijection. Then the following statements are equivalent:

(1) $f$ is an IF $\delta$-continuous function.

(2) $\text{int}_\delta(f(U)) \leq f(\text{int}_\delta(U))$ for each IF set $U$ in $X$.

*Proof.* $(1) \Rightarrow (2)$. Let $U$ be an IF set in $X$. Then $f(U)$ is an IF set in $Y$. By Theorem 2.4, $f^{-1}(\text{int}_\delta(f(U))) \leq \text{int}_\delta(f^{-1}(f(U)))$. Since $f$ is one-to-one,

$$f^{-1}(\text{int}_\delta(f(U))) \leq \text{int}_\delta(f^{-1}(f(U))) = \text{int}_\delta(U).$$

Since $f$ is onto,

$$\text{int}_\delta(f(U)) = f(f^{-1}(\text{int}(f(U)))) \leq f(\text{int}(U)).$$

$(2) \Rightarrow (1)$. Let $V$ be an IF set in $Y$. Then $f^{-1}(V)$ is an IF set in $X$. By the hypothesis, $\text{int}_\delta(f(f^{-1}(V))) \leq f(\text{int}_\delta(f^{-1}(V)))$. Since $f$ is onto,

$$\text{int}_\delta(V) = \text{int}_\delta(f(f^{-1}(V))) \leq f(\text{int}_\delta(f^{-1}(V))).$$

Since $f$ is one-to-one,

$$f^{-1}(\text{int}_\delta(V)) \leq f^{-1}(f(\text{int}_\delta(f^{-1}(V)))) = \text{int}_\delta(f^{-1}(V)).$$

Hence by Theorem 2.4, $f$ is an IF $\delta$-continuous function.

Recall that a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be *fuzzy weakly $\delta$-continuous* if for each fuzzy point $x_\alpha$ in $X$ and each fuzzy open $q$-neighborhood $V$ of $f(x_\alpha)$, there exists an fuzzy open $q$-neighborhood $U$ of $x_\alpha$ such that $f(\text{int}(\text{cl}(U))) \leq \text{cl}(V)$ (See [14]). We define a similar definition in the intuitionistic fuzzy topological spaces as follows.

**Definition 2.8.** A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be *intuitionistic fuzzy weakly $\delta$-continuous* if for each intuitionistic fuzzy point $x_{(\alpha,\beta)}$ in $X$ and each intuitionistic fuzzy open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$, there exists an intuitionistic fuzzy open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that

$$f(\text{int}(\text{cl}(U))) \leq \text{cl}(V).$$

**Theorem 2.9.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent:

(1) $f$ is an IF weakly $\delta$-continuous function.

(2) $f(\text{cl}_\delta(A)) \leq \text{cl}_\theta(f(A))$ for each IF set $A$ in $X$.

(3) $\text{cl}_\delta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\theta(B))$ for each IF set $B$ in $Y$.

(4) $f^{-1}(\text{int}_\theta(B)) \leq \text{int}_\delta(f^{-1}(B))$ for each IF set $B$ in $Y$.

*Proof.* $(1) \Rightarrow (2)$. Let $x_{(\alpha,\beta)} \in \text{cl}_\delta(A)$, and let $V$ be an IF open $q$-neighborhood of $f(x_{(\alpha,\beta)})$ in $Y$. Since $f$ is an IF weakly $\delta$-continuous function, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $f(\text{int}(\text{cl}(U))) \leq \text{cl}(V)$. Since $\text{int}(\text{cl}(V))$ is an IF regular open $q$-neighborhood of $x_{(\alpha,\beta)}$ and $x_{(\alpha,\beta)} \in \text{cl}_\delta(A)$, we have $Aq\text{int}(\text{cl}(V))$. Thus $f(A)qf(\text{int}(\text{cl}(V)))$. Since $f(\text{int}(\text{cl}(V))) \leq \text{cl}(V)$, we have $f(A)q\text{cl}(V)$. Thus $f(x_{(\alpha,\beta)}) \in \text{cl}_\theta(f(A))$. Hence $f(\text{cl}_\delta(A)) \leq \text{cl}_\theta(f(A))$.

