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CONDENSATION IN DENSITY DEPENDENT ZERO RANGE PROCESSES

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ABSTRACT. We consider zero range processes with density dependent jump rates g given by $g = g(n,k) = g_1(n)g_2(k/n)$ with $g_1(x) = x^{-\alpha}$ and

$$g_2(x) = \begin{cases} x^{-\alpha} \text{ if } a < x \\ Mx^{-\alpha} \text{ if } x \le a. \end{cases} \tag{0.1}$$

In this case, with $1/2 < a < 1$ and $\alpha > 0$, we show that non-complete condensation occurs with maximum cluster size an. More precisely, for any $\epsilon > 0$, there exists $M^* > 0$ such that, for any $0 < M \le M^*$, the maximum cluster size is between $(a-\epsilon)n$ and $(a+\epsilon)n$ for large n. This provides a simple example of non-complete condensation under perturbation of rates which are deep in the range of perfect condensation (e.g. $\alpha >> 1$) and supports the instability of the condensation transition.

1. INTRODUCTION

Suppose there are m indistinguishable particles distributed over n sites on one dimensional lattice, and suppose a nondecreasing function g defined on nonnegative integers and a transition matrix $\{P_{ij}\}_{i,j=1}^n$ are given. We say there is a k-cluster at site i, if k particles occupy site i. The zero range process introduced by Spitzer in 1970 describes the following dynamics: A k-cluster at site i waits an exponential amount of time with parameter $g(k)$, picks site j with probability P_{ij}, and gives one particle to site j [13], [14].

If m and n are fixed, and $\{P_{ij}\}_{i,j=1}^n$ is irreducible, then there is a unique invariant measure of the zero range process which is independent of the transition probability $\{P_{ij}\}_{i,j=1}^n$. Let $Z \doteq (Z_1, Z_2, \cdots, Z_n)$ denote the steady state of a zero range process corresponding the invariant measure, and let Z_n^* be the size of the maximum cluster,

$$Z_n^* = \max_{1\le i\le n} Z_i. \tag{1.1}$$

If g is decreasing, larger clusters wait more time than smaller ones to give a particle to other sites, which means that large clusters rather gain particles from small clusters and consequently there are growing phenomena. One can guess that for rapidly decreasing rate functions the

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maximum cluster draws all particles. As a matter of fact, the size of the maximum cluster depends on how fast the rate functions decrease.

Recently, it has a great attention after the discovery of existence of condensation transition, the phenomenon that positive fraction of all particles form a giant cluster (see Definition 1.1). Considering the jump rates given by

$$g(k) = 1 + \frac{\beta}{k},\ \beta > 0, \tag{1.2}$$

Jeon et al. [12] and Evans [4] independently discovered the occurrence of non-complete condensation transition in zero range processes.
Condensation is an important phenomenon which can be applied to various areas. In mathematics, Armendáriz and Loulakis, and Beltran and Landim studied several aspects of condensation recently among others [1], [2]. It can be also applied to the areas such as sandpile dynamics, interface growth, granular systems, network flows, transport processes, and macroeconomics. For the significance of the condensation transition and applications, see the survey paper by Evans and Hanney [5].

To study the condensation phenomena in detail, let us adopt the following definition from [10].

Definition 1.1. (1) *A condensation event occurs if Z_n^*/n converges to a constant c, $0 < c \le 1$, in probability as n tends to infinity.*
(2) *A complete condensation event occurs if Z_n^*/n converges to 1 in probability as n tends to infinity.*
(3) *A perfect condensation event occurs if $n - Z_n^*$ converges to 0 in probability as n tends to infinity.*

Note that "condensation" implies that the maximum cluster size is a positive fraction of the number of total particles, "complete condensation" means that the maximum cluster size is of order n, and "perfect condensation" indicates that all particles coalesce into a single cluster. The condensation transition is conceptually similar to the emergence of a giant component in a random graph and gelation for coagulation fragmentation equations [8], [12].

In [12], Jeon et al. also considered the cases that the rate function given by

$$g(k) = k^{-\alpha}, \qquad -\infty < \alpha < \infty, \tag{1.3}$$

and showed that if the transition matrix $\{P_{ij}\}_{i,j=1}^n$ is symmetric and irreducible, then two types of transitions occur. Assume that $m = n$, i.e., the density $\rho = m/n = 1$.

