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COUNTING PROBLEMS IN GENERALIZED PAPER FOLDING SEQUENCES

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Abstract. In this paper, we discuss numbers of downwards and upwards in generalized paper folding sequences. We compute the exact number of downwards and upwards in R_p^n and $(R_pR_q)^n$ by using the properties of recursive sequences where n, p and q are natural numbers with $p \geq 2$ and $q \geq 2$.

1. Introduction and Preliminaries

When we fold a sheet of paper and unfold it, the paper has some creases. Dekking [4] used 0 for a crease that makes the paper upward and 1 for a crease that makes the paper downward. Note that a paper folding sequence is the sequence of 0s and 1s obtained by unfolding a sheet of paper which has been folded many times.
Paper folding sequences have been studied extensively by Allouche, Bates, Bunder, Tognetti, France and Poorten in [1, 2, 5] since Davis and Knuth introduced its concept in [3]. Dekking [4] showed how the automatic structure of the paper folding sequences lead to self-similarity of the curves. Lee, Kim and Choi [6] showed the trace of paper folding sequences using $(0, 1)$ codes and $(0, 1)$ matrices. In this paper, we introduce generalized paper folding sequences and compute the exact number of 0s and 1s in generalized paper folding sequences.

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When we fold a sheet of paper, we may fold it left over right or right over left. We use R when we fold a sheet of paper left over right and L when we fold a sheet of paper right over left. When we fold a sheet of paper left over right and rotate it 180° angles, the creases are the same as that of the paper folding right over left.
Let $p, q, n \in \mathbb{N}$ with $p \geq 2$ and $q \geq 2$. If we fold a sheet of paper in p left over right, we get a generalized paper folding sequence and denote it by R_p. If we iterate R_p process n times, then we get another generalized paper folding sequence and denote it by R_p^n. Similarly, if we fold a sheet of paper in p left over right and then fold the result in q left over right, we get a paper folding sequence and denote it by R_pR_q. If we iterate R_pR_q process n times, then we get another generalized paper folding sequence and denote it by $(R_pR_q)^n$.

Example 1.1. *Some examples of generalized paper folding sequences are given as follows :*

(1) R_4^3 : 0 0 0 0 1 1 1 0 0 0 0 0 1 1 1 1 0 0 0 1 1 1 1 1 0 0
0 1 1 1 1 1 0 0 0 0 1 1 1 0 0 0 0 0 1 1 1 1 0 0 0 1
1 1 1 1 0 0 0 1 1 1 1

(2) $(R_2R_3)^2$: 0 0 0 1 1 0 0 0 1 1 1 0 0 0 0 1 1 1 0 0 1 1 1 1
0 0 0 1 1 1 0 0 1 1 1

(3) $(R_3R_2)^2$: 0 0 1 0 0 1 1 1 0 1 1 1 0 0 1 0 0 0 1 1 0 1 1 1
0 0 1 0 0 1 1 1 0 1 1

Let X be a paper folding sequence. We define X^c the paper folding sequence obtained by reversing the order and swapping 0s and 1s in X. $|X|$ denotes the number of all 0s and 1s in X. $|X|_0$ and $|X|_1$ denote the number of all 0s in X and all 1s in X, respectively. The following lemma can be easily obtained by the definitions of $|X|, |X|_0, |X|_1$ and X^c.

Lemma 1.2. *Let X be a paper folding sequence. Then we have*
(1) $|X| = |X|_0 + |X|_1$
(2) $|X^c|_0 = |X|_1$
(3) $|X^c|_1 = |X|_0$
(4) $|X^c| = |X|$.

2. Number of 0s and 1s in R_p^n

Davis and Knuth [3] proved the following theorem and it provided us with impetus to probe the problems related to number of downwards and upwards in generalized paper folding sequences.

Theorem 2.1. *Let $p \in \mathbb{N}$ with $p \geq 2$. If R_p and X are paper folding sequences, then*

$$R_pX = \begin{cases} (X^c\,1\,X\,1\,X^c\,1\,X\,1\,\cdots\,1\,X^c\,1\,X) & \text{if } p \text{ is even} \\ (X\,1\,X^c\,1\,X\,1\,X^c\,1\,\cdots\,1\,X^c\,1\,X) & \text{if } p \text{ is odd.} \end{cases} \tag{2.1}$$

First, we compute the number of 0s and 1s in R_p^n using Theorem 2.1 and the properties of a recursive sequence.

