# NEW SUMMATION FORMULAE FOR THE GENERALIZED HYPERGEOMETRIC FUNCTIONS OF HIGH ORDER 

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#### Abstract

The aim of this paper is to provide two interesting summation formulae with the argument unity for the generalized hypergeometric function of higher order. The results are obtained with the help of two new summation formulae very recently obtained by Kim et al.. Summation formulae obtained earlier by Carlson and re-derived by Exton turn out to be special cases of our main findings.


## 1. Introduction and results required

The generalized hypergeometric function with $p$ numeratorial and $q$ denominatorial parameters is defined by the series [8]

$$
{ }_{p} F_{q}\left[\begin{array}{c}
a_{1}, \cdots, a_{p}  \tag{1}\\
b_{1}, \cdots, b_{q}
\end{array} ; z\right]=\sum_{n=0}^{\infty} \frac{\left(a_{1}\right)_{n} \cdots\left(a_{p}\right)_{n}}{\left(b_{1}\right)_{n} \cdots\left(b_{q}\right)_{n}} \frac{z^{n}}{n!},
$$

where $(\alpha)_{n}$ denotes the Pochhammer symbol (or shifted factorial) defined for any $\alpha \in \mathbb{C}$ by

$$
(\alpha)_{n}=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}= \begin{cases}1 & n=0  \tag{2}\\ \alpha(\alpha+1) \cdots(\alpha+n-1) & n \in \mathbb{N}\end{cases}
$$

while by convention $(0)_{0} \equiv 1$. Here is

$$
\Gamma(s)=\int_{0}^{\infty} x^{s-1} \mathrm{e}^{-x} \mathrm{~d} x, \quad \Re\{s\}>0
$$

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the celebrated Euler's Gamma function.
It is well-known that whenever a generalized hypergeometric function reduces to Gamma-function the results are very important in view of applications. Thus, the well known and classical summation theorems such those of Gauss, Gauss' second, Bailey and Kummer for the series ${ }_{2} F_{1}$; Watson, Dixon, Whipple and Saalschütz for the series ${ }_{3} F_{2}$ play the key role in the theory of hypergeometric and generalized hypergeometric series. Recently good progress has been done in generalizing the above mentioned classical summation theorems, see [5, 6, 7], and the references therein.

Very recently, extensions of the above mentioned classical summation theorems were given by Kim et al. [4]; the following two of them we shall need in the sequel:
(3) ${ }_{3} F_{2}\left[\begin{array}{c}a, b, d+1 \\ c+1, d\end{array} ; 1\right]=\frac{\Gamma(c+1) \Gamma(c-a-b)}{\Gamma(c-a+1) \Gamma(c-b+1)}\left[(c-a-b)+\frac{a b}{d}\right]$
provided $\Re\{c-a-b\}>0, \Re\{d\}>0$, and

$$
{ }_{3} F_{2}\left[\begin{array}{c}
a, b, d+1 \\
a-b+2, d
\end{array} ;-1\right]=\frac{\sqrt{\pi} \Gamma(a-b+2)}{2^{a}(1-b)}\left[\frac{(a-b+1) / d-1}{\Gamma\left(\frac{1}{2} a\right) \Gamma\left(\frac{1}{2} a-b+\frac{3}{2}\right)}\right.
$$

$$
\begin{equation*}
\left.+\frac{1-(a / d)}{\Gamma\left(\frac{1}{2} a+\frac{1}{2}\right) \Gamma\left(\frac{1}{2} a-b+1\right)}\right] \tag{4}
\end{equation*}
$$

provided $\frac{1}{2}<\Re\{b\}<1, \Re\{d\}>0$. Results (3) and (4) are easily seen to be extensions of Gauss' summation theorem:

$$
{ }_{2} F_{1}\left[\begin{array}{cc}
a, & b \\
c & ; 1
\end{array}\right]=\frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}
$$

provided $\Re\{c-a-b\}>0$, and Kummer's summation theorem

$$
{ }_{2} F_{1}\left[\begin{array}{c}
a, b \\
a-b+1
\end{array} ;-1\right]=\frac{\sqrt{\pi} \Gamma(a-b+1)}{2^{a} \Gamma\left(\frac{1}{2} a+\frac{1}{2}\right) \Gamma\left(\frac{1}{2} a-b+1\right)}
$$

provided $\Re\{b\}<1$, respectively.
We remark in passing that the generalized hypergeometric functions occur in many theoretical and practical applications, such as mathematics, theoretical physics, engineering and statistics. For detailed discussions of the functions, including the convergence of its series representation, we refer to Slater [9], Exton [2] or Srivastava and Choi [10] for example.

Numerous cases are known when the function ${ }_{q+1} F_{q}[ \pm 1]$ can be sum med in compact form. For a list of many of these results see $[9$, Appendix

III]. The process of resolving hypergeometric series with even and odd parts is widely used, see [11], for instance.

The aim of this short research note is to provide two interesting summation formulae with the unit argument for the generalized hypergeometric function of higher order. The results are obtained with the help of (3) and (4). Summaton formulae due to Carlson and Exton follow as special cases of our main findings.