Theorem 1.2. *(Jeon, March, Pittel [12])*
(a) If $\alpha > 1$, then $n - Z_n^*$ converges to 0 in probability.
(b) If $\alpha = 1$, then $n - Z_n^*$ weakly converges to a Poisson distribution with the parameter equal to 1.

(c) If $0 < \alpha < 1$, *then* $(n - Z_n^*)/n^{1-\alpha}$ *converges to 1 in probability.*
(d) If $\alpha = 0$, *then* $Z_n^*/\log n$ *converges to* $\log 2$ *in probability.*
(e) If $\alpha < 0$, *then* $Z_n^* \log(\log n)/\log n$ *converges to* $-\alpha^{-1}$ *in probability.*

In the above Theorem, perfect condensation occurs for the case (a) and non-perfect complete condensation occurs for the case (b) and (c).

In some models, the dynamics can be described by jump rates expressed by

$$g = g(k,n) = g_1(n)g_2(k/n) \tag{1.4}$$

where g_1 represents the scaling part which depends only on the total number of particles, and g_2 is a function of the density k/n [5], [6]. Note that in this model, the invariant measure doesn't depend on g_1. More recently, Jeon considered a case in which the jump rate is given by

$$g(k) = g(k,n) = \begin{cases} M/k^{\alpha} \text{ if } (1/2)^{2l} < k/n \le (1/2)^{2l+1} \\ 1/k^{\alpha} \text{ if } (1/2)^{2l+1} < k/n \le (1/2)^{2l+2}, \end{cases} \tag{1.5}$$

for $l = 0, 1, \cdots$ with $\alpha > 0$, which is a density-dependent dyadic periodic perturbation of (1.3) [9]. In this case Jeon showed that non-complete condensation occurs. This implies that even the rates are deep in the range of perfect condensation(e.g. $\alpha >> 1$), a small perturbation can cause a significant amount of reduction of the maximum cluster size and showed the instability of the condensation phenomena. (For the instability of random perturbation, see [3], [7].)

In this paper, we derive a much simpler example of rate functions which exhibits similar characteristics. Indeed, we perturb (1.3) to consider the case that

$$g(k) = \begin{cases} k^{-\alpha} \text{ if } a < k/n \\ Mk^{-\alpha} \text{ if } k/n \le a. \end{cases} \tag{1.6}$$

Under this g with $\alpha > 0$ and $1/2 < a < 1$, we are able to show that non-complete condensation occurs with maximum cluster size close to an, in the sense that for any $\epsilon > 0$, there exists $M^* > 0$ in (1.6) such that, for any $0 < M \le M^*$ the maximum cluster size is between $(a-\epsilon)n$ and $(a+\epsilon)n$ for large n. (See Theorem 3.5 and 3.6.) Our result provides another simple example of non-complete condensation under perturbation, even the rates are deep in the range of perfect condensation, which also supports the instability of the condensation phenomena. In this paper, we considered only the case $1/2 < a < 1$. We believe the situation is true for all $0 < a < 1$, which we leave as an open problem.

This study is organized as follows: Section 2 briefly introduces the zero range processes and invariant measures; and Section 3 presents proofs of the main theorems.

2. Zero Range Process

A zero range process is a stochastic process defined on $\Omega_n^* = \{0, 1, 2, \dots\}^{N_n}$, where $N_n = \{1, 2, \cdots, n\}$. Any $\eta \doteq (\eta(1), \eta(2), \cdots, \eta(n)) \in \Omega_n^*$ represents the distribution of particles along the n sites, and this suggests that there is a $\eta(i)$-cluster at site i. Suppose there is a jump rate, a nonnegative function g defined on nonnegative integers. Assume further that an irreducible stochastic matrix $\{P_{ij}\}_{1 \le i,j \le n}$ satisfying $\sum_{j=1}^n P_{ij} = 1$ for all i is given. Then zero range process is the stochastic process with the following generator. For $\eta \in \Omega_n^*$,

$$(L_n f)(\eta) = \sum_{i=1}^n \sum_{j=1}^n P_{ij} g(\eta(i)) \{f(\eta^{i,j}) - f(\eta)\},$$

where f is any bounded function on Ω_n^*, and $\eta^{i,j}$ is given by

$$\begin{cases} \text{if } \eta(i) = 0, \text{ then } \eta^{i,j} = \eta \\ \text{if } \eta(i) \neq 0, \text{ then } \eta^{i,j}(k) = \begin{pmatrix} \eta(i) - 1 & \text{if } k = i \\ \eta(j) + 1 & \text{if } k = j \\ \eta(k) & \text{otherwise.} \end{pmatrix} \end{cases}$$