Theorem 2.2. *If p is an even number with $p \geq 2$ and $n \in \mathbb{N}$, then*

$$|R_p^n|_0 = \frac{1}{2}(p^n - p) \quad \text{and} \quad |R_p^n|_1 = \frac{1}{2}(p^n + p - 2). \tag{2.2}$$

Proof. Since p is even and $R_p^n = R_pR_p^{n-1}$, Theorem 2.1 gives

$$R_p^n = \big((R_p^{n-1})^c\,1\,R_p^{n-1}\,1\,\cdots\,1\,(R_p^{n-1})^c\,1\,R_p^{n-1}\big). \tag{2.3}$$

Note that $(R_p^{n-1})^c$ and R_p^{n-1} appear $\frac{p}{2}$ times and $\frac{p}{2}$ times in (2.3), respectively. In addition, 1 appears $p-1$ times in (2.3).
By (2.3) and Lemma 1.2, we have

$$\begin{aligned} |R_p^n| &= \frac{p}{2}|(R_p^{n-1})^c| + \frac{p}{2}|R_p^{n-1}| + (p-1) \\ &= \frac{p}{2}|R_p^{n-1}| + \frac{p}{2}|R_p^{n-1}| + (p-1) \\ &= p|R_p^{n-1}| + (p-1). \end{aligned} \tag{2.4}$$

By adding 1 on both sides of (2.4), we get

$$\begin{aligned} |R_p^n| + 1 &= p|R_p^{n-1}| + p \\ &= p(|R_p^{n-1}| + 1) \\ &= p^2(|R_p^{n-2}| + 1) \\ &= \cdots \\ &= p^n(|R_p^0| + 1) \\ &= p^n, \end{aligned} \tag{2.5}$$

since $|R_p^0| = 0$. Thus

$$|R_p^n| = p^n - 1. \tag{2.6}$$

Now, we compute the number of 0s in R_p^n. By (2.4), (2.6) and Lemma 1.2, we get

$$
\begin{aligned}
|R_p^n|_0 &= \frac{p}{2}|(R_p^{n-1})^c|_0 + \frac{p}{2}|R_p^{n-1}|_0 \\
&= \frac{p}{2}|R_p^{n-1}|_1 + \frac{p}{2}|R_p^{n-1}|_0 \\
&= \frac{p}{2}|R_p^{n-1}| \qquad (2.7)\\
&= \frac{p}{2}(p^{n-1}-1) \\
&= \frac{1}{2}(p^n-p).
\end{aligned}
$$

Since the number of 1s can be computed by subtracting the number of 0s from the total number of creases, we have

$$
\begin{aligned}
|R_p^n|_1 &= |R_p^n| - |R_p^n|_0 \\
&= (p^n-1) - \frac{1}{2}(p^n-p) \qquad (2.8)\\
&= \frac{1}{2}(p^n+p-2).
\end{aligned}
$$

Thus we complete the proof. □

Now, we compute the number of 0s and 1s in R_p^n when p is odd with $p \geq 3$. In this case, we use a different property of a recursive sequence that is not used in Theorem 2.2.

Theorem 2.3. *If p is an odd number with $p \geq 3$ and $n \in \mathbb{N}$, then*

$$
(2.9) \quad |R_p^n|_0 = \frac{1}{2}(p^n - np + n - 1) \quad \text{and} \quad |R_p^n|_1 = \frac{1}{2}(p^n + np - n - 1).
$$

Proof. Since p is odd and $R_p^n = R_p R_p^{n-1}$, Theorem 2.1 gives

$$
(2.10) \qquad R_p^n = \big(R_p^{n-1}\ 1\ (R_p^{n-1})^c\ 1\ R_p^{n-1}\ 1\ \cdots\ 1\ (R_p^{n-1})^c\ 1\ R_p^{n-1}\big).
$$

Note that $(R_p^{n-1})^c$ and R_p^{n-1} appear $\frac{p-1}{2}$ times and $\frac{p+1}{2}$ times in (2.10), respectively. In addition, 1 appears $p-1$ times in (2.10).
By (2.10) and Lemma 1.2, we have

$$
\begin{aligned}
|R_p^n| &= \frac{p-1}{2}|(R_p^{n-1})^c| + \frac{p+1}{2}|R_p^{n-1}| + (p-1) \\
&= \frac{p-1}{2}|R_p^{n-1}| + \frac{p+1}{2}|R_p^{n-1}| + (p-1) \qquad (2.11)\\
&= p|R_p^{n-1}| + (p-1).
\end{aligned}
$$

By adding 1 on both sides of (2.11), we get

$$
\begin{aligned}
|R_p^n| + 1 &= p|R_p^{n-1}| + p \\
&= p(|R_p^{n-1}| + 1) \\
&= p^2(|R_p^{n-2}| + 1) \qquad (2.12)\\
&= \cdots \\
&= p^n(|R_p^0| + 1) \\
&= p^n,
\end{aligned}
$$

since $|R_p^0| = 0$. Thus

$$|R_p^n| = p^n - 1. \qquad (2.13)$$

Now, we compute the number of 0s in R_p^n. By (2.11), (2.13) and Lemma 1.2, we get

$$
\begin{aligned}
|R_p^n|_0 &= \frac{p-1}{2}|(R_p^{n-1})^c|_0 + \frac{p+1}{2}|R_p^{n-1}|_0 \\
&= \frac{p-1}{2}|R_p^{n-1}|_1 + \frac{p+1}{2}|R_p^{n-1}|_0 \\
&= |R_p^{n-1}|_0 + \frac{p-1}{2}\big(|R_p^{n-1}|_1 + |R_p^{n-1}|_0\big) \qquad (2.14)\\
&= |R_p^{n-1}|_0 + \frac{p-1}{2}|R_p^{n-1}| \\
&= |R_p^{n-1}|_0 + \frac{p-1}{2}(p^{n-1} - 1) \\
&= |R_p^{n-1}|_0 + \frac{1}{2}(p^n - p^{n-1} - p + 1).
\end{aligned}
$$

