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Numerical Invariants on a Ring $F[[X, Y]]/\langle XY(X+Y)\rangle$ for a Field F

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Abstract

We provide a complete proof that the existence of a system of parameters, which is a reduction of the maximal ideal, is essential in Koh-Lee's conjecture.

Key words : Column-Row Invariants, Hypersurface Ring, Minimal Reduction of a Maximal Ideal

1. Basic Definitions and Facts

In this section, we rewrite a definition and some facts as a background for readers' convenience[1].

Throughout this paper, we assume that $(A, \mathfrak{m})$ is a commutative Noetherian local ring, and all modules are unitary. We first state the following result[2], on which the definitions of our invariants are based:

Theorem 1.1.[2] Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then there is an integer $t \geq 1$ such that for each finitely generated A-module M of infinite projective dimension, the ideal generated by the entries of the map φ_i is not contained in $\mathfrak{m}^t$ for all $i > 1 + depth(A)$ ($i > depth(A)$, if A is Cohen-Macaulay), where

$$(F_\bullet, \varphi_\bullet): \cdots \rightarrow A^{n_{j+1}} \xrightarrow{\varphi_{j+1}} A^{n_j} \xrightarrow{\varphi_j} A^{n_{j-1}} \rightarrow \cdots \rightarrow A^{n_0} \rightarrow 0$$

is a minimal (free) resolution of M.

Using the argument of the above theorem, some new numerical invariants of local rings were introduced[2]. They are $col(A)$ and $row(A)$ associated with the columns and rows, respectively, of the maps in infinite minimal resolutions. Two more invariants[3] were defined, say $crs(A)$ and $drs(A)$, which are associated with the Cyclic modules determined by Regular Sequences and their Matlis duals, respectively. We remark here that the 'dual' invariants $row(A)$ and $drs(A)$, which correspond to $col(A)$ and $crs(A)$ respectively, can be defined using the finitely generated modules of finite injective dimension and the vanishing of Ext-modules.

More precisely, we have the following definitions :

Definition 1.2.[3] In defining $row(A)$ and $drs(A)$ below, we assume that A is a Cohen-Macaulay ring. We denote by $\varphi_i(M)$ the i-th map in a minimal resolution of a finitely generated A-module M. We also use the usual notation $Soc(M) := Hom_A(A/\mathfrak{m}, M)$ to denote the socle of M.

i) $col(A) := \inf\{t \geq 1$: for each finitely generated A-module M of infinite projective dimension, each column of $\varphi_i(M)$ contains an element outside $\mathfrak{m}^t$, for all $i > 1 + depth(A)\}$

$row(A) := \inf\{t \geq 1$: for each finitely generated A-module M of infinite projective dimension, each row of $\varphi_i(M)$ contains an element outside $\mathfrak{m}^t$, for all $i > 1 + depth(A)\}$

ii) $crs(A) := \inf\{t \geq 1 : Soc(A/(x)) \not\subset \mathfrak{m}^t(A/(x))$ for some maximal regular sequence $x = x_1, \cdots, x_d\}$.

$drs(A) := \inf\{t \geq 1 : Soc((A/(x))^\vee) \not\subset \mathfrak{m}^t((A/(x))^\vee)$ for some system of parameters $x = x_1, \cdots, x_d\}$.

When A is regular local, we interpret the above definition as $col(A) = row(A) = 1$.

These invariants are related as follows:

Proposition 1.3.[3] Let $(A, \mathfrak{m})$ be a Noetherian local ring.

i) $1 \leq col(A) \leq crs(A) < \infty$.

ii) If A is Cohen-Macaulay, then
$1 \leq row(A) \leq drs(A) < \infty$.

iii) A is a regular local ring if and only if any,

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equivalently all, of the invariants in i) and ii) is 1.

In view of the above proposition, it seems natural to consider the following conjecture:

Koh-Lee Conjecture.[3] Let (A,m) be a Noetherian local ring such that some system of parameters is a reduction of m. Then

(i) $col(A) = crs(A)$,

(ii) if A is Cohen-Macaulay, $row(A) = drs(A)$.

