Honam Mathematical J. **34** (2012), No. 3, pp. 381–390
http://dx.doi.org/10.5831/HMJ.2012.34.3.381

ON THE QUATERNIONIC GENERAL HELICES IN EUCLIDEAN 4-SPACE

Dae Won Yoon

Abstract. In this paper, we give some characterizations for a quaternionic general helix by means of curvatures of a curve in a 4-dimensional Euclidean space.

1. Introduction

The quaternion number system was discovered by Hamilton in 1843. Quaternions form an extension of the field of complex numbers having the property that the commutative law fails for multiplication, despite the fact that every non-zero element has a multiplicative inverse. As a set, the quaternions $\mathbb{Q}$ are equal to a 4-dimensional vector space $\mathbb{R}^4$ over real numbers. Every element in $\mathbb{Q}$ has an expression of the form $a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4$, where a, b, c, d are real numbers and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are basis on $\mathbb{R}^4$. The basis element $\mathbf{e}_4 = 1$ will be the identity element of $\mathbb{Q}$, meaning that multiplication by $\mathbf{e}_4 = 1$ does nothing, and for this reason, elements of $\mathbb{Q}$ are usually written $a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4$ ($\mathbf{e}_4 = 1$), supposing the basis element $\mathbf{e}_4 = 1$.

In [2] K. Baharathi and M. Nagaraj studied a quaternionic curve in $\mathbb{R}^3$ and $\mathbb{R}^4$ and gave the Frenet formula for it. Also, A. C. Coken and A.

Received June 17, 2012. Accepted July 4, 2012.
2000 Mathematics Subject Classification. 53A04.
Key words and phrases. General helix, Quaternionic curve, Curvature, Frenet frame.

Tuna([5]) studied the differential geometry of a quaternionic curve in a 4-dimensional semi-Euclidean space $\mathbb{R}^4_2$ with index 2.

A curve of constant slope or general helix in a 3-dimensional Euclidean space $\mathbb{R}^3$ is defined by the property that the tangent makes a constant angle with a fixed direction. A general helix is characterized by the fact that the ratio τ/κ is constant along the curve, where κ and τ denote the curvature and the torsion, respectively. Similar characterization of a helix in a 4-dimensional Euclidean space $\mathbb{R}^4$ was given by Magden([8]). He characterizes a helix if and only if the function

$$\frac{k_1^2}{k_2^2} + \left(\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right)^2$$

is a constant, where k_1, k_2 and k_3 are the first, second and third curvatures of a curve in $\mathbb{R}^4$, respectively, and they are nowhere zero. On the other hand, corresponding characterizations of time-like general helix in a 4-dimensional Minkowski space $\mathbb{R}^4_1$ were given in [9]. Also, C. Camci and et al.([3]) have given some characterizations of general helix by using harmonic curvatures of a curve in $\mathbb{R}^n$. For the study of a quaternionic curve, M. A. Gungor and M. Tosun([7]) investigated quaternionic rectifying curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ and A. C. Coken and A. Tuna([5]) studies quaternionic inclined curves in $\mathbb{R}^4_2$.

Our main aim in the present work is to study the quaternionic general helix in $\mathbb{R}^4$.

2. Preliminaries

A real quaternionic is defined with $q = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4$ such that

(i) $\mathbf{e}_i \times \mathbf{e}_i = -\mathbf{e}_4, \quad (1 \leq i \leq 3)$

(ii) $\mathbf{e}_i \times \mathbf{e}_j = -\mathbf{e}_j \times \mathbf{e}_i = \mathbf{e}_k \quad (1 \leq i, j \leq 3)$,

where (ijk) is even permutation of (123).

Let p and q be any two elements of $\mathbb{Q}$. Then the product of p and q is defined by

$$p \times q = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q,$$

where S_r and $\mathbf{V}_r$ denote scalar and vector part of $r \in \mathbb{Q}$ and we have used the inner product and the cross product in Euclidean space.
On the other hand, the conjugate of $q = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4 \in \mathbb{Q}$ is denoted by αq and given by

$$\alpha q = S_q - \mathbf{V}_q = d\mathbf{e}_4 - a\mathbf{e}_1 - b\mathbf{e}_2 - c\mathbf{e}_3.$$

From this, we define the symmetric non-degenerate real-valued bilinear form h as follows:

$$h : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$$
$$(p, q) \to h(p, q) = \frac{1}{2}(p \times \alpha q + q \times \alpha p).$$

It is called the quaternionic inner product. The norm of q is

$$||q||^2 = q \times \alpha q = \alpha q \times q = a^2 + b^2 + c^2 + d^2.$$

The concept of a spatial quaternion will be made use throughout our work. A number q is said to be a spatial quaternion if $q + \alpha q = 0$. It is a temporal quaternion if $q - \alpha q = 0$. Therefore, any quaternion q can be written as the form $q = \frac{1}{2}(q + \alpha q) + \frac{1}{2}(q - \alpha q)$. The spatial part of q is $\frac{1}{2}(q - \alpha q)$ and is a spatial quaternion, while $\frac{1}{2}(q + \alpha q)$ the temporal part of q and is temporal quaternion ([2]).