$(2) \Rightarrow (3)$. Let $B$ be an IF set in $Y$. Then $f^{-1}(B)$ is an IF set in $X$. By (2), $f(\text{cl}_\delta(f^{-1}(B))) \leq \text{cl}_\theta(f(f^{-1}(B))) \leq \text{cl}_\theta(B)$. Hence $\text{cl}_\delta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\theta(B))$.

$(3) \Rightarrow (1)$. Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $V$ be an IF open $q$-neighborhood of $f(x_{(\alpha,\beta)})$ in $Y$. Since $\text{cl}(V) \leq \text{cl}(V)$, $\text{cl}(V)\widetilde{q}(\text{cl}(V))^c$. Thus $f(x_{(\alpha,\beta)}) \notin \text{cl}_\theta((\text{cl}(V))^c)$. By (3), $f(x_{(\alpha,\beta)}) \notin \text{cl}_\delta(f^{-1}((\text{cl}(V))^c))$. Then there exists an intuitionistic fuzzy regular open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $\text{int}(\text{cl}(U))\widetilde{q}f^{-1}((\text{cl}(V))^c)$. Thus $\text{int}(\text{cl}(U)) \leq f^{-1}(\text{cl}(V))$. Therefore, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $f(\text{int}(\text{cl}(U))) \leq \text{cl}(V)$. Hence $f$ is an IF weakly $\delta$-continuous function.

$(3) \Rightarrow (4)$. Let $B$ be an IF set in $Y$. Then $B^c$ is an IF set in $Y$. By (3), $\text{cl}_\delta(f^{-1}(B^c)) \leq f^{-1}(\text{cl}_\theta(B^c))$. Hence we have $\text{int}_\delta(f^{-1}(B)) = (\text{cl}_\delta(f^{-1}(B^c))) \geq (f^{-1}(\text{cl}_\theta(B^c)))^c = \text{int}_\theta(f^{-1}(B))$.

$(4) \Rightarrow (3)$. Similarly.

**Theorem 2.10.** A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is IF weakly $\delta$-continuous if and only if for each IF point $x_{(\alpha,\beta)}$ in $X$ and each IF $\theta$-neighborhood $N$ of $f(x_{(\alpha,\beta)})$, the IF set $f^{-1}(N)$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$.

*Proof.* Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $N$ be an IF $\theta$-neighborhood of $f(x_{(\alpha,\beta)})$ in $Y$. Then there exists an IF open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$ such that $\text{cl}(V) \leq N$. Since $f$ is an IF weakly $\delta$-continuous function, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $f(\text{int}(\text{cl}(U)) \leq \text{cl}(V)$. Since $\text{cl}(V) \leq N$, $\text{int}(\text{cl}(U)) \leq f^{-1}(N)$. Hence $f^{-1}(N)$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$.

Conversely, let $x_{(\alpha,\beta)}$ be an IF point in $X$ and let $V$ be an IF open $q$-neighborhood of $f(x_{(\alpha,\beta)})$. Since $\text{cl}(V) \leq \text{cl}(V)$, $\text{cl}(V)$ is an IF $\theta$-neighborhood of $f(x_{(\alpha,\beta)})$. By the hypothesis, $f^{-1}(\text{cl}(V))$ is an IF $\delta$-neighborhood of $x_{(\alpha,\beta)}$. Then there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $\text{int}(\text{cl}(V)) \leq f^{-1}(\text{cl}(V))$. Thus $\text{int}(\text{cl}(V)) \leq f^{-1}(\text{cl}(V))$. Hence $f$ is IF almost strongly $\delta$-continuous.

**Theorem 2.11.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be an IF weakly $\delta$-continuous function. Then the following statements are true:

(1) $f^{-1}(V)$ is an IF $\theta$-closed set in $X$ for each IF $\delta$-closed set $V$ in $Y$.

(2) $f^{-1}(V)$ is an IF $\theta$-open set in $X$ for each IF $\delta$-open set $V$ in $Y$.

