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A NEW LOWER BOUND FOR THE VOLUME PRODUCT OF A CONVEX BODY WITH CONSTANT WIDTH AND POLAR DUAL OF ITS p-CENTROID BODY

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Abstract. In this paper, we prove that if K is a convex body in E^n and E_i and E_o are inscribed ellipsoid and circumscribed ellipsoid of K respectively with $\alpha E_i = E_o$, then $\left[(\alpha)^{\frac{n}{p}+1}\right]^n \omega_n^2 \geq V(K)V(\Gamma_p^* K) \geq \left[\left(\frac{1}{\alpha}\right)^{\frac{n}{p}+1}\right]^n \omega_n^2$. Lutwak and Zhang[6] proved that if K is a convex body, $\omega_n^2 = V(K)V(\Gamma_p K)$ if and only if K is an ellipsoid. Our inequality provides very elementary proof for their result and this in turn gives a lower bound of the volume product for the sets of constant width.

1. Introduction

Let V denote the n-dimensional Lebesgue measure and ω_n denote the n-dimensional volume of closed unit ball in E^n. For a star body K in E^n denote a polar body of K by K^* and its p-centroid body by $\Gamma_p K$ and call $\Gamma_p^* K$, the polar body of $\Gamma_p K$.

Lutwak and Zhang[6] determined upper bound of $V(K)V(\Gamma_p K)$ for the convex body K and proved that

$$\omega_n^2 = V(K)V(\Gamma_p K), \tag{1}$$

if and only if $K$is an ellipsoid.

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In this paper, we develop bound for the volume product of a convex set and polar body of its p-centroid body. And then we focus on convex bodies of constant width. Convex bodies of constant width are studied by many researchers on convex geometry[1, 2, 3, 4, 5, 8]. Convex body of constant width is a convex body of same width in every direction. Among others Reuleaux triangles and regular spheres are the typical ones. However, regular sphere is the only one example of constant width which is centrally symmetric convex body.

Our main theorems are the followings:

Theorem 1. *If K is a convex body in E^n and E_i and E_o are ellipsoids such that $E_i \subset K \subset E_o$ and $\alpha E_i = E_o$, then*

$$\left[(\alpha)^{\frac{n}{p}+1}\right]^n \omega_n^2 \geq V(K)V(\Gamma_p^* K) \geq \left[\left(\frac{1}{\alpha}\right)^{\frac{n}{p}+1}\right]^n \omega_n^2. \tag{2}$$

Theorem 2. *Let K be a convex body of constant width in E^n. Then for each real $p \geq n$,*

$$V(K)V(\Gamma_p^* K) \geq \left((\sqrt{2}-1)^2\right)^n \omega_n^2.$$

2. Preliminaries

For each direction $u \in S^{n-1}$, we define the *support function* $h(K,u)$ on S^{n-1} of the convex body K by

$$h(K,u) = \sup\{u \cdot x | x \in K\}$$

and the *radial function* $\rho(K,u)$ on S^{n-1} of the convex body K by

$$\rho(K,u) = \sup\{\lambda > 0 | \lambda u \in K\}.$$

Let S^{n-1} denote the unit sphere centered at the origin in E^n, and write O_{n-1} for the $(n-1)$-dimensional volume of S^{n-1}. For the closed unit ball B in E^n, ω_n is defined by

$$\omega_n = \pi^{n/2}/\Gamma(1+\frac{n}{2}), \qquad \text{and} \qquad O_{n-1} = n\omega_n.$$

where Γ denotes Gamma function.

The *polar body* of a convex body K, denoted by K^*, is a convex body defined by

$$K^* = \{y | x \cdot y \ \leq 1 \text{ for all } \ x \in K\}.$$

It is easily verified that for convex bodies K_1, K_2 in E^n, $K_1 \subset K_2$ implies $K_2^* \subset K_1^*$.

For a convex body K in E^n and real $p \geq 1$, the *p-centroid body*, $\Gamma_p K$, of K is the convex body defined by the support function $h(\Gamma_p K, x)$:

$$h(\Gamma_p K, x) = \left(\frac{\omega_n^2 \omega_2 \omega_{p-1}}{\omega_{n+p}}\right)^{\frac{1}{p}} \left(\frac{1}{V(K)}\right)^{\frac{1}{p}} \left(\int_{S^{n-1}} |x \cdot v|^p \rho(K, v)^{n+p} dv\right)^{\frac{1}{p}} \tag{3}$$

for all $x \in E^n$.

From the definition of the p-centroid body, it is easy to see that if E is a centered ellipsoid, then

$$\Gamma_p E = E. \tag{4}$$

One can also easily check that for a convex body K in E^n and a positive real number r

$$\Gamma_p(rK) = r\Gamma_p K. \tag{5}$$

The width $w(K, u)$ of a convex body K in a direction $u \in S^{n-1}$ is defined by $w(K, u) = h(K, u) + h(K, -u)$, which is, of course, the distance between the two supporting planes of K orthogonal to u. The convex body K is said to be of constant width if $w(K, u)$ is a constant for all $u \in S^{n-1}$.

3. Main Results

Theorem 3. *If K is a convex body in E^n and E_i and E_o are ellipsoids such that $E_i \subset K \subset E_o$ and $\alpha E_i = E_o$, then*

$$\left[(\alpha)^{\frac{n}{p}+1}\right]^n \omega_n^2 \geq V(K)V(\Gamma_p^* K) \geq \left[\left(\frac{1}{\alpha}\right)^{\frac{n}{p}+1}\right]^n \omega_n^2. \tag{6}$$

Proof :

For the left hand side inequality, let E_i and E_o be the ellipsoids such that $E_i \subset K \subset E_o$ and $\alpha E_i = E_o$. Then $K \subset \alpha E_i$.

