J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2013.20.4.287
Volume 20, Number 4 (November 2013), Pages 287–298

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

# DYNAMIC RISK MEASURES AND G-EXPECTATION

Ju Hong Kim

Abstract. A standard deviation has been a starting point for a mathematical definition of risk. As a remedy for drawbacks such as subadditivity property discouraging the diversification, coherent and convex risk measures are introduced in an axiomatic approach. Choquet expectation and $g$-expectations, which generalize mathematical expectations, are widely used in hedging and pricing contingent claims in incomplete markets. The each risk measure or expectation give rise to its own pricing rules. In this paper we investigate relationships among dynamic risk measures, Choquet expectation and dynamic $g$-expectations in the framework of the continuous-time asset pricing.

## 1. Introduction

Various kinds of risk measures have been proposed and discussed to measure or quantify the market risks in theoretical and practical perspectives. A starting point for a mathematical definition of risk is simply as standard deviation. Markowitz [19] used the standard deviation to measure the market risk in his portfolio theory but his method doesn't tell the difference between the positive and the negative deviation. Artzner et al. [2, 3] proposed a coherent risk measure in an axiomatic approach, and formulated the representation theorems. Frittelli [12] proposed sublinear risk measures to weaken coherent axioms. Heath [16] firstly studied the convex risk measures and Föllmer & Schied [9, 10, 11] and Frittelli & Rosazza Gianin [13] extended them to general probability spaces. They had weakened the conditions of positive homogeneity and subadditivity by replacing them with convexity.

There exist stochastic phenomena like Allais paradox [1] and Ellsberg paradox [8] which can not be dealt with linear mathematical expectation in economics. Choquet [6] introduced a nonlinear expectation called Choquet expectation which applied to many areas such as statistics, economics and finance. But Choquet expectation has a difficulty in defining a conditional expectation. Peng [21] introduced

Received by the editors October 4, 2013. Revised November 15, 2013. Accepted Nov. 16, 2013.
2010 *Mathematics Subject Classification.* 60G42, 60G44, 60H10.
*Key words and phrases.* coherent risk measure, dynamic risk measure, $g$-expectation, Choquet expectation, nonlinear expectation.

a nonlinear expectation, $g$-expectation which is a solution of a nonlinear backward stochastic differential equation. It's easy to define conditional expectation with Peng's $g$-expectation. In this paper, we show that Choquet expectation is equal to $g$-expectation under some conditions via $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation $\mathcal{E}$ satisfying $\mathcal{E}^\mu$-domination and translability condition.

The coherent (or convex) risk measure which is a static risk measures is defined in section 2. Peng's $g$-expectation, Choquet expectation and dynamic risk measure are introduced in section 3. The relationships between Choquet expectation and $g$-expectation are given as in the literature in section 4. It is shown that Choquet expectation is equal to $g$-expectation under some conditions via $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation $\mathcal{E}$ in section 5.

## 2. Static Risk Measures

Risk measures are introduced to measure or quantify investors' risky positions such as financial contracts or contingent claims. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T$ be a fixed horizon time. Assume that $\mathcal{X} = L^p(\Omega, \mathcal{F}, P)$, with $1 \le p \le +\infty$ is the space of financial positions to be quantified or measured. $L^p(\Omega, \mathcal{F}, P)$ is endowed with its norm topology for $p \in (1, +\infty)$ and with the weak topology $\sigma(L^\infty, L^1)$ for $p = +\infty$.

**Definition 2.1.** A *coherent risk measure* $\rho : \mathcal{X} \to \mathbb{R}$ is a mapping satisfying for $X, Y \in \mathcal{X}$

(1) $\rho(X) \ge \rho(Y)$ if $X \le Y$ (monotonicity),
(2) $\rho(X+m) = \rho(X) - m$ for $m \in \mathbb{R}$ (translation invariance),
(3) $\rho(X+Y) \le \rho(X) + \rho(Y)$ (subadditivity),
(4) $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \ge 0$ (positive homogeneity).

The subadditivity and the positive homogeneity can be relaxed to a weaker quantity, i.e. convexity

$$\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y) \quad \forall\lambda \in [0,1],$$

which means diversification should not increase the risk.