Koh and Lee proved that the conjecture is in the affirmative in a few cases:(1) (i) If $depth(A) = 0$, then $col(A) = crs(A)$, and (ii) If $\dim A = 0$, then $row(A) = drs(A)$[3]. (2) Let (A,m) be a non-regular Cohen-Macaulay local ring such that some system of parameters is a reduction of m. Then A is of minimal multiplicity, i.e., $mult(A) = 1 + edimA - \dim A$, if and only if all of its four invariants are equal to 2[3]. (3) If (A,m) is a hypersurface ring, then $col(A) = mult(A)$. Moreover, if some system of parameters is a reduction of m, then all four invariants of A are equal to $mult(A)$[3]. (4) Let $R = k[[t^e, t^{e+1}, t^{(e-1)e-1}]]$, where k is a field and $e \geq 4$. Then (i) $col(R) = 2 = crs(R)$, (ii) $row(R) = e-1 = drs(R)$[1].

Remark 1.4. S. Ding[4] defined the numerical invariant called the index of a Gorenstein local ring R: $index(R)$ is the smallest positive integer t such that $\delta(R/m^t) \neq 0$, where $\delta(M)$ is defined to be the maximal rank of free summands of X_M in a minimal Cohen-Macaulay approximation of a finite R-module M. One[5] conjectures the following:(Ding's Conjecture[5]) Let R be a Gorenstein local ring. Is $index(R)$ the minimum of all integers n for which there exists a system of parameters x of R such that $m^t \subseteq (x)$? This minimum number is called the generalized Loewy length of R, and denoted by $\ell\ell(R)$. Thus the question can be written simply as $index(R) = \ell\ell(R)$. It is shown that if Koh-Lee conjecture holds, then Ding's conjecture does[6].

2. On a Ring $F[[X,Y]]/\langle XY(X+Y)\rangle$ for a Field F

M. Hashimoto and A. Shida[7] showed that Ding's conjecture may not hold unless the residue field of R is infinite; their example is $R = F[[X,Y]]/(XY(X+Y))$, where F is a field with two elements. (We note here that Ding's conjecture is still open with the additional condition that the residue field is infinite.) Using this example R, it is remarked with a simple explanation[3] that Koh-Lee conjecture does not hold in R, which has no a system of parameter that is a reduction of m. In this section, we provide a complete and elementary proof that the existence of a system of parameters, which is a reduction of the maximal ideal, is essential in Koh-Lee's conjecture since its proof is simply sketched[3], and not perfect.

Throughout this section, we assume that F is a field with two elements, and $F[[X,Y]]/\langle XY(X+Y)\rangle$, unless specified. We note that R is a hypersurface ring of dimension 1.

We will show the following main theorem:

Main Theorem. Let $R = F[[X,Y]]/\langle XY(X+Y)\rangle$. Then, $col(R) = 3$, and $crs(R) = 4$, i.e., Koh-Lee conjecture does not hold in R, which implies that the assumption about a minimal reduction is essential.

To prove the above theorem, we need some preparation. We first show that R has no a system of parameter that is a reduction of m. We first recall that a system of parameters $x = x_1, \cdots, x_n$ of a local ring (A,m) is a reduction of the maximal ideal m if the ideal (x) is a reduction of m, i.e. $(x)m^r = m^{r+1}$ for some r. (It is known that the minimal reduction of m exists if a residue field A/m is infinite[8].)

Notation. Let R be a ring of power series, and $f \in R$. We denote

$$mdeg(f) = \min\{i : a_i \neq 0, \text{ where } f(x) = \sum_{i=0}^{\infty} a_i x^i \ \}.$$

Proposition 2.1. Let m be a maximal ideal in $F[[X,Y]]/\langle XY(X+Y)\rangle$. Then for any $f \in m$, there is no nonnegative integer r such that $m^{r+1} = (f)m^r$, i.e., there is no minimal reduction of the maximal ideal m.

Proof. Let x and y denote the images of X and Y, respectively in $F[X,Y]/\langle XY(X+Y)\rangle$. Suppose there is $f \in m$ such that $m^{r+1} = (f)m^r$ for some nonnegative integer r. Then $mdeg(f) = 1$. There are three cases:

Case1: $f = x + f^*$, where $mdeg(f^*) \geq 2$.

Since $y^{r+1} \in m^{r+1} = (f)m^r$, $y^{r+1} = fg$ for some $g \in m^r$. Then $y^{r+1} = (x + f^*)g$, and hence $y^{r+1} = f^*(0,y)g(0,y)$ by putting $x = 0$. Note that either $f^*(0,y)g(0,y) = 0$ or $mdeg(f^*(0,y)g(0,y)) \geq r+2$. But $mdeg(y^{r+1}) = r+1$. This is a contradiction.