*Proof.* (1) Let $B$ be an IF $\theta$-closed set in $Y$. Then $\text{cl}_\theta(B) = B$. Since $f$ is an IF weakly $\delta$-continuous function, by Theorem 2.9, $\text{cl}_\delta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\theta(B)) = f^{-1}(B)$. Hence $f^{-1}(B)$ is an IF $\delta$-closed set in $X$.

(2) Trivial.

**Theorem 2.12.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a bijection. Then the following statements are equivalent:

(1) $f$ is an IF weakly $\delta$-continuous function.

(2) $\text{int}_\theta(f(A)) \leq f(\text{int}_\delta(A))$ for each IF set $A$ in $X$.

*Proof.* $(1) \Rightarrow (2)$. Let $A$ be an IF set in $X$. Then $f(A)$ is an IF set in $Y$. By Theorem 2.9-(4), $f^{-1}(\text{int}_\theta(f(A))) \leq \text{int}_\delta(f^{-1}(f(A)))$. Since $f$ is one-to-one,

$$f^{-1}(\text{int}_\theta(f(A))) \leq \text{int}_\delta(f^{-1}(f(A))) = \text{int}_\delta(A).$$

Since $f$ is onto,

$$\text{int}_\theta(f(A)) = f(f^{-1}(\text{int}_\theta(f(A)))) \leq f(\text{int}_\delta(A)).$$

Hence $\text{int}_\theta(f(A)) \leq f(\text{int}_\delta(A))$.

(2) $\Rightarrow$ (1). Let $B$ be an IF set in $Y$. Then $f^{-1}(B)$ is an IF set in $X$. By (2) $\text{int}_\theta(f(f^{-1}(B))) \leq f(\text{int}_\delta(f^{-1}(B)))$. Since $f$ is onto,

$$\text{int}_\theta(B) = \text{int}_\theta(f(f^{-1}(B)) \leq f(\text{int}_\delta(f^{-1}(B)).$$

$f$ is one-to-one,

$$f^{-1}(\text{int}_\theta(B) \leq f^{-1}(f(\text{int}_\delta(f^{-1}(B))) = \text{int}_\delta(f^{-1}(B)).$$

By Theorem 2.9, $f$ is an IF weakly $\delta$-continuous function.

## 3. IF Almost Continuous and Almost Strongly $\theta$-continuous Functions

**Definition 3.1** ( [7]). A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be *intuitionistic fuzzy almost continuous* if for any intuitionistic fuzzy regular open set $V$ in $Y$, $f^{-1}(V)$ is an intuitionistic fuzzy open set in $X$.

**Theorem 3.2** ( [12]). A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is IF almost continuous if and only if for each IF point $x_{(\alpha,\beta)}$ in $X$ and for any IF open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that

$$f(U) \leq \text{int}(\text{cl}(V)).$$

**Theorem 3.3.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent:

(1) $f$ is an IF almost continuous function.

(2) $f(\text{cl}(U)) \leq \text{cl}_\delta(f(U))$ for each IF set $U$ in $X$.

(3) $f^{-1}(V)$ is an IF closed set in $X$ for each IF $\delta$-closed set $V$ in $Y$.

(4) $f^{-1}(V)$ is an IF open set in $X$ for each IF $\delta$-open set $V$ in $Y$.

*Proof.*

(1) $\Rightarrow$ (2). Let $x_{(\alpha,\beta)} \in \text{cl}(U)$. Suppose that $f(x_{(\alpha,\beta)}) \notin \text{cl}_\delta(f(U))$. Then there exists an IF open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$ such that $V\tilde{q}f(U)$. Since $f$ is an IF almost continuous function, $f^{-1}(V)$ is an IF open set in $X$. Since $Vqf(x_{(\alpha,\beta)})$, we have $f^{-1}(V)qx_{(\alpha,\beta)}$. Thus $f^{-1}(V)$ is an IF open $q$-neighborhood of $x_{(\alpha,\beta)}$. Since $x_{(\alpha,\beta)} \in \text{cl}(U)$, by Theorem 1.5, we have $f^{-1}(V)qU$. Thus $f(f^{-1}(V))qf(U)$. Since $f(f^{-1}(V)) \leq V$, we have $Vqf(U)$. This is a contradiction. Hence $f(\text{cl}(U)) \leq \text{cl}_\delta(f(U))$.

