A CLASS OF INFINITE SERIES FROM GENERALIZED EISENSTEIN SERIES

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Abstract. B. C. Berndt has found a transformation formula for a large class of functions, which comes from a transformation formula for a more general class of Eisenstein series. In this paper, using his formula which is 'twisted' by change of variables, we find a class of infinite series identities.

1. Introduction

B. C. Berndt [3] found a transformation formula(Theorem 1.1) for a large class of functions which originally comes from generalized Eisenstein series. In this paper, using his fromula, we derive identities about infinite series. Some of the results are found in the Notebooks of Ramanujan [7] or have been proved by Berndt([2, 3]). Specially, we obtain elegant symmetric identities(Corollary 2.4–Corollary 2.6) and find relations with sums of Bernoulli's polynomials(Corollary 2.11, Corollary 2.12).

We shall use the following notation. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors. For a complex number w, the branch of the argument is given by $-\pi \leq \arg w < \pi$. Let $e(w) = e^{2\pi i w}$ and let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$, the upper half-plane. For $\tau \in \mathbb{H}$ and an arbitrary complex number s, define

$$A_N(\tau, s; r, h) := \sum_{Nm+r_1>0} \sum_{n-h_2>0} \frac{e\left(Nmh_1 + ((Nm+r_1)\tau + r_2)(n-h_2)\right)}{(n-h_2)^{1-s}},$$

where N is a positive integer. Let

$$H_N(\tau, s; r, h) := A_N(\tau, s; r, h) + e\left(\frac{s}{2}\right) A_N(\tau, s; -r, -h).$$

The characteristic function of the integers modulo N is denoted by λ_N, i.e.,

$$\lambda_N(m) = \begin{cases} 1, & \text{if } m \equiv 0 \pmod{N}, \\ 0, & \text{otherwise.} \end{cases}$$

For real x, y and Re $s > 1$, let

$$\psi(x, y, s) := \sum_{n+y>0} \frac{e(nx)}{(n+y)^s}.$$

For a real number x, let $\{x\} = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x. Let $V\tau = V(\tau) = \frac{a\tau+b}{c\tau+d}$ denote a modular transformation with $c > 0$ and $c \equiv 0 \pmod{N}$ for every $\tau \in \mathbb{H}$. The vectors R and H are defined by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

We now state a twist version of which theorem in Berndt's paper [3].

Theorem 1.1. ([3]). *Let $Q = \{\tau \in \mathbb{H} \mid \operatorname{Re}(\tau) > -d/c\}$, $\varrho_N = c\{R_2\} - Nd\{R_1/N\}$ and $c = c'N$. Then for $\tau \in Q$ and all s,*

$$\begin{aligned}
(c\tau + d)^{-s} H_N(V\tau, s; r, h) &= H_N(\tau, s; R, H) \\
&\quad - \lambda_N(r_1)e(-r_1h_1)(c\tau+d)^{-s}\Gamma(s)(-2\pi i)^{-s}\left(\psi(h_2, r_2, s) + e\left(\frac{s}{2}\right)\psi(-h_2, -r_2, s)\right) \\
&\quad + \lambda_N(R_1)e(-R_1H_1)\Gamma(s)(-2\pi i)^{-s}\left(\psi(H_2, R_2, s) + e\left(-\frac{s}{2}\right)\psi(-H_2, -R_2, s)\right) \\
&\quad + (2\pi i)^{-s}L_N(\tau, s; R, H),
\end{aligned}$$

where

$$L_N(\tau, s; R, H) := \sum_{j=1}^{c'} e(-H_1(Nj + N[R_1/N] - c) - H_2([R_2] + 1 + [(Njd + \varrho_N)/c] - d))$$

$$\cdot \int_C u^{s-1} \frac{e^{-(c\tau+d)(Nj - N\{R_1/N\})u/c}}{e^{-(c\tau+d)u} - e(cH_1 + dH_2)} \frac{e^{\{(Njd + \varrho_N)/c\}u}}{e^u - e(-H_2)}\, du,$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$(e^{-(c\tau+d)u} - e(cH_1 + dH_2))(e^u - e(-H_2))$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

If s is an integer, then we can evaluate the integration in Theorem 1.1 by using the residue theorem. Note that after evaluation of $L_N(\tau, s; R, H)$ for an integer s, the transformation formula in Theorem 1.1 will be valid for all $\tau \in \mathbb{H}$ by analytic continuation.