## 2. Main results

Theorem. The following summation formulae hold true:
${ }_{5} F_{4}\left[\begin{array}{c}a, a+\frac{1}{2}, b, b+\frac{1}{2}, d+1 \\ a-b+1, a-b+\frac{3}{2}, d, \frac{1}{2}\end{array} ; 1\right]=\frac{\Gamma(2(a-b+1))}{2(1-2 b)}\left\{\frac{(1-4 b+2 a b / d) \Gamma(1-4 b)}{\Gamma(1-2 b) \Gamma(2(a-2 b+1))}\right.$

$$
\begin{equation*}
\left.+\frac{\sqrt{\pi}}{4^{a}}\left[\frac{\left(a-b+\frac{1}{2}\right) / d-1}{\Gamma(a) \Gamma(a-2 b+3 / 2)}+\frac{1-(a / d)}{\Gamma(a+1 / 2) \Gamma(a-2 b+1)}\right]\right\} \tag{5}
\end{equation*}
$$

provided $\Re\{b\}<1 / 4, \Re\{d\}>0$, and

$$
\left.\begin{array}{rl}
{ }_{5} F_{4} & {\left[\begin{array}{c}
a+\frac{1}{2}, a+1, b+\frac{1}{2}, b+1, d+\frac{3}{2} \\
a-b+\frac{3}{2}, a-b+2, d+\frac{1}{2}, \frac{3}{2}
\end{array}\right]}
\end{array}\right] \quad \begin{aligned}
& 2 a b(1-2 b)(2 d+1)
\end{aligned} \frac{d(a-b+1) \Gamma(2(a-b+1))}{\Gamma(1-2 b) \Gamma(2(a-2 b+1))} .\left\{\begin{array}{l}
(1-4 b+2 a b / d) \Gamma(1-4 b)
\end{array}\right.
$$

$$
\begin{equation*}
\left.-\frac{\sqrt{\pi}}{4^{a}}\left[\frac{(a-b+1 / 2) / d-1}{\Gamma(a) \Gamma\left(a-2 b+\frac{3}{2}\right)}+\frac{1-(a / d)}{\Gamma\left(a+\frac{1}{2}\right) \Gamma(a-2 b+1)}\right]\right\} \tag{6}
\end{equation*}
$$

provided $\Re\{b\}<1 / 4, \Re\{d\}>0$.
Proof. In order to derive the results (5) and (6) we consider the sum

$$
\begin{align*}
& { }_{q+1} F_{q}\left[\begin{array}{c}
2 a_{1}, \cdots, 2 a_{q+1} ; 1 \\
2 b_{1}, \cdots, 2 b_{q}
\end{array} ; 1\right]{ }_{q+1} F_{q}\left[\begin{array}{c}
2 a_{1}, \cdots, 2 a_{q+1} ;-1 \\
2 b_{1}, \cdots, 2 b_{q}
\end{array} ;-1\right] \\
& \quad=2 \sum_{n=0}^{\infty} \frac{\left(2 a_{1}\right)_{2 n} \cdots\left(2 a_{q+1}\right)_{2 n}}{\left(2 b_{1}\right)_{2 n} \cdots\left(2 b_{q}\right)_{2 n}} \frac{1}{(2 n)!} \tag{7}
\end{align*}
$$

since evidently all the odd indexed terms cancel and only the evenindexed addends survive. Using the identities

$$
(2 a)_{2 n}=4^{n}(a)_{n}\left(a+\frac{1}{2}\right)_{n}, \quad(2 n)!=(1)_{2 n}=4^{n} n!\left(\frac{1}{2}\right)_{n}
$$

the expression (7) can be rewritten in terms of generalized hypergeometric functions of order $2 q+2$ as

$$
{ }_{2 q+2} F_{2 q+1}\left[\begin{array}{c}
a_{1}, a_{1}+\frac{1}{2}, \cdots, a_{q+1}, a_{q+1}+\frac{1}{2} \\
b_{1}, b_{1}+\frac{1}{2}, \cdots, b_{q}, b_{q}+\frac{1}{2}, \frac{1}{2}
\end{array} ; 1\right]
$$

$$
=\frac{1}{2}\left\{{ }_{q+1} F_{q}\left[\begin{array}{c}
2 a_{1}, \cdots, 2 a_{q+1}  \tag{8}\\
2 b_{1}, \cdots, 2 b_{q}
\end{array} ; 1\right]+{ }_{q+1} F_{q}\left[\begin{array}{c}
2 a_{1}, \cdots, 2 a_{q+1} \\
2 b_{1}, \cdots, 2 b_{q}
\end{array} ;-1\right]\right\} .
$$

Now, if in (8) one takes $q=2, a_{1}=a, a_{2}=b, a_{3}=d+1 / 2, b_{1}=a-b+1$ and $b_{2}=d$, then we get

$$
\begin{aligned}
& { }_{5} F_{4}\left[\begin{array}{c}
a, a+\frac{1}{2}, b, b+\frac{1}{2}, d+1 \\
a-b+1, a-b+\frac{3}{2}, d, \frac{1}{2} ; 1
\end{array}\right] \\
& \quad=\frac{1}{2}\left\{{ }_{3} F_{2}\left[\begin{array}{c}
2 a, 2 b, 2 d+1 \\
2(a-b+1), 2 d
\end{array} ; 1\right]+{ }_{3} F_{2}\left[\begin{array}{c}
2 a, 2 b, 2 d+1 \\
2(a-b+1), 2 d
\end{array} ;-1\right]\right\} .
\end{aligned}
$$

Finally, it is easy to see that the ${ }_{3} F_{2}[ \pm 1]$ terms appearing on the right hand side can be evaluated with the aid of (3) and (4) so that the assertion (5) follows after some reduction.