Suppose its initial configuration is $\eta = (\eta(1), \eta(2), \cdots, \eta(n))$. Then $\eta(i)$-cluster waits for an exponential amount of time with parameter $g(\eta(i))$, picks site j with probability P_{ij}, and allocate one particle to j site. As a result, $\eta(i)$ decreases to $\eta(i) - 1$, and $\eta(j)$ increases to $\eta(j) + 1$. Since these dynamics do not permit the creation or annihilation of particles, the total number of particles are preserved. Let

$$\Omega_n^m = \{\eta \in \Omega_n^* : \sum_{i=1}^n \eta(i) = m\}, \qquad 1 \le m < \infty, \tag{2.1}$$

then there is a unique invariant measure, say ν_n^m, on Ω_n^m, and Spitzer showed that the invariant measure can be expressed by a simple factorized form [13], [14].

Lemma 2.1. *(Spitzer [14]) For any jump rate g, and for any $\eta \in \Omega_n^m$, let*

$$\mu_n^m(\eta) = \prod_{i=1}^n \{g!(\eta(i))\}^{-1}, \tag{2.2}$$

where $g!(l) = g(l)g(l-1)g(l-2)\cdots g(1)$, with the convention $g!(0) = 1$. Let

$$\nu_n^m(\eta) = \frac{1}{\Gamma} \mu_n^m(\eta), \tag{2.3}$$

where Γ is the normalizing constant given by $\Gamma = \mu_n^m(\Omega_n^m) = \sum_{\eta \in \Omega_n^m} \mu_n^m(\eta)$. Then ν_n^m is the invariant measure corresponding to g.

3. MAIN THEOREMS AND PROOFS

In this section, if there are no confusions or no differences in the estimates, we will drop the largest integer function $[\cdot]$ from the notation. For example, we mean $(\epsilon n)!$ by $[\epsilon n]!$ and $\binom{n}{\epsilon n} = \binom{n}{[\epsilon n]}$. Furthermore, if $n = m$, to simplify the notation, we will indicate $\Omega_n^m, \mu_n^m, \nu_n^m$ as Ω_n, μ_n, ν_n, respectively.

Now, let us consider the density dependent jump rate $g = g(k,n) = g_1(n)g_2(k/n)$ which is given by $g_1(x) = 1/x^\alpha$ and g_2 by

$$g_2(x) = \begin{cases} \dfrac{1}{x^\alpha}, \text{ if } x > a \\ \dfrac{M}{x^\alpha}, \text{ if } x/n \leq a, \end{cases} \tag{3.1}$$

for $a, 0 < a < 1$. Then g becomes (1.6), and it is a simple perturbation of (1.3). Note that, in (1.3), if $\alpha > 1$ then perfect condensation occurs from Theorem 1.2.

Before we state and prove the main theorems, let us introduce some useful Lemmas which can be found in [9], [10]. Let $Z \doteq (Z_1, Z_2, \cdots, Z_n)$ be the steady state of a zero range process corresponding the invariant measure, and let Z_n^* be the size of the maximum cluster, that is,

$$Z_n^* = \max_{1\leq i\leq n} Z_i. \tag{3.2}$$

Let $|\Omega_n^m|$ be the number of elements in Ω_n^m. Since Ω_n^m is the set of nonnegative integers satisfying the equation

$$x_1 + x_2 + \cdots + x_n = m,$$

elementary combinatorics provides

$$|\Omega_n^m| = \binom{n+m-1}{n-1}. \tag{3.3}$$

We also have the next lemma from elementary combinatorics.