Since

$$\alpha^n V(E_i) \geq V(K), \tag{7}$$

$$\left(\frac{1}{V(K)}\right)^{\frac{1}{n}} \geq \left(\frac{1}{V(E_i)}\right)^{\frac{1}{n}} \alpha^{-\frac{n}{p}}. \tag{8}$$

But $\rho(K) \geq \rho(E_i)$. By the definition (3) of $h(\Gamma_p K, x)$

$$h(\Gamma_p K) \geq \alpha^{-\frac{n}{p}} h(\Gamma_p(E_i)). \tag{9}$$

So

$$\Gamma_p^* K \subset \alpha^{\frac{n}{p}} \Gamma_p^* E_i$$

or

$$V(\Gamma_p^* K) \leq \alpha^{\frac{n^2}{p}} V(\Gamma_p^* E_i). \tag{10}$$

By (7) and(10), we have

$$\left[(\alpha)^{\frac{n}{p}+1}\right]^n \omega_n^2 \geq V(K)V(\Gamma_p^* K)$$

which follows from $V(E_i)V(\Gamma_p^* E_i) = \omega_n^2$.

For the right hand side inequality, we have $E_o \subset \alpha K$ which implies $V(E_o) \leq \alpha^n V(K)$. So

$$\left(\frac{1}{V(K)}\right)^{\frac{1}{p}} \leq \alpha^{\frac{n}{p}} \left(\frac{1}{V(E_o)}\right)^{\frac{1}{p}}. \tag{11}$$

Using $\rho(K, x) \leq \rho(E_o, x)$ and (11), we have

$$h(\Gamma_p K, x) \leq \alpha^{\frac{n}{p}} h(\Gamma_p E_o, x),$$

which implies

$$\left(\frac{1}{\alpha}\right)^{\frac{n}{p}} \Gamma_p^* E_o \subset \Gamma_p^* K.$$

So we have

$$\left(\frac{1}{\alpha}\right)^{\frac{n^2}{p}} V(\Gamma_p^* E_o) \leq V(\Gamma_p^* K). \tag{12}$$

From (12) and the relation that $V(E_i) \leq V(K)$, we get

$$V(K)V(\Gamma_p^* K) \geq \left(\frac{1}{\alpha}\right)^{\frac{n^2}{p}} V(E_i)V(\Gamma_p^* E_o). \tag{13}$$

Since $E_o = \alpha E_i$,

$$\frac{1}{\alpha}\Gamma_p^* E_i = \Gamma_p^* E_o.$$

Hence we have

$$V(\Gamma_p^* E_o) = \left(\frac{1}{\alpha}\right)^n V(\Gamma_p^* E_i). \tag{14}$$

Substituting (14) for (13), we obtain

$$V(K)V(\Gamma_p^* K) \geq [(\frac{1}{\alpha})^{\frac{n}{p}+1}]^n \omega_n^2.$$

which follows from $V(E_i)V(\Gamma_p^* E_i) = \omega_n^2$. $\square$

The following corollary provides very elementary but short proof for (1) compared with the proof by Lutwak and Zhang in [6].

Corollary 1. *$V(K)V(\Gamma_p^* K) = \omega_n^2$ if and only if K is an ellipsoid.*

Proof : Note that $V(K)V(\Gamma_p^* K) = \omega_n^2$ if and only if α=1 in (6) if and only if K is an ellipsoid. $\square$

Theorem 4. *Let K be a convex body of constant width in E^n. Then for each real $p \geq n$,*

$$V(K)V(\Gamma_p^* K) \geq \left((\sqrt{2}-1)^2\right)^n \omega_n^2.$$

Proof : Melzak [7] proved that any set of width 1 has a unique cocentric circumscribed ball B_{r_0} of radius r_o and inscribed ball B_{r_i} of radius r_i such that $1 - [\frac{n}{2(n+1)}]^{\frac{1}{2}} \leq r_i$ and $[\frac{n}{2(n+1)}]^{\frac{1}{2}} \geq r_o$.

So it is easy to check that any convex body of constant width b in E^n has the concentric inscribed ball B_i and circumscribed ball B_o with their radii, r_i and r_o, respectively, satisfying $B_i \subset K \subset B_o$ and

$$r_i + r_o = b \quad \text{and} \quad b\left(1 - \sqrt{\frac{n}{2n+2}}\right) \leq r_i \leq r_o \leq b\sqrt{\frac{n}{2n+2}}. \tag{15}$$

Now let

$$\lambda = \frac{\sqrt{\frac{n}{2n+2}}}{1 - \sqrt{\frac{n}{2n+2}}}.$$

Then $\lambda B_i = B_o$. Let $\alpha = \sqrt{\frac{n}{2n+2}}$. Then, for $n \geq 2$, $\sqrt{\frac{n}{n+1}} < 1$, and so $0 < \alpha < \frac{1}{\sqrt{2}}$. Therefore $\lambda < \sqrt{2}+1$. This means that we may choose balls B_i and B_o such that $B_i \subset K \subset B_o$ and $B_o = (\sqrt{2}+1)B_i$. Now by Theorem 3, we have

$$V(K)V(\Gamma_p^* K) \geq \left[\left(\sqrt{2}-1\right)^{\frac{n}{p}+1}\right]^n \omega_n^2.$$

So for $p \geq n$

$$V(K)V(\Gamma_p^* K) \geq \left[\left(\sqrt{2}-1\right)^2\right]^n \omega_n^2.$$

$\square$

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