Case2: $f=y+f^*$, where $mdeg(f^*)\geq 2$.
Since $x^{r+1}\in m^{r+1}=(f)m^r$, $x^{r+1}=fg$ for some $g\in m^r$. Then $x^{r+1}=(y+f^*)g$, and hence $x^{r+1}=f^*(x,0)g(x,0)$ by putting $y=0$. Note that either $f^*(0,y)g(0,y)=0$ or $mdeg(f^*(x,0)g(x,0))$ $r+2$. But $mdeg(x^{r+1})=r+1$. This is a contradiction.
Case3: $f=x+y+f^*$, where $mdeg(f^*)\geq 2$.
Let $x+y=z$. Note that $xz(x+z)=xy(x+y)=0$ and $F[[x,z]]=F[[x,x+y]]=F[[x,y]]$.
Then

$$f=z+f^*\text{ in }F[[x,z]].$$

This is the same case as the above case. Thus this is also a contradiction.

It is known that $col(A)=row(A)$ and $crs(A)=drs(A)$ over a Gorenstein local ring A[3]. Since $R=F[[X,Y]]/\langle XY(X+Y)\rangle$ is a hypersurface ring, which is Gorenstein, we will compute the value of $drs(R)$ as it is easy to handle with.

In the following, we show $drs(R)>3$.

Theorem 2.2. Let m be a maximal ideal in $F[[X,Y]]$. Then for any $f\in m$, $\langle X,Y\rangle^3\not\subseteq\langle f,XY(X+Y)\rangle$. In particular, if $R=F[[X,Y]]/\langle XY(X+Y)\rangle$ and m_R is a maximal ideal of R, then m_R^3 for any $r\in m_R^3$, i.e., $drs(R)>3$.

Proof. We will prove $\langle X,Y\rangle^3\not\subseteq\langle f,XY(X+Y)\rangle$ for all , $f\in m-m^2, f\in m^2-m^3$, $f\in m^3-m^4$ or $f\in m^4$.
Step 1. Let $f\in m-m^2$. Then $mdeg(f)=1$. There are two cases:
Case1: $f=X+f^*$, where $mdeg(f^*)\geq 2$.
Suppose $Y^3\in\langle f,XY(X+Y)\rangle$. Then

$$Y^3=(X+f^*)g_1+XY(X+Y)g_2\text{ with }g_1,g_2\in F[[X,Y]].$$

If $g_1=0$, then g_2 must have a nonzero constant term since $\deg Y^3=3$, and so $Y^3=XY(X+Y)$, which is a contradiction. If $mdeg(g_1)\leq 1$, then the minimum degree of the right side is equal or less than 2 while the minimum degree of Y^3 is 3. Thus, we must have $mdeg(g_1)\geq 2$. Also, from $Y^3=(X+f^*)g_1+XY(X+Y)g_2$ we know $Y^3=f^*(0,Y)g_1(0,Y)$. But, $mdeg(Y^3)=3$ and $mdeg(f^*(0,Y)g_1(0,Y))\geq 4$ unless $f^*(0,Y)g_1(0,Y)=0$. This is a contradiction. Thus $Y^3\not\in\langle f,XY(X+Y)\rangle$. Similarly, if $f=Y+f^*$, then we can show $X^3\not\in\langle f,XY(X+Y)\rangle$.
Case2: $f=X+Y+f^*$, where $mdeg(f^*)\geq 2$.
Suppose $X^3\in\langle f,XY(X+Y)\rangle$. Then

$$X^3=(X+Y+f^*)g_1+XY(X+Y)g_2\text{ with }g_1,g_2\in F[[X,Y]]$$

Let $X+Y=Z$. Then $Y=X+Z$ and

$$X^3=(Z+f^*)g_1+XZ(X+Z)g_2\text{ in }F[[X,Z]]$$

This is the same case as the above case, which is a contradiction. Thus $X^3\not\in\langle f,XY(X+Y)\rangle$.
Step 2. Let $f\in m^2-m^3$. Then $mdeg(f)=2$. There are three cases:
Case1: $f=Xf'+f^*$, where $mdeg(Xf')=2=\deg(Xf')$ and $mdeg(f^*)\geq 3$.
Suppose $Y^3\in\langle f,XY(X+Y)\rangle$. Then