Lemma 3.1. *For fixed l, $\binom{n}{l}$ is an increasing function of n. Moreover, for fixed n, the function is increasing for $l < \frac{n}{2}$ and decreasing for $l > \frac{n}{2}$.*

Let

$$A_k = \{\eta \in \Omega_n^m : \max_{1\leq i\leq n} \eta(i) \leq k\} \tag{3.4}$$

be the set of configurations of which the maximum cluster size is less than or equal to k. The next lemma describes a comparison of the invariant measures of different configurations.

Lemma 3.2. *Let* $m = lk+r, 0 \le r < k$, *and let* $\eta_* = (k, k, \cdots, k, r, 0, 0, \cdots, 0) \in A_k$, *where the* k*'s are repeated* l *times. Let* g *be a rate function with corresponding invariant measure* ν_n^m *on* Ω_n^m. *Then for any* $\eta \in A_k$:
(a) $\nu_n^m(\eta_*) \ge \nu_n^m(\eta)$, *if* g *is decreasing.*
(b) $\nu_n^m(\eta_*) \le \nu_n^m(\eta)$, *if* g *is increasing.*

Proof. See Lemma 1.3 in [12]. □

Let μ_n^m be the unnormalized zero range invariant measure on Ω_n^m corresponding to g in (3.1). Then

$$\mu_n^m(\eta) = \prod_{i=1}^{n} \{g!(\eta(i))\}^{-1} = \prod_{i=1}^{n} \frac{(\eta(i)!)^\alpha}{M^l}, \tag{3.5}$$

for some $l \in \{0, 1, 2, \cdots\}$. In this expression, since l depends on η, we can define a function $\phi : \Omega_n^m \to \{0, 1, 2, \cdots\}$, which counts the number of Ms contained in $\mu_n^m(\eta)$ as

$$\phi(\eta) = l. \tag{3.6}$$

From now on, we assume $m = n$ so that the density $\rho = m/n = 1$.

Proposition 3.3. *Suppose that* g *is given by (3.1). For* $a \ge 1/2$,

$$\nu_n(A_{\frac{an}{4}}) \to 0, \ \text{as } n \to \infty, \tag{3.7}$$

that is, the size of the largest cluster is greater than $an/4$ *in probability.*

Proof. Consider $\eta_0 = (an, b, 0, \cdots, 0) \in \Omega_n$ with $an + b = n$ (recall, in this notation 'an' is actually '$[an]$'). From Lemma 3.2, $\mu_n(\eta_0) \ge \mu_n(\eta)$, for any $\eta \in A_{an}$, since $\phi(\eta) = n$ for all $\eta \in A_{an}$, and $\phi(\eta_0) = n$. Therefore, without loss of generality, we may assume $M = 1$.

Let $\eta_1 = (an/4, an/4, \cdots, an/4, d, 0, \cdots, 0)$ with suitable d ($d \le an/4$) which makes $\eta_1 \in \Omega_n$. Similarly, we see that $\mu_n(\eta_1) \ge \mu_n(\eta)$ for any $\eta \in A_{an/4}$. Note that the number of elements of the set $A_{an/4}$ is bounded by that of Ω_n, which is bounded by $\binom{2n-1}{n-1}$ from (3.3) and $\binom{2n-1}{n-1}$ is bounded by $\binom{2n}{n}$. Therefore,

$$\nu_n(A_{an/4}) = \frac{\mu_n(A_{an/4})}{\mu_n(\Omega_n)} \tag{3.8}$$

$$\le \frac{\mu_n(\eta_1)\binom{2n}{n}}{\mu_n(\eta_0)}, \tag{3.9}$$

Since $\phi(\eta_0) = \phi(\eta_1)$ and from Stirling's formula, we have

$$\nu_n(A_{an/4}) \le \frac{P(n)(an/4)^{\alpha n}}{(an)^{\alpha n}} 4^n \tag{3.10}$$

$$\le P(n)(\frac{4}{4^\alpha})^n. \tag{3.11}$$

Here, the degree of the polynomial $P(n)$ is independent of n. Now, the final term tends to 0 as n tends to infinity, since $\alpha > 1$. □

The next Proposition proves that the probability that the maximum cluster size is between $an/4$ and $(a-\epsilon)n$ tends to 0 as n tends to infinity.