## 3. Peng's $g$-expectation and Choquet Expectation

Let $(W_t)_{t\ge0}$ a standard $d-$dimensional Brownian motion and $(\mathcal{F}_t)_{t\ge0}$ the augmented filtration associated with the one generated by $(W_t)_{t\ge0}$. Let $L^2_{\mathcal{F}}(T;\mathbb{R}^n)$ be the space of the adapted processes $(\xi_t)_{t\in[0,T]}$ such that

$$E\left[\int_0^T \|\xi_s\|^2 ds\right] < +\infty.$$

where $\|\cdot\|$ represents the Euclidean norm on $\mathbb{R}^n$.

Suppose that for $t \in [0,T]$, $L^2(\mathcal{F}_t) := L^2(\Omega, \mathcal{F}_t, P)$ is the space of real-valued, $\mathcal{F}_t$-measurable and square integrable random variables endowed with the $L^2$-norm $\|\cdot\|_2$ topology.

Let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ a function that $g \mapsto g(t,y,z)$ is measurable for each $(y,z) \in \mathbb{R} \times \mathbb{R}^n$ and satisfy the following conditions

$$|g(t,y,z) - g(t,\bar{y},\bar{z})| \leq K(|y-\bar{y}| + |z-\bar{z}|) \quad \forall t \in [0,T], \forall (y,z), (\bar{y},\bar{z}) \in \mathbb{R} \times \mathbb{R}^n, \text{ for some } K > 0, \tag{3.1a}$$

$$\int_0^T |g(t,0,0)|^2\, dt < \infty, \tag{3.1b}$$

$$\text{For each } (t,y) \in [0,T] \times \mathbb{R}, g(t,y,0) = 0. \tag{3.1c}$$

**Theorem 3.1** ([20])**.** *For every terminal condition $X \in L^2(\mathcal{F}_T)$ the following backward stochastic differential equation*

$$-dy_t = g(t,y_t,z_t)\, dt - z_t dW_t, \quad 0 \leq t \leq T \tag{3.2a}$$

$$y_T = X \tag{3.2b}$$

*has a unique solution $(y_t, z_t)_{t\in[0,T]} \in L^2_{\mathcal{F}}(T;\mathbb{R}) \times L^2_{\mathcal{F}}(T;\mathbb{R}^n)$.*

**Definition 3.2.** For each $X \in L^2(\mathcal{F}_T)$ and for each $t \in [0,T]$ $g-$expectation of $X$ and the conditional $g-$expectation of $X$ under $\mathcal{F}_t$ is respectively defined by

$$\mathcal{E}_g[X] := y_0, \quad \mathcal{E}_g[X|\mathcal{F}_t] := y_t,$$

where $y_t$ is the solution of the BSDE (3.2).

Since $g$-expectation and conditional $g$-expectation can be considered as the extension of classic mathematical expectation and conditional mathematical expectation, they preserve most properties of classic mathematical expectation and conditional mathematical expectation except the linearity.

**Definition 3.3.** A real-valued set function $c : \mathcal{F} \to [0,1]$ is called *capacity* if it satisfies (1) $c(A) \leq c(B)$ for $A \subset B$, (2) $c(\emptyset) = 0$ and $c(\Omega) = 1$.

**Definition 3.4.** A capacity is called *submodular* or 2-*alternating* if

$$c(A \cup B) + c(A \cap B) \leq c(A) + c(B).$$

**Definition 3.5.** Two measurable functions $X$ and $Y$ on $(\Omega, \mathcal{F})$ are called *comonotone* if there exists a measurable function $Z$ on $(\Omega, \mathcal{F})$ and increasing functions $f$ and $g$ on $\mathbb{R}$ such that

$$X = f(Z) \text{ and } Y = g(Z).$$

A risk measure $\rho$ on $L^p(\mathcal{F}_T)$ is called *comonotonic* if

$$\rho(X+Y) = \rho(X) + \rho(Y)$$

whenever $X$ and $Y$ are comonotonic.

Define the Choquet integral of the loss as

$$\rho(X) := \int (-X) dc.$$

Then $\rho : \mathcal{X} \to \mathbb{R}$ satisfies monotonicity, translation invariance and positive homogeneity, and other properties according to the given conditions.