$$Y^3=(Xf'+f^*)g_1+XY(X+Y)g_2\text{ with }g_1,g_2\in F[[X,Y]].$$

If $g_1=0$, then g_2 must have a nonzero constant term since $\deg Y^3=3$, and so $Y^3=XY(X+Y)$, which is a contradiction. If $mdeg(g_1)=0$, then the minimum degree of the right side is 2 while the minimum degree of Y^3 is 3. Thus, we must have $mdeg(g_1)\geq 1$. Also, from $Y^3=(Xf'+f^*)g_1+XY(X+Y)g_2$ we know $Y^3=f^*(0,Y)g_1(0,Y)$. But, $mdeg(Y^3)=3$ and $mdeg(f^*(0,Y)$ $g_1(0,Y))\geq 4$ unless $f^*(0,Y)g_1(0,Y)=0$. This is a contradiction. Thus, $Y^3\not\in\langle f,XY(X+Y)\rangle$. Similarly, if $f=Yf'+f^*$, then we can show $X^3\not\in\langle f,XY(X+Y)\rangle$.
Case2: $f=X^2+Y^2+f^*$, where $mdeg(f^*)\geq 3$.
If $X^3\in\langle f,XY(X+Y)\rangle$, then

$$\begin{aligned}X^3&=(X^2+Y^2+f^*)g_1+XY(X+Y)g_2\\&=((X+Y)^2+f^*)g_1+XY(X+Y)g_2\text{ with}\\&g_1,g_2\in F[[X,Y]]\end{aligned}$$

Let $X+Y=Z$. Then $Y=X+Z$ and

$$X^3=(Z^2+f^*)g_1+XZ(X+Z)g_2\text{ in }F[[X,Z]]$$

This is the same case as the above case. Thus,

$X^3 \notin \langle f, XY(X+Y)\rangle$.

Case3: $f = X^2+XY+Y^2+f^*$, where $mdeg(f^*) \geq 3$. Suppose $X^2Y \in \langle f, XY(X+Y)\rangle$. Then

$$X^2Y = (X^2+XY+Y^2+f^*)g_1 + XY(X+Y)g_2 \text{ with } g_1, g_2 \in F[[X,Y]].$$

If $g_1 = 0$ or $mdeg(g_1) \geq 2$, then g_2 must have a nonzero constant term since $\deg Y^3 = 3$, and so $X^2Y = XY(X+Y)$, which is a contradiction. If $mdeg(g_1) = 0$, the minimum degree of the right side is 2 while the minimum degree of Y^3 is 3. Thus, we must have $mdeg(g_1) = 1$. There are two subcases:

subcase $g_1 = X+g_1^*$ and $[g_2 = 0$ or $mdeg(g_2) \geq 1]$, where $mdeg(g_1^*) \geq 2$.

Since either $XY(X+Y)g_2 = 0$ or $mdeg(XY(X+Y)g_2) \geq 4$, we must have $X^2Y = (X^2+XY+Y^2)X$ since $\deg Y^3 = 3$. This is a contradiction. Similarly, we can show both $g_1 = Y+g_1^*$ and $g_1 = X+Y+g_1^*$ are also contradiction when $g_2 = 0$ or $mdeg(g_2) = 1$

subcase $g_1 = X+g_1^*$ and $mdeg(g_2) = 0$, where $mdeg(g_1^*) \geq 2$.

Then we must have $X^2Y = (X^2+XY+Y^2)X + XY(X+Y)$, which is a contradiction. Similarly, we can show both $g_1 = Y+g_1^*$ and $g_1 = X+Y+g_1^*$ are also contradiction when $mdeg(g_2) = 0$.

Thus, $X^2Y \notin \langle f, XY(X+Y)\rangle$.

Step 3. Let $f \in m^3 - m^4$. Then $mdeg(f) = 3$. There are three cases:

Case1: $f = Xf' + f^*$, where $mdeg(Xf') = 3 = \deg(Xf')$ and $mdeg(f') \geq 4$.

Suppose $Y^3 \in \langle f, XY(X+Y)\rangle$, then

$$Y^3 = (Xf' + f^*)g_1 + XY(X+Y)g_2 \text{ with } g_1, g_2 \in F[[X,Y]].$$

We may assume that $g_1 \neq 0$. Then $Y^3 = f^*(0,Y)g_1(0,Y)$. But, $mdeg(Y^3) = 3$ and $mdeg(f^*(0,Y)g_1(0,Y)) \geq 4$ unless $f^*(0,Y)g_1(0,Y) \neq 0$ This is a contradiction. Thus, $Y^3 \notin \langle f, XY(X+Y)\rangle$. Similarly, if $f = Yf' + f^*$, then we can show $X^3 \notin \langle f, XY(X+Y)\rangle$.