Similarly, bearing in mind the identities
$(2 a)_{2 n+1}=a 2^{2 n+1}\left(a+\frac{1}{2}\right)_{n}(a+1)_{n}, \quad(2 n+1)!=(1)_{2 n+1}=4^{n} n!\left(\frac{3}{2}\right)_{n}$,
we arrrive at the following relationship:

$$
{ }_{2 q+2} F_{2 q+1}\left[\begin{array}{c}
a_{1}+\frac{1}{2}, a_{1}+1, \cdots, a_{q+1}+\frac{1}{2}, a_{q+1}+1 \\
b_{1}+\frac{1}{2}, b_{1}+1, \cdots, b_{q}+\frac{1}{2}, b_{q}+1, \frac{3}{2}
\end{array} ; 1\right]
$$

(9) $=\frac{b_{1} \cdots b_{q}}{4 a_{1} \cdots a_{q+1}}\left\{{ }_{q+1} F_{q}\left[\begin{array}{c}2 a_{1}, \cdots, 2 a_{q+1} \\ 2 b_{1}, \cdots, 2 b_{q}\end{array} ; 1\right]-{ }_{q+1} F_{q}\left[\begin{array}{c}2 a_{1}, \cdots, 2 a_{q+1} \\ 2 b_{1}, \cdots, 2 b_{q}\end{array} ;-1\right]\right\}$.

Taking once more $q=2, a_{1}=a, a_{2}=b, a_{3}=d+\frac{1}{2}, b_{1}=a-b+1$ and $b_{2}=d$, we arrive at

$$
\begin{aligned}
& { }_{5} F_{4}\left[\begin{array}{c}
a+\frac{1}{2}, a+1, b+\frac{1}{2}, b+1, d+\frac{3}{2} \\
a-b+\frac{3}{2}, a-b+2, d+\frac{1}{2}, \frac{3}{2}
\end{array} ; 1\right] \\
(10) & =\frac{d(a-b+1)}{2 a b(2 d+1)}\left\{{ }_{3} F_{2}\left[\begin{array}{c}
2 a, 2 b, 2 d+1 \\
2(a-b+1), 2 d
\end{array} ; 1\right]-{ }_{3} F_{2}\left[\begin{array}{c}
2 a, 2 b, 2 d+1 \\
2(a-b+1), 2 d
\end{array} ;-1\right]\right\},
\end{aligned}
$$

such that yields (6) by virtue of (3) and (4).
The following two special cases of our main findings have been obtained by Carlson [1] and re-derived by Exton [3]:

$$
\begin{aligned}
& { }_{4} F_{3}\left[\begin{array}{c}
\frac{1}{2} a, \frac{1}{2} a+\frac{1}{2}, \frac{1}{2} b, \frac{1}{2} b+\frac{1}{2} \\
\frac{1}{2} a-\frac{1}{2} b+\frac{1}{2}, \frac{1}{2} a-\frac{1}{2} b+1, \frac{1}{2} ; 1
\end{array}\right] \\
& \quad=\frac{\Gamma(a-b+1)}{2^{a+1} \Gamma\left(\frac{1}{2} a-b+1\right)}\left[\frac{\Gamma\left(\frac{1}{2}-b\right)}{\Gamma\left(\frac{1}{2} a-b+\frac{1}{2}\right)}+\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} a+\frac{1}{2}\right)}\right]
\end{aligned}
$$

provided $\Re\{b\}<\frac{1}{2}$ and under the same constraint

$$
\begin{aligned}
& { }_{4} F_{3}\left[\begin{array}{c}
\frac{1}{2} a+\frac{1}{2}, \frac{1}{2} a+1, \frac{1}{2} b+\frac{1}{2}, \frac{1}{2} b+1 \\
\frac{1}{2} a-\frac{1}{2} b+1, \frac{1}{2} a-\frac{1}{2} b+\frac{3}{2}, \frac{3}{2}
\end{array} ; 1\right] \\
& \quad=\frac{\Gamma(a-b+2)}{2^{a+1} \Gamma\left(\frac{1}{2} a-b+1\right)}\left[\frac{\Gamma\left(\frac{1}{2}-b\right)}{\Gamma\left(\frac{1}{2} a-b+\frac{1}{2}\right)}-\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} a+\frac{1}{2}\right)}\right] .
\end{aligned}
$$

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