(2) $\Rightarrow$ (3). Let $V$ be an IF $\delta$-closed set in $Y$. Then $f^{-1}(V)$ is an IF set in $X$. By the hypothesis,

$$f(\text{cl}(f^{-1}(V)))) \leq \text{cl}_\delta(f(f^{-1}(V))) \leq \text{cl}_\delta(V) = V.$$

Thus $\text{cl}(f^{-1}(V)) \leq f^{-1}(V)$. Hence $f^{-1}(V)$ is an IF closed set in $X$.

(3) $\Rightarrow$ (4). Let $V$ be an IF $\delta$-open set in $Y$. Then $V^c$ is an IF $\delta$-closed set in $Y$. By the hypothesis, $f^{-1}(V^c) = (f^{-1}(V))^c$ is an IF closed set in $X$. Hence $f^{-1}(V)$ is an IF open set in $X$.

(4) $\Rightarrow$ (1). Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $V$ be an IF open $q$-neighborhood of $f(x_{(\alpha,\beta)})$ in $Y$. Then $\text{int}(\text{cl}(V))$ is an IF regular open $q$-neighborhood $f(x_{(\alpha,\beta)})$. By Theorem 1.2, $\text{int}(\text{cl}(V))$ is an IF $\delta$-open set in $Y$. By the hypothesis, $f^{-1}(\text{int}(\text{cl}(V)))$ is IF open in $X$. Since $\text{int}(\text{cl}(V))qf(x_{(\alpha,\beta)})$, we have $x_{(\alpha,\beta)}qf^{-1}(\text{int}(\text{cl}(V)))$. Thus $x_{(\alpha,\beta)}$ does not belong to the set $(f^{-1}(\text{int}(\text{cl}(V))))^c$. Put $B = (f^{-1}(\text{int}(\text{cl}(V))))^c$. Since $B$ is an IF closed set and $x_{(\alpha,\beta)} \notin B = \text{cl}(B)$, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $U\tilde{q}B$. Then $x_{(\alpha,\beta)}qU \leq B^c = f^{-1}(\text{int}(\text{cl}(V)))$. Thus $f(U) \leq \text{int}(\text{cl}(V))$. Hence, $f$ is an IF almost continuous function.

**Theorem 3.4.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent:

(1) $f$ is an IF almost continuous function.

(2) $\text{cl}(f^{-1}(V)) \leq f^{-1}(\text{cl}_\delta(V))$ for each IF set $V$ in $Y$.

(3) $\text{int}_\delta(f^{-1}(V)) \leq f^{-1}(\text{int}(V))$ for each IF set $V$ in $Y$.

*Proof.* (1) $\Rightarrow$ (2). Let $V$ be an IF set in $Y$. Then $f^{-1}(V)$ is an IF set in $X$. By Theorem 3.3,

$$f(\text{cl}(f^{-1}(V))) \leq \text{cl}_\delta(f(f^{-1}(V))) \leq \text{cl}_\delta(V).$$

Thus $\text{cl}(f^{-1}(V)) \leq f^{-1}(\text{cl}_\delta(V))$.

(2) $\Rightarrow$ (1). Let $U$ be an IF set in $X$. Then $f(U)$ is an IF set in $Y$. By the hypothesis, $\text{cl}(f^{-1}(f(U))) \leq f^{-1}(\text{cl}_\delta(f(U)))$. Then

$$\text{cl}(U) \leq \text{cl}(f^{-1}(f(U))) \leq f^{-1}(\text{cl}_\delta(f(U))).$$

Thus $f(\text{cl}(U)) \leq \text{cl}_\delta(f(U))$. By Theorem 3.3, $f$ is an IF almost continuous function.