Case2: $f = X^3+Y^3+f^*$, where $mdeg(f^*) \geq 4$.

If $X^3 \in \langle f, XY(X+Y)\rangle$, then

$$X^3 = (X^3+Y^3+f^*)g_1 + XY(X+Y)g_2 \text{ with } g_1, g_2 \in F[[X,Y]].$$

Let $X+Y = Z$. Then $Y = X+Z$ and

$$X^3 = (Z(Z^2+ZX+X^2)+f^*)g_1 + XZ(X+Z)g_2 \text{ in } F[[X,Y]].$$

This is the same case as the above case. Thus, $X^3 \notin \langle f, XY(X+Y)\rangle$.

Similarly, if $f = X^3+X^2Y+XY^2+Y^3+f^* = (X+Y)^3+f^*$, then we can show $X^3 \notin \langle f, XY(X+Y)\rangle$.

Case3: $f = X^3+X^2Y+Y^3+f^*$, where $mdeg(f^*) \geq 4$. Suppose $X^3 \in \langle f, XY(X+Y)\rangle$. Then

$$X^3 = (X^3+X^2Y+Y^3+f^*)g_1 + XY(X+Y)g_2 \text{ with } g_1, g_2 \in F[[X,Y]]$$

We may assume that $g_1 \neq 0$ or $g_2 \neq 0$. If $mdeg(g_1) \neq 0$ and $mdeg(g_2) \neq 0$, the minimum degree of right side is equal or greater than 4 while the minimum degree of X^3 is 3. Thus we must have $mdeg(g_1) = 0$ or $mdeg(g_2) = 0$. There are three subcases:

Subcase $mdeg(g_1) = 0$ and $mdeg(g_2) \neq 0$. Then we have $X^3 = X^3+X^2Y+Y^3$, which is a contradiction.

Subcase $mdeg(g_1) \neq 0$ and $mdeg(g_2) = 0$. Then we have $X^3 = XY(X+Y)$, which is a contradiction.

Subcase $mdeg(g_1) = 0$ and $mdeg(g_2) = 0$. Then we have $X^3 = X^3+X^2Y+Y^3+XY(X+Y)$, which is a contradiction.

Thus, $X^3 \notin \langle f, XY(X+Y)\rangle$. Similarly, if $f = X^3+XY^2+Y^3+f^*$, then we can show $X^3 \notin \langle f, XY(X+Y)\rangle$.

Step 4. $f \in m^4$. Then $mdeg(f) \geq 4$. If $X^3 \in \langle f, XY(X+Y)\rangle$,

$$X^3 = fg_1 + XY(X+Y)g_2 \text{ with } g_1, g_2 \in F[[X,Y]].$$

Since either $fg_1 = 0$ or $mdeg(fg_1) \geq 4$, $mdeg(g_2) = 0$ and hence we have $X^3 = XY(X+Y)$, which is a contradiction. Thus, $X^3 \notin \langle f, XY(X+Y)\rangle$.

In all, $\langle X, Y\rangle^3 \not\subset \langle f, XY(X+Y)\rangle$ for all $f \in m$.

The next theorem shows that $drs(R)$ is bounded by 4.

Theorem 2.3. $\langle X, Y\rangle^4 \subset \langle Y-X^2, XY(X+Y)\rangle$ in $F[[X, Y]]$. In particular, if $R = F[[X, Y]]/\langle XY(X+Y)\rangle$ and m_R is a maximal ideal of R, then $m_R^4 \subseteq (y-x^2)$ in R, i.e., $drs(R) \leq 4$.