Now, we consider a quaternionic curve in $\mathbb{R}^3$. A 3-dimensional Euclidean space $\mathbb{R}^3$ is identified with the space of spatial quaternion $\{\beta \in \mathbb{Q} \mid \beta + \alpha\beta = 0\}$ in an obvious manner. Let $I = [0, 1]$ be an interval in the real line $\mathbb{R}$ and $s \in I$ be the arc-length parameter along the smooth curve

$$\beta \ : I \in \mathbb{R} \to \mathbb{Q}$$
$$s \to \beta(s) = \sum_{i=1}^{3} \beta_i(s)\mathbf{e}_i, \quad (1 \leq i \leq 3).$$

The tangent vector $\beta'(s) = \mathbf{t}(s)$ has unit length $||\mathbf{t}(s)|| = 1$ for all s. It follows

$$\mathbf{t}' \times \alpha\mathbf{t} + \mathbf{t} \times \alpha\mathbf{t}' = 0,$$

which implies $\mathbf{t}'$ is orthogonal to $\mathbf{t}$ and $\mathbf{t}' \times \alpha\mathbf{t}$ is a spatial quaternion.

Let $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ be the Frenet frame of $\beta(s)$. Then Frenet formula is given by

$$\begin{aligned} \mathbf{t}' &= k\mathbf{n}_1 \\ \mathbf{n}_1' &= -k\mathbf{t} + r\mathbf{n}_2 \\ \mathbf{n}_2' &= -r\mathbf{n}_1 \end{aligned}$$

where $\mathbf{t}$, $\mathbf{n}_1$, $\mathbf{n}_2$ are the unit tangent, the unit principal normal and the unit binormal vector of a quaternionic curve β, respectively. The functions k, r are called the principal curvature and the torsion of β, respectively([2]).

3. Characterization of quaternionic general helices in $\mathbb{Q}$

In this section, we give the necessary and sufficient condition for quaternionic general helices.

Let

$$\begin{aligned} \gamma\ &: I \subset \mathbb{R} \to \mathbb{Q} \\ &s \to \gamma(s) = \sum_{i=1}^{4} \gamma_i(s)\mathbf{e}_i, \quad \mathbf{e}_4 = 1 \end{aligned}$$

be a smooth curve in $\mathbb{R}^4$ defined by over an interval I with the arc-length parameter s. The tangent $\mathbf{T}(s) = \gamma'(s) = \sum_{i=1}^{4} \gamma_i'(s)\mathbf{e}_i$ has unit length. Let $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ be the Frenet apparatus of the space curve in $\mathbb{R}^4$.

Then the Frenet formula are ([2])

$$
\begin{aligned}
\mathbf{T}'(s) &= K\mathbf{N}_1(s) \\
\mathbf{N}_1'(s) &= -K\mathbf{T}(s) + k\mathbf{N}_2(s) \\
\mathbf{N}_2'(s) &= -k\mathbf{N}_1(s) + (r-K)\mathbf{N}_3(s) \\
\mathbf{N}_3'(s) &= -(r-K)\mathbf{N}_2(s),
\end{aligned}
\tag{3.1}
$$

where $\mathbf{N}_1 = \mathbf{t} \times \mathbf{T},\ \mathbf{N}_2 = \mathbf{n}_1 \times \mathbf{T},\ \mathbf{N}_3 = \mathbf{n}_2 \times \mathbf{T}$ and $K = ||\mathbf{T}'(s)||$.

This Frenet formula of the curve γ is obtained by making use the Frenet formula for a curve β in $\mathbb{R}^3$. Moreover, there are relationships between curvatures of the curve γ and β, These relations can be explained that the torsion k of γ is the principal curvature of the curve β and the bitorsion of γ is $(r-K)$, where r is the torsion of β and K is the principal curvature of β. These relationships are only determined for quaternions.

Theorem 3.1. *Let $\gamma = \gamma(s)$ be a unit speed quaternionic curve in $\mathbb{Q}$ with non-zero curvatures $K(s), k(s)$ and $r(s) - K(s)$. Then γ is a general helix in $\mathbb{Q}$ if and only if the function*

$$
\left(\frac{K}{k}\right)^2 + \frac{1}{(r-K)^2}\left(\left(\frac{K}{k}\right)'\right)^2
\tag{3.2}
$$

is a constant.