Recursively, we obtain from (2.14) that

$$
\begin{aligned}
|R_p^n|_0 - |R_p^{n-1}|_0 &= \frac{1}{2}(p^n - p^{n-1} - p + 1) \\
|R_p^{n-1}|_0 - |R_p^{n-2}|_0 &= \frac{1}{2}(p^{n-1} - p^{n-2} - p + 1) \\
&\ \vdots \qquad (2.15)\\
|R_p^1|_0 - |R_p^0|_0 &= \frac{1}{2}(p^1 - p^0 - p + 1).
\end{aligned}
$$

Note that $|R_p^0|_0 = 0$. By adding all left terms and all right terms of (2.15), respectively, we get

$$|R_p^n|_0 = |R_p^n|_0 - |R_p^0|_0 = \frac{1}{2}(p^n - np + n - 1). \qquad (2.16)$$

Since the number of 1s can be computed by subtracting the number of 0s from the total number of creases, we have

$$
\begin{aligned}
|R_p^n|_1 &= |R_p^n| - |R_p^n|_0 \\
&= (p^n - 1) - \frac{1}{2}(p^n - np + n - 1) \qquad (2.17)\\
&= \frac{1}{2}(p^n + np - n - 1).
\end{aligned}
$$

Thus we complete the proof. □

3. Number of 0s and 1s in $(R_pR_q)^n$

In this section, we compute the number of 0s and 1s in $(R_pR_q)^n$. First, we estimate the number of 0s and 1s in $(R_pR_q)^n$ when p and q are even.

Theorem 3.1. *Let p and q be even numbers with $p \geq 2$ and $q \geq 2$. For $n \in \mathbb{N}$, we have*

$$
|(R_pR_q)^n|_0 = \frac{1}{2}((pq)^n - p) \quad \text{and} \quad |(R_pR_q)^n|_1 = \frac{1}{2}((pq)^n + p - 2). \qquad (3.1)
$$

Proof. Since p and q are even, Theorem 2.1 gives

$$
\begin{aligned}
&(R_pR_q)^n \\
=\ & R_p\big(R_q(R_pR_q)^{n-1}\big) \qquad (3.2)\\
=\ & \Big(\big(R_q(R_pR_q)^{n-1}\big)^c \; 1 \; R_q(R_pR_q)^{n-1} \; 1 \; \cdots \; 1 \; R_q(R_pR_q)^{n-1}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&R_q(R_pR_q)^{n-1} \\
=\ & R_q\big((R_pR_q)^{n-1}\big) \qquad (3.3)\\
=\ & \Big(\big((R_pR_q)^{n-1}\big)^c \; 1 \; (R_pR_q)^{n-1} \; 1 \; \cdots \; 1 \; (R_pR_q)^{n-1}\Big).
\end{aligned}
$$

$\big(R_q(R_pR_q)^{n-1}\big)^c$ and $R_q(R_pR_q)^{n-1}$ appear $\frac{p}{2}$ times and $\frac{p}{2}$ times, respectively, and 1 appears $p-1$ times in (3.2). In addition, $\big((R_pR_q)^{n-1}\big)^c$ and $(R_pR_q)^{n-1}$ appear $\frac{q}{2}$ times and $\frac{q}{2}$ times, respectively, and 1 appears $q-1$ times in (3.3). By (3.2), (3.3) and Lemma 1.2, we have

$$
\begin{aligned}
|(R_pR_q)^n| &= \frac{p}{2}\big|\big(R_q(R_pR_q)^{n-1}\big)^c\big| + \frac{p}{2}|R_q(R_pR_q)^{n-1}| + (p-1) \\
&= \frac{p}{2}|R_q(R_pR_q)^{n-1}| + \frac{p}{2}|R_q(R_pR_q)^{n-1}| + (p-1) \qquad (3.4)\\
&= p|R_q(R_pR_q)^{n-1}| + (p-1)
\end{aligned}
$$

and

$$\begin{aligned} |R_q(R_pR_q)^{n-1}| &= \frac{q}{2}|\big((R_pR_q)^{n-1}\big)^c| + \frac{q}{2}|(R_pR_q)^{n-1}| + (q-1) \\ &= \frac{q}{2}|(R_pR_q)^{n-1}| + \frac{q}{2}|(R_pR_q)^{n-1}| + (q-1) \\ &= q|(R_pR_q)^{n-1}| + (q-1). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we get

$$\begin{aligned} |(R_pR_q)^n| &= p|R_q(R_pR_q)^{n-1}| + (p-1) \\ &= p\big(q|(R_pR_q)^{n-1}| + (q-1)\big) + (p-1) \\ &= pq|(R_pR_q)^{n-1}| + pq - 1. \end{aligned} \tag{3.6}$$