Proposition 3.4. *Suppose that g is given by (3.1). For any small $\epsilon > 0$, and $a > \frac{1}{2}$,*

$$\nu_n(A_{(a-\epsilon)n} \setminus A_{\frac{an}{4}}) \to 0, \text{ as } n \to \infty. \tag{3.12}$$

Proof. For $A_{(a-\epsilon)n}$, without loss of generality, we may assume $M = 1$, since $\phi(\eta) = n$ for all $\eta \in A_{(a-\epsilon)n}$. Let

$$B_k = A_k \setminus A_{k-1} \tag{3.13}$$

be the set of configurations of which the maximum cluster size is exactly k. For $\eta \in \Omega_n$, define γ by

$$\gamma(\eta) = |\{i : \eta(i) \geq 1\}|, \tag{3.14}$$

where $|A|$ denotes the number of elements in A. Note that γ indicates the total number of occupied sites. Let $C_{k,l}$ be the set of configurations in B_k with exactly l occupied sites. That is,

$$C_{k,l} = \{\eta \in B_k : \gamma(\eta) = l\}. \tag{3.15}$$

For any $\eta \in B_k$, it contains a k cluster, and there are $n-k$ remaining particles. Therefore, for $k < n$, B_k can be expressed by

$$B_k = \bigcup_{l=1}^{n-k+1} C_{k,l}.$$

Note that there are $\binom{n}{l}$ ways of choosing l occupied sites, there are l ways of locating the largest k-cluster on the l sites. Since the number of ways locating the remaining $n-k$ particles on the $l-1$ sites without empty site is bounded by $P(n)\binom{n}{l}$ for some polynomial $P(n)$, $|C_{k,l}|$ is bounded by $P(n)\binom{n}{l}\binom{n}{l}$.

Now, choose $\eta_0 = (an, b, 0, 0, \cdots, 0)$ with $an + b = n$ and $b < an$. For k, $an/4 \leq k \leq (a-\epsilon)n$,

(1) If $k \leq b$, then choose

$$\eta_1 = (k, an-k, b, 0, \cdots, 0) \in \Omega_n,$$

then, from Lemma 3.2, $\mu_n(\eta_1) \geq \mu_n(\eta)$ for any $\eta \in C_{k,l}$. Moreover,

$$\frac{\mu_n(\eta_1)}{\mu_n(\eta_0)} = \left(\frac{k!(an-k)!}{(an)!}\right)^{\alpha} \tag{3.16}$$

$$\leq \frac{P(n)(k^k(an-k)^{an-k})^{\alpha}}{(an)^{\alpha an}} \tag{3.17}$$

$$= P(n)\left(\frac{(k/n)^{k/n}(a-k/n)^{a-k/n}}{(a)^a}\right)^{\alpha n}. \tag{3.18}$$

Let

$$f(x) = x^x(a-x)^{a-x}, \tag{3.19}$$

and let $g(x) = \ln(f(x))$, then

$$g'(x) = \ln x + 1 - \ln(a-x) - 1 = \ln\left(\frac{x}{a-x}\right).$$

Since $g'(x) = 0$ implies $x = a/2$, $f(x)$ has minimum at $x = a/2$ and f is strictly decreasing for $0 < x < a/2$, and strictly increasing for $a/2 < x < a$. Since $\lim_{x\to 0} f(x) = \lim_{x\to a} f(x) = a^a$, and $a/4 \leq k/n \leq a - \epsilon$, there exists $\gamma_1, 0 < \gamma_1 < 1$ such that

$$\frac{\mu_n(\eta_1)}{\mu_n(\eta_0)} \leq P(n)\gamma_1^n. \tag{3.20}$$

(2) If $b < k \leq b + \epsilon_0 n$, for small ϵ_0 satisfying $0 < \epsilon_0 < \min(\epsilon, a - 1/2)$, choose

$$\tilde{\eta}_0 = (an - \epsilon_0 n, b + \epsilon_0 n, 0, \cdots, 0) \in \Omega_n,$$

and

$$\eta_1 = (k, an - k - \epsilon_0 n, b + \epsilon_0 n, 0, \cdots, 0) \in \Omega_n,$$