(2) $\Rightarrow$ (3). Let $V$ be an IF set in $Y$. Then $V^c$ is an IF set in $Y$. By the hypothesis, $\text{cl}(f^{-1}(V^c)) \leq f^{-1}(\text{cl}_\delta(V^c))$. Thus

$$\begin{aligned} f^{-1}(\text{int}_\delta(V)) &= f^{-1}((\text{cl}_\delta(V^c))^c) = (f^{-1}(\text{cl}_\delta((V^c))))^c \\ &\leq (\text{cl}(f^{-1}(V^c)))^c = (\text{cl}((f^{-1}(V))^c))^c \\ &= \text{int}(f^{-1}(V)). \end{aligned}$$

(3) $\Rightarrow$ (2). Let $V$ be an IF set in $Y$. Then $V^c$ is an IF set in $Y$. By the hypothesis, $f^{-1}(\text{int}_\delta(V^c)) \leq \text{int}(f^{-1}(V^c))$. Thus

$$\begin{aligned} \text{cl}(f^{-1}(V)) &= (\text{int}((f^{-1}(V))^c))^c = (\text{int}(f^{-1}(V^c)))^c \\ &\leq (f^{-1}(\text{int}_\delta(V^c)))^c = f^{-1}((\text{int}_\delta(V^c))^c) \\ &= f^{-1}(\text{cl}_\delta(V)). \end{aligned}$$

Hence $\text{cl}(f^{-1}(V)) \leq f^{-1}(\text{cl}_\delta(V))$ .

**Corollary 3.5.** A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is IF almost continuous if and only if for each IF point $x_{(\alpha,\beta)}$ in $X$ and each IF $\delta$-neighborhood $N$ of $f(x_{(\alpha,\beta)})$, the IF set $f^{-1}(N)$ is an IF $q$-neighborhood of $x_{(\alpha,\beta)}$.

*Proof.* Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $N$ be an IF $\delta$-neighborhood of $f(x_{(\alpha,\beta)})$. Then there exists an IF regular open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$ such that $V \leq N$. Since $f$ is an IF almost continuous function, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $f(U) \leq \text{int}(\text{cl}(V)) = V \leq N$. Thus there exists an IF open set $U$ such that $x_{(\alpha,\beta)} q U \leq f^{-1}(N)$. Hence $f^{-1}(N)$ is an IF $q$-neighborhood of $x_{(\alpha,\beta)}$.

Conversely, let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $V$ be an IF $q$-neighborhood of $f(x_{(\alpha,\beta)})$. Then $\text{int}(\text{cl}(V))$ is an IF regular open $q$-neighborhood of $f(x_{(\alpha,\beta)})$. Also, $\text{int}(\text{cl}(V))$ is an IF $\delta$-neighborhood of $f(x_{(\alpha,\beta)})$. By the hypothesis, $f^{-1}(\text{int}(\text{cl}(V)))$ is an IF $q$-neighborhood of $x_{(\alpha,\beta)}$. Since $f^{-1}(\text{int}(\text{cl}(V)))$ is an IF $q$-neighborhood of $x_{(\alpha,\beta)}$, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $U \leq f^{-1}(\text{int}(\text{cl}(V)))$. Thus there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $f(U) \leq \text{int}(\text{cl}(V))$. Hence $f$ is an IF almost continuous function.

**Theorem 3.6.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a bijection. Then the following statements are equivalent:

(1) $f$ is an IF almost continuous function.

(2) $f(\text{int}_\delta(U)) \leq \text{int}(f(U))$ for each IF set $U$ in $X$.

*Proof.* Trivial by Theorem 3.4.

Recall that a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be a *fuzzy almost strongly $\theta$-continuous* function if for each fuzzy point $x_\alpha$ in $X$ and each fuzzy open $q$-neighborhood $V$ of $f(x_\alpha)$, there exists an fuzzy open $q$-neighborhood $U$ of $x_\alpha$ such that $f(\text{cl}(U)) \leq \text{int}(\text{cl}(V))$ (See [14]).

**Definition 3.7.** A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be *intuitionistic fuzzy almost strongly $\theta$-continuous* if for each intuitionistic fuzzy point $x_{(\alpha,\beta)}$ in $X$ and each intuitionistic fuzzy open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$, there exists an intuitionistic fuzzy open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that

$$f(\text{cl}(U)) \leq \text{int}(\text{cl}(V)).$$

**Theorem 3.8.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent:

(1) $f$ is an IF almost strongly $\theta$-continuous function.