By adding 1 on both sides of (3.6), we have

$$\begin{aligned} |(R_pR_q)^n| + 1 &= pq|(R_pR_q)^{n-1}| + pq \\ &= pq\big(|(R_pR_q)^{n-1}| + 1\big) \\ &= (pq)^2\big(|(R_pR_q)^{n-2}| + 1\big) \\ &= \cdots \\ &= (pq)^n\big(|(R_pR_q)^0| + 1\big) \\ &= (pq)^n, \end{aligned} \tag{3.7}$$

since $|(R_pR_q)^0| = 0$. Thus

$$|(R_pR_q)^n| = (pq)^n - 1 \tag{3.8}$$

and

$$\begin{aligned} |R_q(R_pR_q)^{n-1}| &= q|(R_pR_q)^{n-1}| + (q-1) \\ &= q\big((pq)^{n-1} - 1\big) + (q-1) \\ &= p^{n-1}q^n - 1. \end{aligned} \tag{3.9}$$

Now, we compute the number of 0s and 1s in $(R_pR_q)^n$.
By (3.4), (3.9) and Lemma 1.2, we have

$$\begin{aligned} |(R_pR_q)^n|_0 &= \frac{p}{2}|\big(R_q(R_pR_q)^{n-1}\big)^c|_0 + \frac{p}{2}|R_q(R_pR_q)^{n-1}|_0 \\ &= \frac{p}{2}|R_q(R_pR_q)^{n-1}|_1 + \frac{p}{2}|R_q(R_pR_q)^{n-1}|_0 \\ &= \frac{p}{2}|R_q(R_pR_q)^{n-1}| \\ &= \frac{p}{2}(p^{n-1}q^n - 1) \\ &= \frac{1}{2}\big((pq)^n - p\big). \end{aligned} \tag{3.10}$$

By (3.8) and (3.10), we finally have

$$\begin{aligned} |(R_pR_q)^n|_1 &= |(R_pR_q)^n| - |(R_pR_q)^n|_0 \\ &= \big((pq)^n - 1\big) - \frac{1}{2}\big((pq)^n - p\big) \qquad (3.11) \\ &= \frac{1}{2}\big((pq)^n + p - 2\big). \end{aligned}$$

Therefore we prove (3.1). □

Now, we estimate the number of 0s and 1s in $(R_pR_q)^n$ when p is even and q is odd.

Theorem 3.2. *Let p be an even number with $p \geq 2$ and let q be an odd number with $q \geq 3$. For $n \in \mathbb{N}$, we have*

$$|(R_pR_q)^n|_0 = \frac{1}{2}((pq)^n - p) \quad \textit{and} \quad |(R_pR_q)^n|_1 = \frac{1}{2}((pq)^n + p - 2). \qquad (3.12)$$

Proof. Since p is even and q is odd, Theorem 2.1 gives

$$\begin{aligned} &(R_pR_q)^n \\ &= R_p\big(R_q(R_pR_q)^{n-1}\big) \qquad (3.13) \\ &= \Big(\big(R_q(R_pR_q)^{n-1}\big)^c\ 1\ R_q(R_pR_q)^{n-1}\ 1\ \cdots\ 1\ R_q(R_pR_q)^{n-1}\Big) \end{aligned}$$

and

$$\begin{aligned} &R_q(R_pR_q)^{n-1} \\ &= R_q\big((R_pR_q)^{n-1}\big) \qquad (3.14) \\ &= \Big((R_pR_q)^{n-1}\ 1\ \big((R_pR_q)^{n-1}\big)^c\ 1\ \cdots\ 1\ (R_pR_q)^{n-1}\Big). \end{aligned}$$

$\big(R_q(R_pR_q)^{n-1}\big)^c$ and $R_q(R_pR_q)^{n-1}$ appear $\frac{p}{2}$ times and $\frac{p}{2}$ times, respectively, and 1 appears $p-1$ times in (3.13). In addition, $\big((R_pR_q)^{n-1}\big)^c$ and $(R_pR_q)^{n-1}$ appear $\frac{q-1}{2}$ times and $\frac{q+1}{2}$ times, respectively, and 1 appears $q-1$ times in (3.14). By (3.13), (3.14) and Lemma 1.2, we have

$$\begin{aligned} |(R_pR_q)^n| &= \frac{p}{2}|\big(R_q(R_pR_q)^{n-1}\big)^c| + \frac{p}{2}|R_q(R_pR_q)^{n-1}| + (p-1) \\ &= \frac{p}{2}|R_q(R_pR_q)^{n-1}| + \frac{p}{2}|R_q(R_pR_q)^{n-1}| + (p-1) \qquad (3.15) \\ &= p|R_q(R_pR_q)^{n-1}| + (p-1) \end{aligned}$$