(1) (Constant preserving) $\int \lambda dc = \lambda$ for constant $\lambda$.
(2) (Monotonicity) If $X \leq Y$, then $\int (-X) dc \geq \int (-Y) dc$.
(3) (Positive homogeneity) For $\lambda \geq 0$, $\int \lambda(-X) dc = \lambda \int (-X) dc$.
(4) (Translation invariance) $\int (-X+m)\, dc = \int (-X)\, dc + m, \quad m \in \mathbb{R}$.
(5) (Comonotone additivity) If $X$ and $Y$ are comonotone functions, then

$$\int [(-X) + (-Y)] dc = \int (-X) dc + \int (-Y) dc.$$

(6) (Subadditivity) If $c$ is submodular or concave function, then

$$\int (X+Y)\, dc \leq \int X\, dc + \int Y\, dc.$$

The static risk measures do not account for payoffs or new information according to the time evolution(refer to [25, 26]).

**Definition 3.6.** A dynamic risk measures are defined as the mappings $(\rho_t)_{t \in [0,T]}$ satisfying

(1) $\rho_t : L^p(\mathcal{F}_T) \to L^0(\Omega, \mathcal{F}_t, P)$, for all $t \in [0,T]$,
(2) $\rho_0$ is a static risk measure,
(3) $\rho_T(X) = -X \ P-$a.s., for all $X \in L^p(\mathcal{F}_T)$.

## 4. Nonlinear Expectations and Nonlinear Pricing

To quantify riskiness of financial positions, coherent (or convex) risk measures, Choquet expectation and $g$-expectation are widely used. It depends on practitioner's appropriate choices. The paper [5] shows that the pricing with the coherent risk measure is less than one with the Choquet expectation.

Denote the Choquet expectation $\mathcal{C}(\cdot)$ as $\mathcal{C}_g(\cdot)$ with respect to the capacity $V_g$ defined as

$$V_g(A) := \mathcal{E}_g[I_A] \quad \forall A \in \mathcal{F}_T.$$

**Theorem 4.1** ([5])**.** *If $\mathcal{E}_g[\cdot]$ is a coherent risk measure, then $\mathcal{E}_g[\cdot]$ is bounded by the Choquet expectation $\mathcal{C}_g(\cdot)$, that is*

$$\mathcal{E}_g[X] \leq \mathcal{C}_g(X), \quad X \in L^2(\Omega, \mathcal{F}, P).$$

*But if $\mathcal{E}_g[\cdot]$ is a convex risk measure, then the above inequality does not hold generally.*

**Theorem 4.2** ([15])**.** *Let $g$ be convex function with respect to $z$, independent of $y$ and deterministic. Let $g$ also satisfy (3.1). Then $\rho^g(X) \leq \mathcal{C}_g[-X]$ for $X \in L^2(\mathcal{F}_T)$ if and only if $\rho^g$ is a coherent risk measure. Here $\rho^g(X)$ is defined as $\rho^g(X) := \mathcal{E}_g[-X]$ for $X \in L^2(\mathcal{F}_T)$.*

Note that $\rho^g : L^2(\mathcal{F}_T) \mapsto \mathbb{R}$ is a coherent (or convex) risk measure if and only if $g$ is independent of $y$ and is positively homogeneous and subadditive (or convex) with respect to $z$ (see [23, 14, 22]).

The positive homogeneity and comonotonic additivity hold in the Choquet expectation. The time consistency holds in the $g$-expectation.

$$E[\xi + \eta] = E[\xi] + E[\eta] \quad \forall \xi, \eta \in L^2(\Omega, \mathcal{F}, P).$$

The above equality holds for the Choquet expectation if $\xi$ and $\eta$ are comonotonic. But if $g$ is nonlinear, the above equality does not hold for the $g$-expectation even if $\xi$ and $\eta$ are comonotonic. These facts means that $g$-expectation is more nonlinear than the Choquet expectation on $L^2(\Omega, \mathcal{F}, P)$ [15].

The following Lemmas (4.3) and (4.6), Proposition (4.4), and Theorem (4.5) are from the paper [5].