then, from Lemma 3.2, $\mu_n(\eta_0) \geq \mu_n(\tilde{\eta}_0)$ and $\mu_n(\eta_1) \geq \mu_n(\eta)$ for any $\eta \in C_{k,l}$. Similarly we have

$$\frac{\mu_n(\eta_1)}{\mu_n(\eta_0)} \leq \frac{\mu_n(\eta_1)}{\mu_n(\tilde{\eta}_0)} \leq P(n)\left(\frac{(k/n)^{k/n}(a-\epsilon_0-k/n)^{a-\epsilon_0-k/n}}{(a-\epsilon_0)^{a-\epsilon_0}}\right)^{\alpha n}. \tag{3.21}$$

and there exists $\gamma_2, 0 < \gamma_2 < 1$ such that

$$\frac{\mu_n(\eta_1)}{\mu_n(\tilde{\eta}_0)} \leq P(n)\gamma_2^n. \tag{3.22}$$

(3) If $k > b + \epsilon_0 n$ then choose

$$\eta_1 = (k, an - k + b, 0, \cdots, 0) \in \Omega_n,$$

then, from Lemma 3.2, $\mu_n(\eta_1) \geq \mu_n(\eta)$ for any $\eta \in C_{k,l}$. Moreover,

$$\begin{aligned} \frac{\mu_n(\eta_1)}{\mu_n(\eta_0)} &= \left(\frac{k!(an-k+b)!}{(an)!b!}\right)^{\alpha} && (3.23)\\ &\leq \frac{P(n)(k^k(an-k+b)^{an-k+b})^{\alpha}}{((an)^{an}b^b)^{\alpha}} && (3.24)\\ &= \frac{P(n)(k^k(n-k)^{n-k})^{\alpha}}{((an)^{an}(n-an)^{(n-an)})^{\alpha}} && (3.25)\\ &= P(n)\left(\frac{(k/n)^{k/n}(1-k/n)^{1-k/n}}{(a)^a(1-a)^{1-a}}\right)^{\alpha n} && (3.26) \end{aligned}$$

Let

$$f_1(x) = x^x(1-x)^{1-x}, \qquad (3.27)$$

then f_1 is the case that $a = 1$ in (3.19). Therefore, $f(x)$ has minimum at $x = 1/2$ and f is strictly decreasing for $0 < x < 1/2$, and strictly increasing for $1/2 < x < 1$. Note that $b + \epsilon_0 n < k < (a - \epsilon)n$ implies $n - an + \epsilon_0 n < k < (a - \epsilon)n$, i.e., $1 - a + \epsilon_0 < k/n < a - \epsilon$. Since

$$f_1(a) = f_1(1-a) = a^a(1-a)^{(1-a)},$$

there exists $\gamma_3, 0 < \gamma_3 < 1$ such that

$$\frac{\mu_n(\eta_1)}{\mu_n(\eta_0)} \leq P(n)\gamma_3^n. \qquad (3.28)$$

Let $\gamma = \max\{\gamma_1, \gamma_2, \gamma_3\}$, then $\gamma < 1$ and

$$\begin{aligned} \nu_n\Big(\bigcup_{k=an/4}^{(a-\epsilon)n}\bigcup_l C_{k,l}\Big) &\leq \frac{\mu_n\big(\bigcup_k \bigcup_l C_{k,l}\big)}{\mu_n(\eta_0)} && (3.29)\\ &\leq \frac{\sum_k \sum_l \mu_n(C_{k,l})}{\mu_n(\eta_0)} && (3.30)\\ &\leq \sum_l P(n)\binom{n}{l}\binom{n}{l}\gamma^n && (3.31) \end{aligned}$$

for some polynomial $P(n)$.