and

$$
\begin{aligned}
|R_q(R_pR_q)^{n-1}| &= \frac{q-1}{2}|\big((R_pR_q)^{n-1}\big)^c| + \frac{q+1}{2}|(R_pR_q)^{n-1}| + (q-1)\\
&= \frac{q-1}{2}|(R_pR_q)^{n-1}| + \frac{q+1}{2}|(R_pR_q)^{n-1}| + (q-1) \qquad (3.16)\\
&= q|(R_pR_q)^{n-1}| + (q-1).
\end{aligned}
$$

From (3.15) and (3.16), we get

$$
\begin{aligned}
|(R_pR_q)^n| &= p|R_q(R_pR_q)^{n-1}| + (p-1)\\
&= p\big(q|(R_pR_q)^{n-1}| + (q-1)\big) + (p-1) \qquad (3.17)\\
&= pq|(R_pR_q)^{n-1}| + pq - 1.
\end{aligned}
$$

By adding 1 on both sides of (3.17), we have

$$
\begin{aligned}
|(R_pR_q)^n| + 1 &= pq|(R_pR_q)^{n-1}| + pq\\
&= pq\big(|(R_pR_q)^{n-1}| + 1\big)\\
&= (pq)^2\big(|(R_pR_q)^{n-2}| + 1\big)\\
&= \cdots \qquad (3.18)\\
&= (pq)^n\big(|(R_pR_q)^0| + 1\big)\\
&= (pq)^n,
\end{aligned}
$$

since $|(R_pR_q)^0| = 0$. Thus

$$
|(R_pR_q)^n| = (pq)^n - 1 \qquad (3.19)
$$

and

$$
\begin{aligned}
|R_q(R_pR_q)^{n-1}| &= q|(R_pR_q)^{n-1}| + (q-1)\\
&= q\big((pq)^{n-1} - 1\big) + (q-1) \qquad (3.20)\\
&= p^{n-1}q^n - 1.
\end{aligned}
$$

Now, we compute the number of 0s and 1s in $(R_pR_q)^n$.
By (3.15), (3.20) and Lemma 1.2, we have

$$
\begin{aligned}
|(R_pR_q)^n|_0 &= \frac{p}{2}|\big(R_q(R_pR_q)^{n-1}\big)^c|_0 + \frac{p}{2}|R_q(R_pR_q)^{n-1}|_0\\
&= \frac{p}{2}|\big(R_q(R_pR_q)^{n-1}\big)|_1 + \frac{p}{2}|R_q(R_pR_q)^{n-1}|_0\\
&= \frac{p}{2}|R_q(R_pR_q)^{n-1}| \qquad (3.21)\\
&= \frac{p}{2}(p^{n-1}q^n - 1)\\
&= \frac{1}{2}\big((pq)^n - p\big).
\end{aligned}
$$

By (3.19) and (3.21), we finally have

$$
\begin{aligned}
|(R_pR_q)^n|_1 &= |(R_pR_q)^n| - |(R_pR_q)^n|_0 \\
&= \big((pq)^n - 1\big) - \frac{1}{2}\big((pq)^n - p\big) \qquad (3.22)\\
&= \frac{1}{2}\big((pq)^n + p - 2\big).
\end{aligned}
$$

Therefore we prove (3.12). □

Now, we estimate the number of 0s and 1s in $(R_pR_q)^n$ when p is odd and q is even.

Theorem 3.3. *Let p be an odd number with $p \geq 3$ and let q be an even number with $q \geq 2$. For $n \in \mathbb{N}$, we have*

$$|(R_pR_q)^n|_0 = \frac{1}{2}((pq)^n - p - q + 1) \qquad (3.23)$$

and

$$|(R_pR_q)^n|_1 = \frac{1}{2}((pq)^n + p + q - 3). \qquad (3.24)$$

Proof. Since p is odd and q is even, Theorem 2.1 gives

$$
\begin{aligned}
&(R_pR_q)^n \\
(3.25) \quad =\ & R_p\big(R_q(R_pR_q)^{n-1}\big) \\
=\ & \Big(R_q(R_pR_q)^{n-1}\ 1\ \big(R_q(R_pR_q)^{n-1}\big)^c\ 1\ \cdots\ 1\ R_q(R_pR_q)^{n-1}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&R_q(R_pR_q)^{n-1} \\
(3.26) \quad =\ & R_q\big((R_pR_q)^{n-1}\big) \\
=\ & \Big(\big((R_pR_q)^{n-1}\big)^c\ 1\ (R_pR_q)^{n-1}\ 1\ \cdots\ 1\ (R_pR_q)^{n-1}\Big).
\end{aligned}
$$