We shall use the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \ (|t| < 2\pi)$$

for the Bernoulli polynomials $B_n(x)$, $n \geq 0$. The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. Recall that $B_{2n+1} = 0$, $n \geq 1$, and that $B_{2n+1}(1/2) = 0$, $n \geq 0$. The following formulas are useful [1];

$$B_n(1 - x) = (-1)^n B_n(x),$$

$$\sum_{j=0}^{c-1} B_n\left(\frac{j}{c} + x\right) = c^{1-n}B_n(cx),$$

$$B_n\left(\frac{1}{2}\right) = -(1 - 2^{1-n})B_n, \ n \geq 0.$$

2. A class of infinite series

In this section, we find a class of infinite series from Theorem 1.1 under a modular transformation. Let N be a positive integer, and let r_1 and r_2 be arbitrary real numbers. Put

$$r = \left(r_1, \frac{r_2}{N}\right), \ h = (0,0), \ \tau = \frac{N-1}{N} + \frac{1}{N}z, \ V\tau = \frac{1}{N} - \frac{1}{N}\frac{1}{z}$$

for $\operatorname{Re}(z) \geq 0$. Here, V is a modular transformation corresponding to the matrix

$$\begin{pmatrix} 1 & -1 \\ N & -N+1 \end{pmatrix}.$$

Then we see that $(R_1, R_2) = (r_1 + r_2, -r_1 - r_2 + r_2/N)$. Employing above r, h, τ, we obtain the following results. Let n be an arbitrary integer. For $N \nmid r_1$,

$$(2.1) \quad H_N(V\tau, -2n; r, 0) = \sum_{k=1}^{\infty} \frac{\cosh(\pi i k(2(r_1 + r_2)/N + (1 - 2\{r_1/N\})/z))}{k^{2n+1}\sinh(\pi i k/z)},$$

and
(2.2)

$$H_N(V\tau, -2n - 1; r, 0) = \sum_{k=1}^{\infty} \frac{\sinh(\pi i k(2(r_1 + r_2)/N + (1 - 2\{r_1/N\})/z))}{k^{2n+2}\sinh(\pi i k/z)}.$$

For $N \nmid R_1 = r_1 + r_2$,

$$(2.3) \quad H_N(\tau, -2n; R, 0) = \sum_{k=1}^{\infty} \frac{\cosh(\pi i k(-2r_1/N + (2\{(r_1+r_2)/N\} - 1)z))}{k^{2n+1} \sinh(-\pi i k z)},$$

and
(2.4)

$$H_N(\tau, -2n-1; R, 0) = \sum_{k=1}^{\infty} \frac{\sinh(\pi i k(-2r_1/N + (2\{(r_1+r_2)/N\} - 1)z))}{k^{2n+2} \sinh(-\pi i k z)}.$$

Next, we have, by the residue theorem,

$$
\begin{aligned}
L_N(\tau, -n; R, 0) &= \int_C u^{-n-1} \frac{e^{-z(1-\{R_1/N\})u}}{e^{-zu}-1} \frac{e^{\{\varrho_N/N\}u}}{e^u - 1} du \\
&= -z^{-1} \int_C u^{-n-3} \sum_{\ell=0}^{\infty} B_\ell(1 - \{(r_1+r_2)/N\}) \frac{(-zu)^\ell}{\ell!} \\
&\qquad \cdot \sum_{m=0}^{\infty} B_m(\{1 - N + \varrho_N/N\}) \frac{u^m}{m!} \, du \\
(2.5) \qquad &= -2\pi i \sum_{k=0}^{n+2} \frac{\bar{B}_k((r_1+r_2)/N) \bar{B}_{n+2-k}(\varrho_N/N)}{k!(n+2-k)!} z^{k-1}.
\end{aligned}
$$