**Lemma 4.3.** *For any $X \in L^2(\Omega, \mathcal{F}_T, P)$, there exists unique $\eta \in L^2(\Omega, \mathcal{F}_t, P)$ such that*

$$\mathcal{E}_g[I_A X] = \mathcal{E}_g[I_A \eta] \quad \forall A \in \mathcal{F}_t.$$

*The $\eta$ is called the conditional $g$-expectation of $X$ and it is written as $\mathcal{E}_g[X|\mathcal{F}_t]$. This $\mathcal{E}_g[X|\mathcal{F}_t]$ is exactly the $y_t$ which is the solution of BSDE (3.2).*

**Proposition 4.4.** *Let $\mu = \{\mu_t\}_{t\in[0,T]}$ be a continuous functions. Suppose that $g(t,y,z) = \mu_t|z_t|$ and the process $(z_t)_{t\in[0,T]}$ is one dimensional. Then for any $\xi \in L^2(\Omega,\mathcal{F},P)$, the conditional $g$-expectation satisfies*

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = ess\ sup_{Q\in\mathcal{Q}} E_Q[\xi|\mathcal{F}_t] \ for\ \mu > 0.$$

*where $\mathcal{Q}$ is a set of probability measures defined as*

$$\begin{aligned}\mathcal{Q} \ &:= \ \Big\{Q^v \,\Big|\, \frac{dQ^v}{dP} := e^{-\frac{1}{2}\int_0^T |v_s|^2 ds + \int_0^T v_s dW_s}, \\ &\qquad v_t \ is\ \mathcal{F}_t - adapted \ \ and \ \ |v_t| \le \mu_t,\ a.e.\ t\in[0,T]\Big\}.\end{aligned}$$

**Theorem 4.5** ([5])**.** *Suppose that $g$ satisfies the given Hypotheses. Then there exists a Choquet expectation whose restriction to $L^2(\Omega,\mathcal{F},P)$ is equal to a $g$-expectation if and only if $g$ is independent of $y$ and is linear in $z$, i.e. there exists a continuous function $\nu(t)$ such that*

$$g(y,z,t) = \nu(t)z.$$

**Lemma 4.6.** *Suppose that $g$ is a convex (or concave) function. If $\mathcal{E}_g[\cdot]$ is comonotonic additive on $L^2_+(\Omega,\mathcal{F},P)$ (or $L^2_-(\Omega,\mathcal{F},P)$), then $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is also comonotonic additive on $L^2_+(\Omega,\mathcal{F},P)$ (or $L^2_-(\Omega,\mathcal{F},P)$) for any $t\in[0,T)$.*

**Corollary 4.7.** *Suppose that $g$ is a convex (or concave) function. If $\mathcal{E}_g[\cdot]$ is a Choquet expectation on $L^2_+(\Omega,\mathcal{F},P)$ (or $L^2_-(\Omega,\mathcal{F},P)$), then $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is also a Choquet expectation on $L^2_+(\Omega,\mathcal{F},P)$ (or $L^2_-(\Omega,\mathcal{F},P)$) for any $t\in[0,T)$.*

## 5. $\mathcal{F}_t$-consistent Expectation

In this section, an $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation $\mathcal{E}$ is defined as a nonlinear functional on $L^2(\mathcal{F}_T)$. We'll show that Choquet expectation is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation $\mathcal{E}$ under some conditions.

**Definition 5.1.** A *nonlinear expectation* is defined as a functional $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ satisfying

(1) (Monotonicity) If $X \ge Y$ $P$-a.s., then $\mathcal{E}(X) \ge \mathcal{E}(Y)$. Moreover, under the inequality $X \ge Y$, $\mathcal{E}(X) = \mathcal{E}(Y)$ if and only if $X = Y$ $P$-a.s..
(2) (Constancy) $\mathcal{E}(c) = c \quad \forall c \in \mathbb{R}$.

**Definition 5.2.** An $\{\mathcal{F}_t\}_{t\in[0,T]}$-*consistent expectation* is defined as the nonlinear expectation $\mathcal{E}$ such that if for any $X \in L^2(\mathcal{F}_T)$ and any $t \in [0,T]$ there exists $\eta \in L^2(\mathcal{F}_t)$ satisfying

$$\mathcal{E}[\mathbf{1}_A X] = \mathcal{E}[\mathbf{1}_A \eta] \quad \forall A \in \mathcal{F}_t. \tag{5.1}$$

The $\eta$ satisfying (5.1) is called *conditional* $\{\mathcal{F}_t\}_{t\in[0,T]}$-*consistent* expectation of $X$ under $\mathcal{F}_t$ and denoted by $\mathcal{E}[X|\mathcal{F}_t]$.