(2) $f(\text{cl}_\theta(A)) \leq \text{cl}_\delta(f(A))$ for each IF set $A$ in $X$.

(3) $\text{cl}_\theta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\delta(B))$ for each IF set $B$ in $Y$.

(4) $f^{-1}(\text{int}_\delta(B)) \leq \text{int}_\theta(f^{-1}(B))$ for each IF set $B$ in $Y$.

*Proof.* (1) $\Rightarrow$ (2). Let $x_{(\alpha,\beta)} \in \text{cl}_\theta(A)$. Suppose $f(x_{(\alpha,\beta)}) \notin \text{cl}_\delta(f(A))$. Then there exists an IF open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$ such that $V \tilde{q} f(A)$. Since $f$ is an IF almost strongly $\theta$ continuous function, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $f(\text{cl}(U)) \leq \text{int}(\text{cl}(V)) = V$. Since $f(A) \leq V^c \leq (f(\text{cl}(U)))^c$, we have $A \leq (f^{-1}(f(\text{cl}(U))))^c$. Thus $A \tilde{q} f^{-1}(f(\text{cl}(U)))$. Also, Since $\text{cl}(U) \leq f^{-1}(f(\text{cl}(U)))$, we have $A \tilde{q} \text{cl}(U)$. Since $x_{(\alpha,\beta)} \in \text{cl}_\theta(A)$, we have $A q \text{cl}(U)$. This is a contradiction.

(2) $\Rightarrow$ (3). Let $B$ be an IF set in $Y$. Then $f^{-1}(B)$ is an IF set in $X$. By (2), $f(\text{cl}_\theta(f^{-1}(B))) \leq \text{cl}_\theta(f(f^{-1}(B))) \leq \text{cl}_\theta(B)$. Thus we have $f(\text{cl}_\theta(f^{-1}(B))) \leq \text{cl}_\theta(f(f^{-1}(B))) \leq \text{cl}_\theta(B)$. Hence $\text{cl}_\theta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\delta(B))$.

(3) $\Rightarrow$ (4). Let $B$ be an IF set in $Y$. Then $B^c$ is an IF set in $Y$. By (3), $\text{cl}_\theta(f^{-1}(B^c)) \leq f^{-1}(\text{cl}_\delta(B^c))$ for each IF set $B$ in $Y$. Therefore $f^{-1}(\text{int}_\delta(B)) = (\text{cl}_\theta(f^{-1}(B^c)))^c \geq (f^{-1}(\text{cl}_\delta(B^c)))^c = \text{int}_\theta(f^{-1}(B))$.

(4) $\Rightarrow$ (1). Let $B$ be an IF set in $Y$. Then $B^c$ is an IF set in $Y$. By (4), $f^{-1}(\text{int}_\delta(B^c)) \leq \text{int}_\theta(f^{-1}(B^c))$. Thus $\text{cl}_\theta(f^{-1}(B^c)) \leq f^{-1}(\text{cl}_\delta(B^c))$. Hence $f$ is an IF almost strongly $\theta$-continuous function.

**Theorem 3.9.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent:

(1) $f$ is an IF almost strongly $\theta$-continuous function.

(2) The inverse image of every IF $\delta$-closed set in $Y$ is an IF $\theta$-closed set in $X$.

(3) The inverse image of every IF $\delta$-open set in $Y$ is an IF $\theta$-open set in $X$.

(4) The inverse image of every IF regular open set in $Y$ is an IF $\theta$-open set in $X$.

*Proof.* $(1) \Rightarrow (2)$. Let $B$ be an IF $\delta$-closed set in $Y$. Then $\mathrm{cl}_\delta(B) = B$. Since $f$ is an IF almost strongly $\theta$-continuous function, by Theorem 3.8, $\mathrm{cl}_\theta(f^{-1}(B)) \leq f^{-1}(\mathrm{cl}_\delta(B)) = f^{-1}(B)$. Thus $\mathrm{cl}_\theta(f^{-1}(B)) = f^{-1}(B)$. Hence $f^{-1}(B)$ is an IF $\theta$-closed set in $X$.