Now, for $\epsilon_1 > 0$ sufficiently small,
(a) If $l < \epsilon_1 n$, then, since

$$\binom{n}{l} \leq \binom{n}{\epsilon_1 n},$$

we have

$$\nu_n(\bigcup_{k=an/4}^{(a-\epsilon)n}\bigcup_{l\leq\epsilon_1 n}) \leq n^2P(n)\binom{n}{\epsilon_1 n}\binom{n}{\epsilon_1 n}\gamma^n \tag{3.32}$$

$$\leq P(n)\left(\frac{n!}{(\epsilon n)!(n-\epsilon n)!}\right)^2\gamma^n \tag{3.33}$$

$$\leq P(n)\left(\frac{\gamma}{(\epsilon_1^{\epsilon_1}(1-\epsilon_1)^{1-\epsilon_1})^2}\right)^n, \tag{3.34}$$

for some polynomial $P(n)$ which may vary in each expression. Since

$$\lim_{x\to 0} x^x(1-x)^{1-x} = 1, \tag{3.35}$$

ϵ_1 can be chosen to satisfy

$$(\epsilon_1^{\epsilon_1}(1-\epsilon_1)^{1-\epsilon_1})^2 > \gamma.$$

(b) Now consider the case that $l \geq \epsilon_1 n$. If $n-k-l-2 \leq an$, we choose

$$\eta_1 = (k, n-k-l-2, 1, 1, \cdots, 1, 0, \cdots, 0).$$

Then, from Lemma 3.2, $\mu_n(\eta_1) \geq \mu_n(\eta)$ for any $\eta \in C_{k,l}$. Moreover,

$$\frac{\mu_n(\eta_1)}{\mu_n(\eta_0)} \leq \left(\frac{k!(n-k-l)!}{(an)!b!}\right)^\alpha \tag{3.36}$$

$$= \left(\frac{k!(n-k)!}{(an)!b!(n-k-1)(n-k-2)\cdots(n-k-l)}\right)^\alpha \tag{3.37}$$

$$\leq \left(\frac{n!}{(an)!b!}\frac{1}{(n-k-1)(n-k-2)\cdots(n-k-\epsilon_1 n)}\right)^\alpha \tag{3.38}$$

$$\leq \left(\frac{C}{((\epsilon-\epsilon_1)n)^{\alpha\epsilon_1}}\right)^n. \tag{3.39}$$

Therefore,

$$\nu_n(\bigcup_{k=an/4}^{(a-\epsilon)n}\bigcup_{l\geq\epsilon_1 n} C_{k,l}) = \frac{\mu_n(\bigcup_k\bigcup_l C_{k,l})}{\mu_n(\eta_0)} \tag{3.40}$$

$$\leq \frac{\sum_k\sum_l \mu_n(C_{k,l})}{\mu_n(\eta_0)} \tag{3.41}$$

$$\leq P(n)\binom{n}{l}\binom{n}{l}\left(\frac{C}{((\epsilon-\epsilon_1)n)^{\alpha\epsilon_1}}\right)^n \tag{3.42}$$

$$\leq \left(\frac{C}{n^\delta}\right)^n, \tag{3.43}$$

for large n with constant δ and C which may vary in each expression.

Therefore, from (a) and (b), we conclude that

$$\nu_n(A_{(a-\epsilon)n} \setminus A_{\frac{an}{4}}) \to 0.$$

□

From Proposition 3.3 and Proposition 3.4 we have the theorem.

Theorem 3.5. *Suppose that g is given by (3.1). Then, for any $\epsilon > 0$, there exists $M, 0 < M < 1$, such that ,*

$$Z_n^* \geq (a-\epsilon)n \tag{3.44}$$

for large n.

Now, let us consider the other part, i.e., the upper bound of the size of the largest cluster.

Theorem 3.6. *For any $\epsilon > 0$, there exists an M such that*

$$Z_n^* \leq (a+\epsilon)n \tag{3.45}$$

for large n.

Proof. For any l, $1 \leq l \leq n$, let B_l be the set of configurations with maximum cluster size exactly l. Let $\eta_1 = (an, b, 0, \cdots, 0) \in \Omega_n$, with $an + b = n$, and let $\eta_2 = (l, n-l, 0, \cdots, 0)$. From Lemma 3.2, $\mu_n(\eta_2) \geq \mu_n(\eta)$ for any $\eta \in B_l$. Therefore, for $l \geq (a+\epsilon)n$,

$$\begin{aligned} P\{Z_n \in B_l\} &= \nu_n(B_l) \\ &\leq \frac{\mu_n(B_l)}{\mu_n(\eta_1)} \\ &\leq \frac{n\binom{2n}{n}\mu_n(\eta_2)}{\mu_n(\eta_1)}. \end{aligned}$$