**Definition 5.3.** It is called that $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation $\mathcal{E}$ is *dominated* by $\mathcal{E}^\mu$ $(\mu > 0)$ if

$$\mathcal{E}[X+Y] - \mathcal{E}[X] \le \mathcal{E}^\mu[Y] \quad \forall X, Y \in L^2(\mathcal{F}_T)$$

where $\mathcal{E}^\mu$ is $g$-expectation with $g(t,y,z) = \mu|z|$.

An $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation $\mathcal{E}$ is called to satisfy the *translability condition* if

$$\mathcal{E}[X+\beta|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t] + \beta \quad \forall X \in L^2(\mathcal{F}_T),\ \forall \beta \in L^2(\mathcal{F}_t). \tag{5.2}$$

The following theorem tells us the relationships between conditional $g$-expectation and $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation.

**Theorem 5.4** ([7])**.** *Let $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ be a $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation. If $\mathcal{E}$ is $\mathcal{E}^\mu$-dominated for some $\mu > 0$ and if it satisfies translability condition (5.2), then there exists a unique $g$ which is independent of $y$, satisfies the assumptions (3.1) and $|g(t,z)| \le \mu|z|$ such that*

$$\mathcal{E}[X] = \mathcal{E}_g[X] \quad \text{and} \quad \mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}_g[X|\mathcal{F}_t] \quad \forall X \in L^2(\mathcal{F}_T).$$

**Theorem 5.5** ([11])**.** *For the Choquet integral with respect to a capacity $c$, the following are equivalent.*

(1) *$\rho_0(X) := \int(-X)\,dc$ is a convex risk measure on $L^2(\mathcal{F}_T)$.*
(2) *$\rho_0(X) := \int(-X)\,dc$ is a coherent risk measure on $L^2(\mathcal{F}_T)$.*
(3) *For $\mathcal{Q}_c := \{Q \in \mathcal{M}_{1,f} \,|\, Q[A] \le c(A)\ \forall A \in \mathcal{F}_T\}$,*

$$\int X\,dc = \sup_{Q\in\mathcal{Q}_c} E_Q[X] \quad \text{for } X \in L^2(\mathcal{F}_T). \tag{5.3}$$

(4) *The set function $c$ is submodular. In this case, $\mathcal{Q}_c = \mathcal{Q}_{max}$.*

The set $\mathcal{M}_{1,f} = \mathcal{M}_{1,f}(\Omega,\mathcal{F})$ in Theorem (5.3) is the one of all finitely additive set functions $Q : \mathcal{F} \to [0,1]$ which is normalized to $Q[\Omega] = 1$. The $\mathcal{Q}_{max}$ is defined as

$$\mathcal{Q}_{max} := \left\{ Q \in \mathcal{M}_{1,f} \,\middle|\, \sup_{X \in \mathcal{A}_\rho} E_Q[-X] = 0 \right\}$$

where $\mathcal{A}_\rho$ is defined as

$$\mathcal{A}_\rho := \{X \in L^2(\mathcal{F}_T) \,|\, \rho(X) \leq 0\}.$$

From the viewpoint of Proposition (4.4) and Theorem (4.5), the set $\mathcal{Q}_c$ of (5.3) is unnecessarily too large so that it could be reduced to a suitable set of probability measures for consistency, i.e.

$$\begin{aligned}(5.4)\quad \mathcal{Q}_c := \Big\{ Q^v \in \mathcal{M}_{1,f} \,\Big|\, & Q^v[A] \leq c(A)\ \forall A \in \mathcal{F}_T,\ \frac{dQ^v}{dP} := e^{-\frac{1}{2}\int_0^T |v_s|^2 ds + \int_0^T v_s dW_s}, \\ & v_t \text{ is } \mathcal{F}_t - \text{adapted and } |v_t| \leq \mu_t, \text{ a.e. } t \in [0,T] \\ & \text{for continuous functions } \mu_t > 0 \Big\}.\end{aligned}$$

It can be shown that $\mathcal{Q}_c$ is indeed the set of equivalent martingale measures by the following Proposition (5.6).