Now we let $N \nmid r_1$ and $N \nmid (r_1 + r_2)$. By Theorem 1.1, we see that

$$(2.6) \qquad z^n H_N(V\tau, -n; r, 0) = H_N(\tau, -n; R, 0) + (2\pi i)^n L_N(\tau, -n; R, 0).$$

Putting (2.1), (2.2), (2.3), (2.4) and (2.5) into (2.6), we find that, for any integer n,

$$
\begin{aligned}
&z^{2n} \sum_{k=1}^{\infty} \frac{\cosh(\pi i k(2(r_1+r_2)/N + (1 - 2\{r_1/N\})/z))}{k^{2n+1} \sinh(\pi i k/z)} \\
&= \sum_{k=1}^{\infty} \frac{\cosh(\pi i k(-2r_1/N + (2\{(r_1+r_2)/N\} - 1)z))}{k^{2n+1} \sinh(-\pi i k z)} \\
(2.7) \qquad &\quad - (2\pi i)^{2n+1} \sum_{k=0}^{2n+2} \frac{\bar{B}_k((r_1+r_2)/N) \bar{B}_{2n+2-k}(\varrho_N/N)}{k!(2n+2-k)!} z^{k-1},
\end{aligned}
$$

and

$$
\begin{aligned}
&z^{2n+1} \sum_{k=1}^{\infty} \frac{\sinh(\pi i k(2(r_1+r_2)/N + (1 - 2\{r_1/N\})/z))}{k^{2n+2} \sinh(\pi i k/z)} \\
&= \sum_{k=1}^{\infty} \frac{\sinh(\pi i k(-2r_1/N + (2\{(r_1+r_2)/N\} - 1)z))}{k^{2n+2} \sinh(-\pi i k z)}
\end{aligned}
$$

$$(2.8) \quad -(2\pi i)^{2n+2} \sum_{k=0}^{2n+3} \frac{\bar{B}_k((r_1 + r_2)/N)\bar{B}_{2n+3-k}(\varrho_N/N)}{k!(2n+3-k)!} z^{k-1}.$$

Theorem 2.1. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Suppose γ and γ' are real numbers such that γ and $\gamma + \gamma'$ are not integers. Then, for any integer n,*

$$\alpha^{-n} \sum_{k=1}^{\infty} \frac{\cosh(2(\gamma + \gamma')\pi ik + (1 - 2\{\gamma\})\alpha k)}{k^{2n+1} \sinh(\alpha k)}$$

$$= (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{\cosh(-2\pi i\gamma k - (2\{\gamma + \gamma'\} - 1)\beta k)}{k^{2n+1} \sinh(\beta k)}$$

$$(2.9) \quad -2^{2n+1} \sum_{k=0}^{2n+2} \frac{\bar{B}_k(\gamma + \gamma')\bar{B}_{2n+2-k}(\varrho_N/N)}{k!(2n+2-k)!} (\pi i)^k \alpha^{n-k+1},$$

and

$$\alpha^{-n-1/2} \sum_{k=1}^{\infty} \frac{\sinh(2(\gamma + \gamma')\pi ik + (1 - 2\{\gamma\})\alpha k)}{k^{2n+2} \sinh(\alpha k)}$$

$$= (-\beta)^{-n-1/2} \sum_{k=1}^{\infty} \frac{\sinh(-2\pi i\gamma k - (2\{\gamma + \gamma'\} - 1)\beta k)}{k^{2n+2} \sinh(\beta k)}$$

$$(2.10) \quad -2^{2n+2} \sum_{k=0}^{2n+3} \frac{\bar{B}_k(\gamma + \gamma')\bar{B}_{2n+3-k}(\varrho_N/N)}{k!(2n+3-k)!} (\pi i)^k \alpha^{n-k+3/2}.$$