Since $\binom{2n}{n} \sim P(n)2^n$, with some polynomial $P(n)$, we have, for $M < 1$,

$$\begin{aligned} P\{Z_n \in B_l\} &= \nu_n(B_l) \\ &\leq P(n)2^n M^{(\phi(\eta_1)-\phi(\eta_2))}\left(\frac{l!(n-l)!}{(an)!b!}\right)^\alpha \\ &\leq P(n)2^n M^{\epsilon n}\left(\frac{l!(n-l)!}{n!}\frac{n!}{(an)!b!}\right)^\alpha \\ &\leq P(n)M^{\epsilon n}\lambda^n, \end{aligned}$$

where $\lambda > 0$ is a constant independent of n and l, and $P(n)$ may differ in each expression. Therefore,

$$P\{Z_n \geq (a+\epsilon)n)\} \leq nP(n)M^{\epsilon n}\lambda^n,$$

and if $M < (1/\lambda)^{1/\epsilon}$, the probability that the maximum cluster size is larger than $(a+\epsilon)n$ tends to zero as n tends to infinity. □

Since $\mu_n(\eta)$ is monotone on M, if the above theorem is true for M_0, then it is true for all $M \leq M_0$. Therefore, together with Theorem 3.5 we have

Theorem 3.7. *Suppose that g is given by (3.1). Then, for any $\epsilon > 0$, there exists $M^* > 0$, such that*

$$(a-\epsilon)n \leq Z_n^* \leq (a+\epsilon)n \tag{3.46}$$

for all $M \leq M^*$ and for large n.

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REFERENCES

[1] I. Armendáriz and M. Loulakis, *Thermodynamic Limit for the Invariant Measures in Supercritical Zero Range Processes*, Prob. Th. Rel. Fields, **145** (2009), no.1-2, 175–188.

[2] J. Beltran and C. Landim, *Metastability of reversible condensed zero range processes on a finite set*, Prob. Th. Rel. Fields, **152** (2012), no. 3, 781–807.

[3] L. Molino, P. Chleboun, and S. Grosskinsky, *Condensation in randomly perturbed zero-range processes*, J. Phys. A: Math. Theor., **45** (2012), no 20, 205001.

[4] M. R. Evans, *Phase transitions in one-dimensional nonequilibrium systems*, Braz. J. Phys. **30** (2000), 42–57.

[5] M. R. Evans and T. Hanney , *Nonequilibrium statistical mechanics of the zero-range process and related models*, J. Phys. A: Math. Gen., **38** (2005), 195–240.

[6] S. Grosskinsky and G. Schütz, *Discontinuous condensation transition and nonequivalence of ensembles in a zero-range process*, J. Stat. Phys., **132** (2008), no. 1, 77–108.

[7] S. Grosskinsky, P. Chleboun, and G. Schütz, *Instability of condensation in the zero-range process with random interaction*, Phys. Rev. E: Stat. Nonlin. Soft Matter Phys., **78** (2008), 030101.

[8] I. Jeon, *Existence of gelling solutions for coagulation fragmentation equations*, Comm. Math. Phys., **194** (1998), 541–567.

[9] _, *Phase transition for perfect condensation and instability under the perturbations on jump rates of the Zero Range Process*, J. Phys. A: Math. and Theor., **43** (2010), 235002.

[10] _, *Condensation in perturbed zero range processes*, J. Phys. A: Math. and Theor., **44** (2011), 255002.

[11] I. Jeon and P. March, *Condensation transition for zero range invariant measures*, In Stochastic models. Proceedings of the International Conference on Stochastic Models in Honor of Professor Donald A. Dawson (Luis G. Gorostiza, B. Gail Ivanoff, eds.), 2000, 233–244.

[12] I. Jeon, P. March, and B. Pittel, *Size of the largest cluster under zero-range invariant measures*, Ann. of Prob., **28** (2000), 1162–1194.

[13] T. M. Liggett, *Interacting Particle Systems*, Springer-Verlag, 1985.

[14] F. Spitzer, *Interaction of Markov processes*, Adv. Math., **5** (1970), 246–290.