$(2) \Rightarrow (3)$. Let $B$ be an IF $\delta$-open set in $Y$. Then $B^c$ is an IF $\delta$-closed set in $Y$. By $(4)$, $f^{-1}(B^c) = (f^{-1}(B))^c$ is an IF $\theta$-closed set in $X$. Hence $f^{-1}(B)$ is an IF $\theta$-open set in $X$.

$(3) \Rightarrow (4)$. Immediate since IF regular open sets are IF $\theta$-open sets.

$(4) \Rightarrow (1)$. Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $V$ be an IF open $q$-neighborhood of $f(x_{(\alpha,\beta)})$. Then $\mathrm{int}(\mathrm{cl}(V))$ is an IF regular open $q$-neighborhood of $f(x_{(\alpha,\beta)})$. By $(4)$, $f^{-1}(\mathrm{int}(\mathrm{cl}(V)))$ is an IF $\theta$-open set in $X$. Then

$$x_{(\alpha,\beta)} \notin (f^{-1}(\mathrm{int}(\mathrm{cl}(V))))^c = \mathrm{cl}_\theta((f^{-1}(\mathrm{int}(\mathrm{cl}(V))))^c).$$

Put $\mathrm{int}(\mathrm{cl}(V)) = D$. Suppose $x_{(\alpha,\beta)} \in (f^{-1}(\mathrm{int}(\mathrm{cl}(V))))^c = f^{-1}(D^c)$. Then

$$\begin{aligned} f(x_{(\alpha,\beta)}) \in f(f^{-1}(D^c)) &= f(f^{-1}((\gamma_D, \mu_D))) \\ &= f((f^{-1}(\gamma_D), f^{-1}(\mu_D))) \\ &= (f(f^{-1}(\gamma_D)), f(f^{-1}(\mu_D))) \\ &\subseteq (\gamma_D, \mu_D). \end{aligned}$$

Let $f(x_{(\alpha,\beta)}) = y_{(\alpha_0,\beta_0)}$. Then $\alpha_0 \leq \gamma_D(y)$ and $\beta_0 \geq \mu_D(y)$. Since $V$ is an IF open set, $V \leq \mathrm{int}(\mathrm{cl}(V)) = D$. Thus $\mu_V \leq \mu_D$ and $\gamma_v \geq \gamma_D$. Thus $\alpha_0 \leq \gamma_V(y)$ and $\beta_0 \geq \mu_V(y)$. Since $V$ is an IF open $q$-neighborhood of $f(x_{(\alpha,\beta)})$, we have $f(x_{(\alpha,\beta)})qV$. Thus $y_{(\alpha_0,\beta_0)} \not\leq V^c = (\gamma_V, \mu_V)$. Hence $\alpha_0 > \gamma_V(y)$ and $\beta_0 < \mu_V(y)$. This is a contradiction. Therefore there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $\mathrm{cl}(U)\widetilde{q}(f^{-1}(\mathrm{int}(\mathrm{cl}(V))))^c$, i.e. $\mathrm{cl}(U) \leq f^{-1}(\mathrm{int}(\mathrm{cl}(V)))$. Then $f(\mathrm{cl}(U)) \leq \mathrm{int}(\mathrm{cl}(V))$. Hence $f$ is an IF almost strongly $\theta$-continuous function.

**Theorem 3.10.** A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is IF almost strongly $\theta$-continuous if and only if for each IF point $x_{(\alpha,\beta)}$ in $X$ and each IF $\delta$-neighborhood $N$ of $f(x_{(\alpha,\beta)})$, the IF set $f^{-1}(N)$ is an IF $\theta$-neighborhood of $x_{(\alpha,\beta)}$.