Proof. Let $z = \pi i/\alpha$ in (2.7) and (2.8). Put $r_1/N = \gamma$ and $r_2/N = \gamma'$. $\qquad\square$

Now, we investigate a few special cases of Theorem 2.1. Let γ be a real number with $0 < \gamma < 1$ and $\gamma' = 0$. We see that

$$(2.11) \quad \left\{ \frac{\varrho_N}{N} \right\} = \{\{-N\gamma\} + N\gamma - \gamma\} = 1 - \gamma.$$

Theorem 2.2. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Let γ be a real number with $0 < \gamma < 1$. Then, for any integer n,*

$$\alpha^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cosh((1 - 2\gamma)\alpha k) \cos((1 - 2\gamma)\pi k)}{k^{2n+1} \sinh(\alpha k)}$$

$$= (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cosh((1 - 2\gamma)\beta k) \cos((1 - 2\gamma)\pi k)}{k^{2n+1} \sinh(\beta k)}$$

$$(2.12) \quad +2^{2n+1} \sum_{k=0}^{n+1} \frac{B_{2k}(\gamma) B_{2n+2-2k}(\gamma)}{(2k)!(2n+2-2k)!} \alpha^{n-k+1} (-\beta)^k,$$

and

$$\alpha^{-n-1/2} \sum_{k=1}^{\infty} \frac{(-1)^k \sinh((1-2\gamma)\alpha k) \cos((1-2\gamma)\pi k)}{k^{2n+2} \sinh(\alpha k)}$$

$$= (-1)^n \beta^{-n-1/2} \sum_{k=1}^{\infty} \frac{(-1)^k \cosh((1-2\gamma)\beta k) \sin((1-2\gamma)\pi k)}{k^{2n+2} \sinh(\beta k)}$$

$$(2.13) \qquad + 2^{2n+2} \sum_{k=0}^{n+1} \frac{B_{2k}(\gamma) B_{2n+3-2k}(\gamma)}{(2k)!(2n+3-2k)!} \alpha^{n-k+3/2} (-\beta)^k.$$

Proof. Let $0 < \gamma < 1$ and $\gamma' = 0$ in Theorem 2.1 and equate the real parts. $\qquad\square$

For $\gamma = 1/2$, (2.12) reduces to Theorem 3.1. in [3], p. 335.

Theorem 2.3. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Let γ be a real number with $0 < \gamma < 1$. Then, for any integer n,*

$$\alpha^{-n} \sum_{k=1}^{\infty} \frac{(-1)^k \sinh((1-2\gamma)\alpha k) \sin((1-2\gamma)\pi k)}{k^{2n+1} \sinh(\alpha k)}$$

$$= -(-\beta)^{-n} \sum_{k=1}^{\infty} \frac{(-1)^k \sinh((1-2\gamma)\beta k) \sin((1-2\gamma)\pi k)}{k^{2n+1} \sinh(\beta k)}$$

$$(2.14) \qquad - 2^{2n+1} \pi \sum_{k=0}^{n} \frac{B_{2k+1}(\gamma) B_{2n+1-2k}(\gamma)}{(2k+1)!(2n+1-2k)!} \alpha^{n-k} (-\beta)^k.$$

Proof. Let $0 < \gamma < 1$ and $\gamma' = 0$ in Theorem 2.1 and equate the imaginary parts. $\qquad\square$

We find the following symmetric identities with respect to α and β.