**Proposition 5.6** ([11]). *If $Q << P$ on $\mathcal{F}$, then $Q$ is equivalent to $P$ if and only if* $\frac{dQ}{dP} > 0 \quad P$*-a.s.*

Assume that the capacity $c$ is submodular. Under the new set $\mathcal{Q}_c$ as in (5.4), we define a nonlinear expectation $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ as

$$(5.5) \qquad \mathcal{E}[X] := \int X\, dc = \operatorname{ess}\sup_{Q \in \mathcal{Q}_c} E_Q[X], \quad X \in L^2(\mathcal{F}_T).$$

We will show that the above $\mathcal{E}[X]$ satisfies all the assumptions of Theorem (5.4). It is easy to show that $\mathcal{E}[X]$ satisfies the monotonicity and constancy in the Definition (5.1) but if $X \geq Y$, $\mathcal{E}[X] = \mathcal{E}[Y]$ if and only if $X = Y$ $P$-a.s.. Suppose that $X \geq Y$ and $\mathcal{E}[X] = \mathcal{E}[Y]$. We prove it contrapositively. Suppose $X = Y$ $P$-a.s. does not hold. Let $A = \{w \in \Omega \,|\, X \neq Y\} \in \mathcal{F}$. Then $E_Q[1_A X] > E_Q[1_A Y]$ for each $Q \in \mathcal{Q}_c$ and there exists a $r \in \mathbb{R}$ such that $E_Q[1_A X] > r > E_Q[1_A Y]$. By taking supremum on the left hand side first, we have $\operatorname{ess}\sup_{Q \in \mathcal{Q}_c} E_Q[1_A X] > r > E_Q[1_A Y]$ and so $\operatorname{ess}\sup_{Q \in \mathcal{Q}_c} E_Q[1_A X] > r \geq \operatorname{ess}\sup_{Q \in \mathcal{Q}_c} E_Q[1_A Y]$, it's a contradiction.

We need the stability property of a set $\mathcal{Q}_c$ to show that $\mathcal{E}[X]$ is a $\{\mathcal{F}_t\}_{t \in [0,T]}$-consistent expectation. In the following definitions, the stopping times $\sigma$ and $\tau$ can be replaced by $t \in [0,T]$ without any loss.

**Definition 5.7.** Let $Q_1$ and $Q_2$ be two equivalent probability measures and $\sigma$ be a stopping time. The probability measure

$$\tilde{Q}[A] := E_{Q_1}\left[Q_2[A|\mathcal{F}_\sigma]\right], \quad A \in \mathcal{F}_T,$$

is called the *pasting* of $Q_1$ and $Q_2$ in $\sigma$.

Note that by the monotone convergence theorem for conditional expectation $\tilde{Q}$ is a probability measure and

$$E_{\tilde{Q}}[Y] := E_{Q_1}\left[E_{Q_2}[Y|\mathcal{F}_\sigma]\right], \quad \forall Y \in L^2(\mathcal{F}_T), Y \geq 0.$$

**Definition 5.8.** A set $\mathcal{Q}$ of equivalent probability measures on $(\Omega, \mathcal{F})$ is called *stable* if, for any $Q_1, Q_2 \in \mathcal{Q}$ and the stopping time $\sigma$, also their pasting in $\sigma$ is contained in $\mathcal{Q}$.

**Proposition 5.9** ([11])**.** *The set $\mathcal{Q}_c$ of equivalent martingale measures is stable.*

**Theorem 5.10** ([11])**.** *Let $\mathcal{Q}$ be a set of equivalent probability measures. If $\mathcal{Q}$ is stable, then the following holds for $X \in L^2(\mathcal{F}_T)$*

$$\operatorname*{ess\,sup}_{Q\in\mathcal{Q}} E_Q[X|\mathcal{F}_t] = \sup_{Q\in\mathcal{Q}} E_Q[\operatorname*{ess\,sup}_{Q'\in\mathcal{Q}} E_{Q'}[X|\mathcal{F}_s]|\mathcal{F}_t] \quad \textit{for } t, s \in [0,T] \textit{ with } t \leq s.$$

From the Theorem (5.10), we can easily see that $\mathcal{E}[X]$ is a $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation condition (5.1), $\mathcal{E}[1_A X] = \mathcal{E}[1_A \cdot \mathcal{E}[X|\mathcal{F}_t]]\ \forall A \in \mathcal{F}_t$.