Proof. Let $\gamma(s)$ be a general helix in $\mathbb{Q}$ and the axis of the curve $\gamma(s)$ be the unit vector $\mathbf{U}$. Then, we have

$$
h(\mathbf{T}, \mathbf{U}) = \text{constant}
$$

along the curve. By differentiating this equation with respect to s and using the Frenet formula (3.1) we have

$$\begin{aligned} 0 = \frac{d}{ds}h(\mathbf{T},\mathbf{U}) &= \frac{1}{2}\frac{d}{ds}(\mathbf{T}\times\alpha\mathbf{U}+\mathbf{U}\times\alpha\mathbf{T}) \\ &= \frac{1}{2}(\mathbf{T}'\times\alpha\mathbf{U}+\mathbf{U}\times\alpha\mathbf{T}') \\ &= h(\mathbf{T}',\mathbf{U}) \\ &= Kh(\mathbf{N}_1,\mathbf{U}). \end{aligned}$$

Therefore, the unit vector $\mathbf{U}$ can be written as follows

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + a_2(s)\mathbf{N}_2(s) + a_3(s)\mathbf{N}_3(s), \tag{3.3}$$

where

$$a_1(s) = h(\mathbf{T},\mathbf{U}) = \text{constant},\ a_2(s) = h(\mathbf{N}_2,\mathbf{U}),\ a_3(s) = h(\mathbf{N}_3,\mathbf{U}),$$
$$a_1^2 + a_2^2 + a_3^2 = 1.$$

The differentiation of (3.3) gives

$$(a_1K - a_2k)\mathbf{N}_1 + (a_2' - a_3(r-K))\mathbf{N}_2 + (a_3' + a_2(r-K))\mathbf{N}_3 = 0,$$

which implies

$$a_1K - a_2k = 0,\ a_2' - a_3(r-K) = 0,\ a_3' + a_2(r-K) = 0,$$

that is,

$$\begin{aligned} a_2 &= \frac{K}{k}a_1 = -\frac{1}{r-K}a_3', \\ a_2' &= a_3(r-K). \end{aligned} \tag{3.4}$$

By differentiating the first equation of (3.4) and using the second equation of (3.4), we obtain the ODE for a_3 as follows

$$a_3'' - \frac{(r-K)'}{r-K}a_3' + (r-K)^2a_3 = 0. \tag{3.5}$$

If we change variables in (3.5) as $t = \int_0^s (r-K)ds$, then the equation (3.5) becomes

$$\frac{d^2a_3}{dt^2} + a_3 = 0. \tag{3.6}$$

Thus, the solution of the differential equation (3.6) is given by

$$a_3 = A \cos t(s) + B \sin t(s), \tag{3.7}$$

for some constants A and B. From the first equation of (3.4) and (3.7) we find

$$\begin{aligned} a_2 &= \frac{K}{k} a_1 = A \sin t(s) - B \cos t(s), \\ a_3 &= \frac{1}{r-K} \left(\frac{K}{k} \right)' a_1 = A \cos t(s) + B \sin t(s). \end{aligned}$$

From the above equations we obtain

$$\begin{aligned} A &= a_1 \left(\frac{K}{k} \sin t(s) + \frac{1}{r-K} \left(\frac{K}{k} \right)' \cos t(s) \right), \\ B &= a_1 \left(\frac{1}{r-K} \left(\frac{K}{k} \right)' \sin t(s) - \frac{K}{k} \cos t(s) \right), \end{aligned}$$

which imply

$$A^2 + B^2 = a_1^2 \left(\left(\frac{K}{k} \right)^2 + \frac{1}{(r-K)^2} \left(\left(\frac{K}{k} \right)' \right)^2 \right).$$

Thus, we have

$$\left(\frac{K}{k} \right)^2 + \frac{1}{(r-K)^2} \left(\left(\frac{K}{k} \right)' \right)^2 = \text{constant}. \tag{3.8}$$

Conversely, if the condition (3.2) holds, then we can always find a constant unit vector $\mathbf{U}$ satisfying $h(\mathbf{T}, \mathbf{U}) = \text{constant}$. We consider the unit vector defined by

$$\mathbf{U} = \mathbf{T} + \frac{K}{k} \mathbf{N}_2 + \frac{1}{r-K} \left(\frac{K}{k} \right)' \mathbf{N}_3.$$