*Proof.* Let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $N$ be an IF $\delta$-neighborhood of $f(x_{(\alpha,\beta)})$. Then there exists an an IF regular open $q$-neighborhood $V$ of $f(x_{(\alpha,\beta)})$ such that $V \leq N$. Thus $\mathrm{int}(\mathrm{cl}(V)) \leq N$. Since $f$ is an IF almost strongly $\theta$ continuous function, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $f(\mathrm{cl}(U)) \leq \mathrm{int}(\mathrm{cl}(V))$. Thus $f(\mathrm{cl}(U)) \leq N$. Therefore, there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $\mathrm{cl}(U) \leq f^{-1}(N)$. Hence $f^{-1}(N)$ is an IF $\theta$-neighborhood of $x_{(\alpha,\beta)}$.

Conversely, let $x_{(\alpha,\beta)}$ be an IF point in $X$, and let $V$ be an IF open $q$-neighborhood of $f(x_{(\alpha,\beta)})$. Since $\mathrm{int}(\mathrm{cl}(V))$ is an IF regular open $q$-neighborhood of $f(x_{(\alpha,\beta)})$ and $\mathrm{int}(\mathrm{cl}(V)) \leq \mathrm{int}(\mathrm{cl}(V))$, $\mathrm{int}(\mathrm{cl}(V))$ is an IF $\delta$-neighborhood of $f(x_{(\alpha,\beta)})$. By the hypothesis, $f^{-1}(\mathrm{int}(\mathrm{cl}(V)))$ is an IF $\theta$-neighborhood of $x_{(\alpha,\beta)}$. Then there exists an IF open $q$-neighborhood $U$ of $x_{(\alpha,\beta)}$ such that $\mathrm{cl}(U) \leq f^{-1}(\mathrm{int}(\mathrm{cl}(V)))$. Therefore $f(\mathrm{cl}(U)) \leq \mathrm{int}(\mathrm{cl}(V))$. Hence $f$ is IF almost strongly $\theta$-continuous.

**Theorem 3.11.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a bijection. Then the following statements are equivalent:

(1) $f$ is an IF almost strongly $\theta$-continuous function.

(2) $\mathrm{int}_\delta(f(A)) \leq f(\mathrm{int}_\theta(A))$ for each IF set $A$ in $X$.

*Proof.* $(1) \Rightarrow (2)$. Let $A$ be an IF set in $X$. Then $f(A)$ is an IF set in $Y$. By Theorem 3.9, $f^{-1}(\mathrm{int}_\delta(f(A))) \leq \mathrm{int}_\theta(f^{-1}(f(A)))$. Since $f$ is one-to-one,

$$f^{-1}(\mathrm{int}_\delta(f(A))) \leq \mathrm{int}_\theta(f^{-1}(f(A))) = \mathrm{int}_\theta(A).$$

Since $f$ is onto,

$$\mathrm{int}_\delta(f(A)) = f(f^{-1}(\mathrm{int}_\delta(f(A)))) \leq f(\mathrm{int}_\theta(A)).$$

$(2) \Rightarrow (1)$. Let $B$ be an IF set in $Y$. Then $f^{-1}(B)$ is an IF set in $X$. By (2), $\mathrm{int}_\delta(f(f^{-1}(B))) \leq f(\mathrm{int}_\theta(f^{-1}(B)))$. Since $f$ is onto,

$$\mathrm{int}_\delta(B) = \mathrm{int}_\delta(f(f^{-1}(B))) \leq f(\mathrm{int}_\theta(f^{-1}(B))).$$

Since $f$ is one-to-one,

$$f^{-1}(\mathrm{int}_\delta(B)) \le f^{-1}(f(\mathrm{int}_\theta(f^{-1}(B)))) = \mathrm{int}_\theta(f^{-1}(B)).$$

By Theorem 3.9, $f$ is an IF almost strongly $\theta$-continuous function.

## 4. Conclusion

We characterized the intuitionistic fuzzy $\delta$-continuous functions in terms of IF $\delta$-closure and IF $\delta$-interior, or IF $\delta$-open and IF $\delta$-closed sets, or IF $\delta$-neighborhoods.

Moreover, we characterized the IF weakly $\delta$-continuous, IF almost continuous, and IF almost strongly $\theta$-continuous functions in terms of closure and interior.

## Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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