Let us show that $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ ($\mu > 0$). Since $\mathcal{E}[X+Y] - \mathcal{E}[X] \leq \operatorname{ess\,sup}_{Q\in\mathcal{Q}_c} E_Q[Y]$ and there exists $g$-expectation $\mathcal{E}^\mu$ with $g(t,y,z) = \mu z$ satisfying $\mathcal{E}^\mu[X] = \operatorname{ess\,sup}_{Q\in\mathcal{Q}_c} E_Q[Y]$ by Theorem (4.5), $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$. Note that $\mathcal{E}^\mu$-dominated nonlinear expectation $\mathcal{E}$ implies that $\mathcal{E}$ is lower semi-continuous [7].

Finally we show that $\{\mathcal{F}_t\}_{t\in[0,T]}$-consistent expectation $\mathcal{E}$ satisfies the translability condition. Let $X \in L^2(\mathcal{F}_T)$ and $\beta \in L^2(\mathcal{F}_t)$. Then by the definition of $\mathcal{E}$ we have

$$\mathcal{E}[(X+\beta)|\mathcal{F}_t] = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_c} E_Q[(X+\beta)|\mathcal{F}_t] = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_c} E_Q[X|\mathcal{F}_t] + \beta = \mathcal{E}[X|\mathcal{F}_t] + \beta.$$

Therefore, the nonlinear expectation $\mathcal{E}$ defined as (5.5) satisfies the all the conditions of Theorem (5.4). Thus the results so far can be summarized in the following Theorem (5.11).

**Theorem 5.11.** *Let the nonlinear expectation $\mathcal{E}$ be defined as (5.5). Then there exists a unique $g$ which is independent of $y$, satisfies the assumptions (3.1) and*

$|g(t,z)| \le \mu|z|$ *such that*

$$\mathcal{E}[X] := \int X\,dc = \mathcal{E}_g[X] \text{ and } \mathcal{E}[X|\mathcal{F}_t] := \int X|_{\mathcal{F}_t}\,dc = \mathcal{E}_g[X|\mathcal{F}_t] \quad \forall X \in L^2(\mathcal{F}_T).$$

Note that the generator $g$ in Theorem (5.11) should be the form of $g(t,y,z) = \mu_t z$ which is linear in $z$ and so $\mathcal{E}_g = \mathcal{E}^\mu$ to be consistent to the results of Theorem (4.5).

In fact, for $g(t,y,z) = \mu_t z$, let us consider the BSDE

$$Y_t = X + \int_t^T \mu_s z_s\,ds - \int_t^T z_s\,dW_s, \quad X \in L^2(\mathcal{F}_T). \tag{5.6}$$

The above differential equation (5.6) is reduced to

$$Y_t = X - \int_t^T z_s d\tilde{W}_s, \quad \tilde{W}_t = W_t - \int_0^t \mu_s\,ds.$$

By Girsanov's Theorem, $(\tilde{W}_t)_{0\le t\le T}$ is a $Q$-Brownian motion under $Q$ defined as

$$\frac{dQ}{dP} = exp\left[-\frac{1}{2}\int_0^T \mu_s^2 ds + \int_0^T \mu_s dW_s\right].$$

Therefore we have the relations

$$\mathcal{E}_g[X] = E_Q[X], \quad \mathcal{E}_g[X|\mathcal{F}_t] = E_Q[X|\mathcal{F}_t]$$

which means that $g$-expectation is a classical mathematical expectation.

**Proposition 5.12** ([23]). *Let the risk measure $\rho_t^g(X)$ be defined as*

$$\rho_t^g(X) := \mathcal{E}_g[-X|\mathcal{F}_t], \quad \forall X \in L^2(\mathcal{F}_T),\ \forall t \in [0,T]$$

*where $g$ satisfies the conditions (3.1). Moreover, if $g$ is sublinear in $(y,z)$, i.e. positively homogeneous in $(y,z)$ and subadditive in $(y,z)$, then $(\rho_t^g)_{t\in[0,T]}$ is a dynamic coherent and time-consistent risk measure.*

Note that if $g$ satisfies both positive homogeneity and subadditivity, $g$ is independent of $y$. The proposition (5.12) and Theorem (4.2) tells us that for Theorem (5.11) to hold the linearity of $g$ is necessary.