Differentiation of $\mathbf{U}$ with the help of (3.8) gives $\mathbf{U}' = 0$, this mean that $\mathbf{U}$ is a constant vector. Consequently, the curve $\gamma(s)$ is a general helix in $\mathbb{Q}$. $\square$

Theorem 3.2. *A unit speed quaternionic curve $\gamma(s)$ in $\mathbb{Q}$ is a general helix if and only if there exists a C^2-function f such that*

$$(r-K)f(s)=\frac{d}{ds}\left(\frac{K}{k}\right),\quad \frac{d}{ds}f(s)=-(r-K)\left(\frac{K}{k}\right). \tag{3.9}$$

Proof. We assume that γ is a general helix. Differentiation of (3.8) gives

$$\left(\frac{K}{k}\right)\left(\frac{K}{k}\right)'+\frac{1}{r-K}\left(\frac{K}{k}\right)'\left(-\frac{(r-K)'}{(r-K)^2}+\frac{1}{r-K}\left(\frac{K}{k}\right)''\right)=0,$$

or equivalently,

$$\left(\frac{K}{k}\right)\frac{d}{ds}\left(\frac{K}{k}\right)+\frac{1}{r-K}\frac{d}{ds}\left(\frac{K}{k}\right)\frac{d}{ds}\left(\frac{1}{r-K}\frac{d}{ds}\left(\frac{K}{k}\right)\right)=0. \tag{3.10}$$

Therefore, we have

$$\frac{1}{r-K}\frac{d}{ds}\left(\frac{K}{k}\right)=-\frac{\left(\frac{K}{k}\right)\frac{d}{ds}\left(\frac{K}{k}\right)}{\frac{d}{ds}\left(\frac{1}{r-K}\frac{d}{ds}\left(\frac{K}{k}\right)\right)}. \tag{3.11}$$

If we define $f=f(s)$ by

$$f(s)=-\frac{\left(\frac{K}{k}\right)\frac{d}{ds}\left(\frac{K}{k}\right)}{\frac{d}{ds}\left(\frac{1}{r-K}\frac{d}{ds}\left(\frac{K}{k}\right)\right)},$$

then the equation (3.11) becomes

$$(r-K)f(s)=\frac{d}{ds}\left(\frac{K}{k}\right). \tag{3.12}$$

From (3.11) it can be written

$$\frac{d}{ds}\left(\frac{1}{r-K}\frac{d}{ds}\left(\frac{K}{k}\right)\right)=-(r-K)\left(\frac{K}{k}\right). \tag{3.13}$$

By combining (3.12) and (3.13), we find

$$\frac{d}{ds}f(s)=-(r-K)\left(\frac{K}{k}\right). \tag{3.14}$$

Conversely, if the equation (3.9) holds, we define a unit constant vector $\mathbf{U}$ by

$$\mathbf{U}=\mathbf{T}+\frac{K}{k}\mathbf{N}_2+f(s)\mathbf{N}_3.$$

It follows $h(\mathbf{T}, \mathbf{U}) = 1$. Thus, γ is a general helix. $\square$

Theorem 3.3. *Let γ be a unit speed quaternionic curve $\gamma(s)$ in $\mathbb{Q}$. Then γ is a general helix if and only if the following condition holds;*

$$\frac{K}{k} = C_1 \cos t + C_2 \sin t, \tag{3.15}$$

where C_1, C_2 are constants and $t(s) = \int_0^s (r - K)ds$.

Proof. Suppose that γ is a general helix. By using Theorem 3.1, let define the C^2-function $t(s)$ and the C^1-functions $m(s)$ and $n(s)$ by

$$t(s) = \int_0^s (r - K)ds, \tag{3.16}$$

$$\begin{aligned} m(s) &= \frac{K}{k} \cos t - f(s) \sin t, \\ n(s) &= \frac{K}{k} \sin t + f(s) \cos t. \end{aligned} \tag{3.17}$$

If we differentiate equation (3.17) with respect to s and take account of (3.16), (3.12) and (3.14), we have $m' = 0$ and $n' = 0$. Therefore, $m = C_1$ and $n = C_2$ are constants. Thus, from (3.17) we obtain

$$\frac{K}{k} = C_1 \cos t + C_2 \sin t.$$

Conversely, suppose that equation (3.15) holds. Then from (3.17) we have

$$f = -C_1 \cos t + C_2 \sin t,$$

which satisfies the condition of Theorem 3.2. Thus, γ is a general helix in $\mathbb{Q}$. $\square$

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Dae Won Yoon
Department of Mathematics Education and RINS, Gyeongsang National University,
Jinju 660-701, Korea.
E-mail: dwyoon@gnu.ac.kr