Corollary 2.4. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. For any positive integer M,*

$$\alpha^{2M} \sum_{k=1}^{\infty} \frac{(-1)^k \cosh((1-2\gamma)\alpha k) \cos((1-2\gamma)\pi k)}{k^{2n+1} \sinh(\alpha k)}$$

$$= \beta^{2M} \sum_{k=1}^{\infty} \frac{(-1)^k \cosh((1-2\gamma)\beta k) \cos((1-2\gamma)\pi k)}{k^{2n+1} \sinh(\beta k)}$$

Proof. Put $n = -2M$ in (2.12). $\qquad\square$

Corollary 2.5. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. For any positive integer M,*

$$\alpha^{2M-1} \sum_{k=1}^{\infty} \frac{(-1)^k \sinh((1-2\gamma)\alpha k) \sin((1-2\gamma)\pi k)}{k^{-4M+3} \sinh(\alpha k)}$$
$$= \beta^{2M-1} \sum_{k=1}^{\infty} \frac{(-1)^k \sinh((1-2\gamma)\beta k) \sin((1-2\gamma)\pi k)}{k^{-4M+3} \sinh(\beta k)}$$

Proof. Put $n = -2M + 1$ in (2.14). $\qquad\square$

Corollary 2.6. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then*

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{\cosh((1-2\gamma)\alpha k) \cos((1-2\gamma)\pi k)}{\sinh(\alpha k)} + \alpha(\gamma^2 - \gamma + \frac{1}{6})$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{\cosh((1-2\gamma)\beta k) \cos((1-2\gamma)\pi k)}{\sinh(\beta k)} + \beta(\gamma^2 - \gamma + \frac{1}{6}).$$

Proof. Let $n = 0$ in (2.12) and use $B_2(x) = x^2 - x + 1/6$. $\qquad\square$

The following corollaries show exact values for some infinite series containing hyperbolic functions.

Corollary 2.7. *For any positive integer M,*

$$\sum_{k=1}^{\infty} \frac{(-1)^k k^{4M+1} \cosh((1-2\gamma)\pi k) \cos((1-2\gamma)\pi k)}{\sinh(\pi k)} = 0.$$

Proof. Put $\alpha = \beta = \pi$ and $n = -2M - 1$ in (2.12) $\qquad\square$

Corollary 2.8. *For any positive integer M,*

$$\sum_{k=1}^{\infty} \frac{(-1)^k k^{4M-1} \sinh((1-2\gamma)\pi k) \sin((1-2\gamma)\pi k)}{\sinh(\pi k)} = 0.$$

Proof. Let $\alpha = \beta = \pi$ and put $n = -2M$ in (2.14). $\qquad\square$

Corollary 2.9.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{\sinh((1-2\gamma)\alpha k)}{\sinh(\alpha k)} + \frac{\sinh((1-2\gamma)\beta k)}{\sinh(\beta k)} \right) \sin((1-2\gamma)\pi k)$$
$$= 2\pi \left(\gamma - \frac{1}{2} \right)^2.$$

Proof. Let $n = 0$ in (2.14). $\qquad\square$

Corollary 2.10.

$$\sum_{k=1}^{\infty} \frac{(-1)^k \ k \ \cosh((1-2\gamma)\pi k) \cos((1-2\gamma)\pi k)}{\sinh(\pi k)} = -\frac{1}{4\pi}.$$

Proof. Let $\alpha = \beta = \pi$ and $n = -1$ in (2.12). $\qquad\qquad\square$

We also see that sums of Bernoulli's polynomials can be expressed by infinite series.

Corollary 2.11. *For any positive integer M,*

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cosh((1-2\gamma)\pi k) \cos((1-2\gamma)\pi k)}{k^{4M-1} \ \sinh(\pi k)}$$

$$= -\frac{1}{2}(2\pi)^{4M-1} \sum_{k=0}^{2M} \frac{B_{2k}(\gamma) B_{4M-2k}(\gamma)}{(2k)!(4M-2k)!}(-1)^k.$$

Proof. Let $\alpha = \beta = \pi$ and replace n with $2M - 1$ in (2.12). $\qquad\square$

Corollary 2.12. *For any positive integer M,*

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sinh((1-2\gamma)\pi k) \sin((1-2\gamma)\pi k)}{k^{4M+1} \ \sinh(\pi k)}$$

$$= -\frac{1}{2}(2\pi)^{4M+1} \sum_{k=0}^{2M} \frac{B_{2k+1}(\gamma) B_{4M+1-2k}(\gamma)}{(2k+1)!(4M+1-2k)!}(-1)^k.$$