Proof. It is enough to show that X^4, Y^4, X^3Y, X^2Y^2, and XY^3 are in $\langle Y-X^2, XY(X+Y)\rangle$. First, we claim $X^4 \in \langle Y-X^2, XY(X+Y)\rangle$. Indeed, $1+X$ is a unit in $F[[X, Y]]$ and

$$
\begin{aligned}
&(Y-X^2)(X^2+XY(1+X)^{-1}) + XY(X+Y)(1+X)^{-1} \\
&= X^4 + X^2Y + X^2Y(1+X)^{-1} + X^3Y(1+X)^{-1} \\
&= X^4 + X^2Y(1+(1+X)^{-1} + X(1+X)^{-1}) \\
&= X^4 + X^2Y(1+(1+X)^{-1}(1+X)) \\
&= X^4
\end{aligned}
$$

Thus $X^4 \in \langle Y-X^2, XY(X+Y)\rangle$. Note that

$$
\begin{aligned}
&Y^2 = (Y-X^2)(Y+X^2) + X^4, \\
&X^3Y = (Y-X^2)X^3 + X^5, \\
&X^2Y^2 = XY(X+Y)X - X^3Y, \quad \text{and} \\
&XY^3 = XY(X+Y)Y - X^2Y^2.
\end{aligned}
$$

Thus Y^4, X^3Y, X^2Y^2, and XY^3 are also in $\langle Y-X^2, XY(X+Y)\rangle$. This completes the proof.

To complete the proof of our main theorem, we need the following fact:

Theorem 2.4.[3] If (A, m) is a hypersurface ring, then $col(A) = mult(A)$. Moreover if A has a system of parameters which is a reduction of m, e.g. A/m is infinite, then all six invariants of A are equal to $mult(A)$.

(Proof of Main Theorem) By the above theorem, $col(R) = mult(R) = 3$. Also, $crs(R) = drs(R) = 4$ by Theorem 2.2, and Theorem 2.3. Hence the existence of a system of parameters, which is a reduction of the maximal ideal, is essential in Koh-Lee's conjecture.

We close this section by proving that if k is a field with more than two elements, then the maximal ideal of $k[[X, Y]]/\langle XY(X+Y)\rangle$ has a minimal reduction.

Theorem 2.5. Let k be a field with $|k| \geq 3$ and m be the maximal ideal in $A = k[[X, Y]]/\langle XY(X+Y)\rangle = k[[x, y]]$. Then $m^3 = (x+\alpha y)m^2$ for $\alpha \in k\ \{0,1\}$, i.e., a minimal reduction of m exists. In particular, $col(A) = 3 = crs(A)$, which implies that Koh-Lee conjecture holds in A.

Proof. Since $x+\alpha y \in m$, $(x+\alpha y)m^2 \subset m^3$. To show that $m^3 \subset (x+\alpha y)m^2$, it is enough to show that x^3, y^3, x^2y, and xy^2 are in $(x+\alpha y)m^2$. First, we claim $x^3 \in (x+\alpha y)m^2$. Indeed, $xy(x+y) = 0$, $1-\alpha$ is a unit in $k[[x, y]]$ and

$$
\begin{aligned}
&(x+\alpha y)(x^2(1+\alpha y^2) + xy(1-\alpha)^{-1}(x^2-\alpha(1+\alpha y^2)) + y^2x^2) \\
&= x^3(1+\alpha y^2) + x^2y(1-\alpha)^{-1}(x^2-\alpha(1+\alpha y^2)) + xy^2x^2 \\
&\quad + \alpha x^2y(1+\alpha y^2) + \alpha xy^2(1-\alpha)^{-1}(x^2-\alpha(1+\alpha y^2)) + \alpha x^2y^3 \\
&= x^3 + \alpha x^2y^2(x+y) + x^2y(1-\alpha)^{-1}(x^2-\alpha(1+\alpha y^2)) - x^2yx^2 \\
&\quad + \alpha x^2y(1+\alpha y^2) - \alpha x^2y(1-\alpha)^{-1}(x^2-\alpha(1+\alpha y^2)) \\
&= x^3 + x^2y(x^2-\alpha(1+\alpha y^2))(1-\alpha)^{-1}(1-\alpha) - x^2y(x^2-\alpha(1+\alpha y^2)) \\
&= x^3.
\end{aligned}
$$

Thus $x^3 \in (x+\alpha y)m^2$. Note that α is a unit in $k[[x, y]]$ and

$$
\begin{aligned}
&y^3 = ((x+\alpha y)(x^2+xy+y^2) - x^3)\alpha^{-1}, \\
&x^2y = ((x+\alpha y)x^2 - x^3)\alpha^{-1}, \quad \text{and} \\
&xy^2 = (x+\alpha y)y^2 - \alpha y^3.
\end{aligned}
$$

Thus y^3, x^2y, and xy^2 are also in $(x+\alpha y)m^2$. In particular, by Theorem 2.4, we have $col(A) = mult(A) = crs(A)$, and $mult(A) = 3$. This completes the proof.

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