## Acknowledgment

This work was supported by the research grant of Sungshin Women's University in 2013.

## References


1. M. Allias: Le comportement de l'homme rationnel devant le risque: Critique des postulates et axiomes de l'ecole americaine. *Econometrica* **21** (1953), 503-546.
2. P. Artzner, F. Delbaen, J.-M. Eber & D. Heath: Thinking coherently. *Risk* **10** (1997), 68-71.
3. P. Artzner, F. Delbaen, J.-M. Eber & D. Heath: Coherent measures of risk. *Mathematical Finance* **9** (1999), 203-228.
4. A. Campana & P. Ferretti: Distortion risk measures and discrete risks. Working paper.
5. Z. Chen, T. Chen & M. Davison: Choquet expectation and Peng's $g$-expectation. *The Annals of Probability* **33** (2005), 1179-1199.
6. G. Choquet: Theory of capacities. *Ann. Inst. Fourier (Grenoble)* **5** (1953), 131-195.
7. F. Coquet, Y. Hu, J. Mémin & S. Peng: Filtration consistent nonlinear expectations and related $g$-expectations. *Probability Theory and Related Fields* **123** (2002), 1-27.
8. D. Ellsberg: Risk, ambiguity, and the Savage axioms. *Quart. J. Econom.* **75** (1961), 643-669.
9. H. Föllmer and A. Schied: Convex measures of risk and trading constraints. *Finance & Stochastics* **6** (2002), 429–447.
10. H. Föllmer & A. Schied: *Robust preferences and convex measures of risk, in: Advances in Finance and Stochastics.* K. Sandmann and P.J. Schönbucher eds., Springer-Verlag, 2002.
11. H. Föllmer & A. Schied: *Stochastic Finance: An introduction in discrete time.* Walter de Gruyter, Berlin, 2004.
12. M. Frittelli: *Representing sublinear risk measures and pricing rules.* http://www.mat.unimi.it/users/frittelli/pdf/Sublinear2000.pdf, 2000.
13. M. Frittelli & E. Rosazza Gianin: Putting order in risk measures. *Journal of Banking & Finance* **26** (2002), 1473-1486.
14. L. Jiang: Convexity, translation invariance and subadditivity for $g$-expectation and related risk measures. *Annals of Applied Provability* **18** (2006), 245-258.
15. K. He, M. Hu & Z. Chen: The relationship between risk measures and choquet expectations in the framework of $g$-expectations. *Statistics and Probability Letters* **79** (2009), 508-512.
16. D. Heath: *Back to the future.* Plenary lecture at the First World Cogress of the Bachelier Society, Paris, June 2000.
17. N. El Karoui, S. G. Peng & M.-C. Quenez: Backward stochastic differential equations in finance. *Math. Finance* **7** (1997), 1-71.
18. S. Kusuoka: On law invariant coherent risk measures. *Adv. Math. Econ.***3** (2001), 83-95.
19. H. Markowitz: Portfolio selection. *The Journal of Finance* **26** (1952), 1443-1471.

20. E. Pardoux & S.G. Peng: Adapted solution of a backward stochastic differential equation. *Systems and Control Letters* **14** (1990), 55-61.
21. S. Peng: *Backward SDE and related g-expectations, in: Backward stochastic differential equations.* N. El Karoui and L. Mazliak eds., Pitman Res. Notes Math. Ser., **364** (1997), 141-159.
22. S. Peng: *Modeling Derivatives Pricing Mechanisms with their Generating Functions.* arXiv:math/0605599 [math.PR], May 2006.
23. E. Rosazza Gianin: Some examples of risk measures via $g$-expectations. *Insurance: Mathematics and Economics* **39** (2006), 19-34
24. S. Wang: Premium calculation by transforming the layer premium density. *Astin Bulletin* **26** (1996), 71-92.
25. F. Riedel: Dynamic coherent risk measures. *Stochastic Processes and their Applications* **112** (2003), 185-200.
26. T. Wang: *A class of dynamic risk measures.* http://web.cenet.org.cn/upfile/57263.pdf (1999).


DEPARTMENT OF MATHEMATICS, SUNGSHIN WOMEN'S UNIVERSITY, SEOUL 136-742, REPUBLIC OF KOREA
*Email address*: jhkkim@sungshin.ac.kr