Proof. Let $\alpha = \beta = \pi$ and replace n with $2M$ in (2.14). $\qquad\square$

For specific values of γ, we obtain many interesting infinite series with hyperbolic functions. Some of them have already been studied by other mathematicians, in special, Ramanujan. The case that $\gamma = 1/4$ in (2.12) gives Theorem 3.1 in [3]. The identity (2.14) with $\gamma = 1/4$ is found in Ramanujan's Notebooks [7] and was also given by Malurkar [5], Nanjundiah [6] and Berndt [2]. Putting $\gamma = 1/4$ in (2.13), we obtain the following proposition.

Proposition 2.13. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer n,*

$$\alpha^{-n-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k \mathrm{csch}((2k+1)\alpha/2)}{(2k+1)^{2n+2}} = (-4)^{-n-1}\beta^{-n-1/2} \sum_{k=1}^{\infty} \frac{(-1)^k \mathrm{sech}(\beta k)}{k^{2n+2}}$$

$$-2^{2n+3}\pi \sum_{k=0}^{n+1} \frac{B_{2k+1}(1/4) B_{2n+2-2k}(1/4)}{(2k+1)!(2n+2-2k)!} \alpha^{n-k+1/2}(-\beta)^k.$$

Corollary 2.14.

$$\sum_{k=0}^{\infty}(-1)^k\text{csch}(\pi(2k+1)/2) = \sum_{k=1}^{\infty}(-1)^k\text{sech}(\pi k) + \frac{1}{2},$$

Proof. Let $n = -1$ and $\alpha = \beta = \pi$ in Proposition 2.13. $\square$

Corollary 2.15. *For any integer $M > 0$,*

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)^{2M}\text{csch}(\pi(2k+1)/2) = (-4)^M\sum_{k=1}^{\infty}(-1)^k k^{2M}\text{sech}(\pi k).$$

Proof. Let $n = -M - 1$ and $\alpha = \beta = \pi$ in Proposition 2.13. $\square$

Let $\left(\frac{\cdot}{3}\right)$ be the Legendre symbol.

Proposition 2.16. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer n,*

$$\alpha^{-n}\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{\sinh(\alpha k/3)}{k^{2n+1}\,\sinh(\alpha k)} = -(-\beta)^{-n}\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{\sinh(\beta k/3)}{k^{2n+1}\,\sinh(\beta k)}$$
$$+ \frac{2^{2n+2}\pi}{\sqrt{3}}\sum_{k=0}^{n}\frac{B_{2k+1}(1/3)B_{2n+1-2k}(1/3)}{(2k+1)!(2n+1-2k)!}\alpha^{n-k}(-\beta)^k.$$

Proof. Let $\gamma = 1/3$ in (2.14). $\square$

Berndt [2] established Proposition 2.16 using the character analogue of his transformation formula.

Proposition 2.17. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer n,*

$$\alpha^{-n}\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{(-1)^k\sinh(2\alpha k/3)}{k^{2n+1}\sinh(\alpha k)} = -(-\beta)^{-n}\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{(-1)^k\sinh(2\beta k/3)}{k^{2n+1}\sinh(\beta k)}$$
$$- \frac{2^{2n+2}\pi}{\sqrt{3}}\sum_{k=0}^{n}\frac{B_{2k+1}(1/6)B_{2n+1-2k}(1/6)}{(2k+1)!(2n+1-2k)!}\alpha^{n-k}(-\beta)^k.$$

Proof. Let $\gamma = 1/6$ in (2.14). $\square$

The following corollary is an immediate consequence of Proposition 2.16 and Proposition 2.17.