$\big(R_q(R_pR_q)^{n-1}\big)^c$ and $R_q(R_pR_q)^{n-1}$ appear $\frac{p-1}{2}$ times and $\frac{p+1}{2}$ times, respectively, and 1 appears $p-1$ times in (3.25). In addition, $\big((R_pR_q)^{n-1}\big)^c$ and $(R_pR_q)^{n-1}$ appear $\frac{q}{2}$ times and $\frac{q}{2}$ times, respectively, and 1 appears $q-1$ times in (3.26). By (3.25), (3.26) and Lemma 1.2, we have

$$
\begin{aligned}
|(R_pR_q)^n| &= \frac{p-1}{2}\big|\big(R_q(R_pR_q)^{n-1}\big)^c\big| + \frac{p+1}{2}|R_q(R_pR_q)^{n-1}| + (p-1) \\
&= \frac{p-1}{2}|R_q(R_pR_q)^{n-1}| + \frac{p+1}{2}|R_q(R_pR_q)^{n-1}| + (p-1) \qquad (3.27)\\
&= p|R_q(R_pR_q)^{n-1}| + (p-1)
\end{aligned}
$$

and

$$
\begin{aligned}
|R_q(R_pR_q)^{n-1}| &= \frac{q}{2}|\big((R_pR_q)^{n-1}\big)^c| + \frac{q}{2}|(R_pR_q)^{n-1}| + (q-1) \\
&= \frac{q}{2}|(R_pR_q)^{n-1}| + \frac{q}{2}|(R_pR_q)^{n-1}| + (q-1) \\
&= q|(R_pR_q)^{n-1}| + (q-1).
\end{aligned} \tag{3.28}
$$

From (3.27) and (3.28), we get

$$
\begin{aligned}
|(R_pR_q)^n| &= p|R_q(R_pR_q)^{n-1}| + (p-1) \\
&= p\big(q|(R_pR_q)^{n-1}| + (q-1)\big) + (p-1) \\
&= pq|(R_pR_q)^{n-1}| + pq - 1.
\end{aligned} \tag{3.29}
$$

By adding 1 on both sides of (3.29), we have

$$
\begin{aligned}
|(R_pR_q)^n| + 1 &= pq|(R_pR_q)^{n-1}| + pq \\
&= pq\big(|(R_pR_q)^{n-1}| + 1\big) \\
&= (pq)^2\big(|(R_pR_q)^{n-2}| + 1\big) \\
&= \cdots \\
&= (pq)^n\big(|(R_pR_q)^0| + 1\big) \\
&= (pq)^n,
\end{aligned} \tag{3.30}
$$

since $|(R_pR_q)^0| = 0$. Thus

$$
|(R_pR_q)^n| = (pq)^n - 1 \tag{3.31}
$$

and

$$
\begin{aligned}
|R_q(R_pR_q)^{n-1}| &= q|(R_pR_q)^{n-1}| + (q-1) \\
&= q\big((pq)^{n-1} - 1\big) + (q-1) \\
&= p^{n-1}q^n - 1.
\end{aligned} \tag{3.32}
$$

Now, we compute the number of 0s and 1s in $(R_pR_q)^n$.

By (3.27), (3.28), (3.31), (3.32) and Lemma 1.2, we have

$$
\begin{aligned}
(3.33)\quad & |(R_pR_q)^n|_0 \\
= \;& \frac{p-1}{2}\big|\big(R_q(R_pR_q)^{n-1}\big)^c\big|_0 + \frac{p+1}{2}|R_q(R_pR_q)^{n-1}|_0 \\
= \;& \frac{p-1}{2}|R_q(R_pR_q)^{n-1}|_1 + \frac{p+1}{2}|R_q(R_pR_q)^{n-1}|_0 \\
= \;& |R_q(R_pR_q)^{n-1}|_0 + \frac{p-1}{2}\Big(|R_q(R_pR_q)^{n-1}|_1 + |R_q(R_pR_q)^{n-1}|_0\Big) \\
= \;& |R_q(R_pR_q)^{n-1}|_0 + \frac{p-1}{2}|R_q(R_pR_q)^{n-1}| \\
= \;& \frac{q}{2}\Big(\big|\big((R_pR_q)^{n-1}\big)^c\big|_0 + |(R_pR_q)^{n-1}|_0\Big) + \frac{p-1}{2}|R_q(R_pR_q)^{n-1}| \\
= \;& \frac{q}{2}\Big(|(R_pR_q)^{n-1}|_1 + |(R_pR_q)^{n-1}|_0\Big) + \frac{p-1}{2}|R_q(R_pR_q)^{n-1}| \\
= \;& \frac{q}{2}|(R_pR_q)^{n-1}| + \frac{p-1}{2}|R_q(R_pR_q)^{n-1}| \\
= \;& \frac{q}{2}(p^{n-1}q^{n-1}-1) + \frac{p-1}{2}(p^{n-1}q^n-1) \\
= \;& \frac{1}{2}\big((pq)^n - p - q + 1\big).
\end{aligned}
$$