Corollary 2.18. *Let M be any positive integer. Then*

$$(2.15) \qquad \sum_{k\equiv 1 \ (\text{mod } 3)}\frac{k^{4M-1}\sinh(\pi k/3)}{\sinh(\pi k)} = \sum_{k\equiv 2 \ (\text{mod } 3)}\frac{k^{4M-1}\sinh(\pi k/3)}{\sinh(\pi k)},$$

$$\sum_{k\equiv1\ (\mathrm{mod}\ 3)}\frac{(-1)^k\ k^{4M-1}\sinh(2\pi k/3)}{\sinh(\pi k)}=\sum_{k\equiv2\ (\mathrm{mod}\ 3)}\frac{(-1)^k\ k^{4M-1}\sinh(2\pi k/3)}{\sinh(\pi k)},$$

(2.16)
$$\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{\sinh(\pi k/3)}{k\ \sinh(\pi k)}=\frac{\pi}{18\sqrt{3}},$$

$$\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{(-1)^k\sinh(2\pi k/3)}{k\ \sinh(\pi k)}=-\frac{2\pi}{9\sqrt{3}},$$

(2.17)
$$\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{\sinh(\pi k/3)}{k^{4M+1}\sinh(\pi k)}$$
$$=\frac{(2\pi)^{4M+1}\pi}{\sqrt{3}}\sum_{k=0}^{2M}\frac{B_{2k+1}(1/3)B_{4M+1-2k}(1/3)}{(2k+1)!(4M+1-2k)!}(-1)^k,$$

$$\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{(-1)^k\sinh(2\pi k/3)}{k^{4M+1}\sinh(\pi k)}$$
$$=-\frac{(2\pi)^{4M+1}\pi}{\sqrt{3}}\sum_{k=0}^{2M}\frac{B_{2k+1}(1/6)B_{4M+1-2k}(1/6)}{(2k+1)!(4M+1-2k)!}(-1)^k.$$

Berndt [2] also has stated (2.15), (2.16), (2.17). We also have a generalized form of the identity in Theorem 3.8. in [3], p. 338.

Theorem 2.19. *Let $\theta=x+y\sqrt{u}$, where x, y and u are integers such that $u>1$, u is square free, and $x^2-uy^2=\varepsilon$ with $\varepsilon\pm1$. Let γ be a real number with $0<\gamma<1$. For any positive integer M,*

$$\theta^{2M}\varepsilon\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{2M+1}}\csc(\pi\theta k)\cos\left(\left(\frac{1}{\theta}-1\right)(1-2r)\pi k\right)$$
$$=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{2M+1}}\csc(\pi\theta k)\cos((\theta-1)(1-2r)\pi k)$$
$$+(2\pi)^{2M+1}\sum_{k=0}^{2M+2}\frac{B_k(\gamma)B_{2M+2-k}(\gamma)}{k!(2M+2-k)!}(-1)^{M+k-1}\theta^{k-1}.$$

Proof. Put $z=\theta=x+y\sqrt{u}$ in (2.6). Let $r_2=0$ and $0<r_1<N$ in (2.6). Replace r_1/N by γ. □

Corollary 2.20. *Let γ be a real number with $0<\gamma<1$. Then*

$$(3+2\sqrt{2})\sum_{k=1}^{\infty}\frac{1}{k^3}\csc(\sqrt{2}\pi k)\cos((2-\sqrt{2})(1-2\gamma)\pi k)$$
$$=-\sum_{k=1}^{\infty}\frac{1}{k^3}\csc(\sqrt{2}\pi k)\cos(\sqrt{2}(1-2\gamma)\pi k)$$

$$-(2\pi)^3 \left(\frac{1-\sqrt{2}}{8}\gamma^4 - \frac{1-\sqrt{2}}{4}\gamma^3 + \frac{15+7\sqrt{2}}{24}\gamma^2 - \frac{1}{12}\gamma + \frac{16-15\sqrt{2}}{360} \right).$$

Proof. Put $\theta = 1 + \sqrt{2}$, $\varepsilon = -1$, $r_1/N = \gamma$ and $M = 1$ in Theorem 2.19. $\qquad\square$

If $\gamma = 1/2$, then Theorem 2.19 delivers Theorem 3.8 in [3]. Corollary 2.20 for $\gamma = 1/2$ had been posed as a problem, which has been solved by Berndt [4].

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