By (3.31) and (3.33), we finally have

$$
\begin{aligned}
|(R_pR_q)^n|_1 &= |(R_pR_q)^n| - |(R_pR_q)^n|_0 \\
&= \big((pq)^n - 1\big) - \frac{1}{2}\big((pq)^n - p - q + 1\big) \qquad (3.34)\\
&= \frac{1}{2}\big((pq)^n + p + q - 3\big).
\end{aligned}
$$

Therefore we prove (3.23) and (3.24). □

Finally, we estimate the number of 0s and 1s in $(R_pR_q)^n$ when p and q are odd. In the proof, we use special properties of recursive sequences that are not used in Theorem 3.1, Theorem 3.2 and Theorem 3.3.

Theorem 3.4. *Let p and q be odd numbers with $p \geq 3$ and $q \geq 3$. For $n \in \mathbb{N}$, we have*

$$(3.35)\qquad |(R_pR_q)^n|_0 = \frac{1}{2}\big((pq)^n - n(p+q-2) - 1\big)$$

and

$$(3.36)\qquad |(R_pR_q)^n|_1 = \frac{1}{2}\big((pq)^n + n(p+q-2) - 1\big).$$

Proof. Since p and q are odd, Theorem 2.1 gives

$$\begin{aligned} &(R_pR_q)^n \\ (3.37) = \quad & R_p(R_q(R_pR_q)^{n-1}) \\ = \quad & \Big(R_q(R_pR_q)^{n-1}\ 1\ \big(R_q(R_pR_q)^{n-1}\big)^c\ 1\ \cdots\ 1\ R_q(R_pR_q)^{n-1}\Big) \end{aligned}$$

and

$$\begin{aligned} & R_q(R_pR_q)^{n-1} \\ (3.38) \qquad = \quad & R_q((R_pR_q)^{n-1}) \\ = \quad & \Big((R_pR_q)^{n-1}\ 1\ \big((R_pR_q)^{n-1}\big)^c\ 1\ \cdots\ 1\ (R_pR_q)^{n-1}\Big). \end{aligned}$$

$\big(R_q(R_pR_q)^{n-1}\big)^c$ and $R_q(R_pR_q)^{n-1}$ appear $\frac{p-1}{2}$ times and $\frac{p+1}{2}$ times, respectively, and 1 appears $p-1$ times in (3.37). In addition, $\big((R_pR_q)^{n-1}\big)^c$ and $(R_pR_q)^{n-1}$ appear $\frac{q-1}{2}$ times and $\frac{q+1}{2}$ times, respectively, and 1 appears $q-1$ times in (3.38). By (3.37), (3.38) and Lemma 1.2, we have

$$\begin{aligned} |(R_pR_q)^n| &= \frac{p-1}{2}|\big(R_q(R_pR_q)^{n-1}\big)^c| + \frac{p+1}{2}|R_q(R_pR_q)^{n-1}| + (p-1) \\ (3.39) \qquad &= \frac{p-1}{2}|R_q(R_pR_q)^{n-1}| + \frac{p+1}{2}|R_q(R_pR_q)^{n-1}| + (p-1) \\ &= p|R_q(R_pR_q)^{n-1}| + (p-1) \end{aligned}$$

and

$$\begin{aligned} |R_q(R_pR_q)^{n-1}| &= \frac{q-1}{2}|\big((R_pR_q)^{n-1}\big)^c| + \frac{q+1}{2}|(R_pR_q)^{n-1}| + (q-1) \\ (3.40) \qquad &= \frac{q-1}{2}|(R_pR_q)^{n-1}| + \frac{q+1}{2}|(R_pR_q)^{n-1}| + (q-1) \\ &= q|(R_pR_q)^{n-1}| + (q-1). \end{aligned}$$

From (3.39) and (3.40), we get

$$\begin{aligned} |(R_pR_q)^n| &= p|R_q(R_pR_q)^{n-1}| + (p-1) \\ (3.41) \qquad &= p\big(q|(R_pR_q)^{n-1}| + (q-1)\big) + (p-1) \\ &= pq|(R_pR_q)^{n-1}| + pq-1. \end{aligned}$$

By adding 1 on both sides of (3.41), we have

$$
\begin{aligned}
|(R_pR_q)^n|+1 &= pq|(R_pR_q)^{n-1}|+pq \\
&= pq\big(|(R_pR_q)^{n-1}|+1\big) \\
&= (pq)^2\big(|(R_pR_q)^{n-2}|+1\big) \\
&= \cdots \\
&= (pq)^n\big(|(R_pR_q)^0|+1\big) \\
&= (pq)^n,
\end{aligned}
\tag{3.42}
$$

since $|(R_pR_q)^0|=0$. Thus

$$
|(R_pR_q)^n| = (pq)^n - 1 \tag{3.43}
$$

and

$$
\begin{aligned}
|R_q(R_pR_q)^{n-1}| &= q|(R_pR_q)^{n-1}|+(q-1) \\
&= q\big((pq)^{n-1}-1\big)+(q-1) \\
&= p^{n-1}q^n-1.
\end{aligned}
\tag{3.44}
$$

Now, we compute the number of 0s and 1s in $(R_pR_q)^n$.
By (3.39) and Lemma 1.2, we have

$$
\begin{aligned}
&|(R_pR_q)^n|_0 \\
&= \frac{p-1}{2}|\big(R_q(R_pR_q)^{n-1}\big)^c|_0+\frac{p+1}{2}|R_q(R_pR_q)^{n-1}|_0 \\
&= \frac{p-1}{2}|R_q(R_pR_q)^{n-1}|_1+\frac{p+1}{2}|R_q(R_pR_q)^{n-1}|_0
\end{aligned}
\tag{3.45}
$$

and

$$
\begin{aligned}
&|(R_pR_q)^n|_1 \\
&= \frac{p-1}{2}|\big(R_q(R_pR_q)^{n-1}\big)^c|_1+\frac{p+1}{2}|R_q(R_pR_q)^{n-1}|_1+(p-1) \\
&= \frac{p-1}{2}|R_q(R_pR_q)^{n-1}|_0+\frac{p+1}{2}|R_q(R_pR_q)^{n-1}|_1+(p-1).
\end{aligned}
\tag{3.46}
$$

From (3.45) and (3.46), we get

$$
\begin{aligned}
&|(R_pR_q)^n|_1-|(R_pR_q)^n|_0 \\
&= |R_q(R_pR_q)^{n-1}|_1-|R_q(R_pR_q)^{n-1}|_0+(p-1).
\end{aligned}
\tag{3.47}
$$

By (3.40) and Lemma 1.2, we have

$$\begin{aligned} & |R_q(R_pR_q)^{n-1}|_0 \\ = & \frac{q-1}{2}|\big((R_pR_q)^{n-1}\big)^c|_0 + \frac{q+1}{2}|(R_pR_q)^{n-1}|_0 \\ = & \frac{q-1}{2}|(R_pR_q)^{n-1}|_1 + \frac{q+1}{2}|(R_pR_q)^{n-1}|_0 \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} & |R_q(R_pR_q)^{n-1}|_1 \\ = & \frac{q-1}{2}|\big((R_pR_q)^{n-1}\big)^c|_1 + \frac{q+1}{2}|(R_pR_q)^{n-1}|_1 + (q-1) \\ = & \frac{q-1}{2}|(R_pR_q)^{n-1}|_0 + \frac{q+1}{2}|(R_pR_q)^{n-1}|_1 + (q-1). \end{aligned} \tag{3.49}$$

From (3.48) and (3.49), we get

$$\begin{aligned} & |R_q(R_pR_q)^{n-1}|_1 - |R_q(R_pR_q)^{n-1}|_0 \\ = & |(R_pR_q)^{n-1}|_1 - |(R_pR_q)^{n-1}|_0 + (q-1). \end{aligned} \tag{3.50}$$

By (3.47) and (3.50), we get

$$\begin{aligned} & |(R_pR_q)^n|_1 - |(R_pR_q)^n|_0 \\ = & |(R_pR_q)^{n-1}|_1 - |(R_pR_q)^{n-1}|_0 + (p-1) + (q-1) \\ = & |(R_pR_q)^{n-2}|_1 - |(R_pR_q)^{n-2}|_0 + 2(p-1) + 2(q-1) \\ = & \cdots \\ = & |(R_pR_q)^0|_1 - |(R_pR_q)^0|_0 + n(p-1) + n(q-1). \end{aligned} \tag{3.51}$$

Since $|(R_pR_q)^0|_1 = |(R_pR_q)^0|_0 = 0$, we get

$$\begin{aligned} |(R_pR_q)^n|_1 - |(R_pR_q)^n|_0 &= n(p-1) + n(q-1) \\ &= n(p+q-2). \end{aligned} \tag{3.52}$$

From (3.43) and Lemma 1.2, we have

$$|(R_pR_q)^n|_1 + |(R_pR_q)^n|_0 = |(R_pR_q)^n| = (pq)^n - 1. \tag{3.53}$$

By combining (3.52) and (3.53), we have

$$|(R_pR_q)^n|_0 = \frac{1}{2}\big((pq)^n - n(p+q-2) - 1\big) \tag{3.54}$$

and

$$|(R_pR_q)^n|_1 = \frac{1}{2}\big((pq)^n + n(p+q-2) - 1\big). \tag{3.55}$$

Therefore we prove (3.35) and (3.36). □

From Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, we obtain the following.

Corollary 3.5. *For any $p, q \in \mathbb{N}$ with $p \geq 2$ and $q \geq 2$, we have*

$$|(R_pR_q)^n| = (pq)^n - 1. \tag{3.56}$$

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