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ON A SYMMETRIC FUNCTIONAL EQUATION

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Abstract. We find a general solution $f : G \to G$ of the symmetric functional equation

$$x + f(y + f(x)) = y + f(x + f(y)), \;\; f(0) = 0$$

where G is a 2-divisible abelian group. We also prove that there exists no measurable solution $f : \mathbb{R} \to \mathbb{R}$ of the equation. We also find the continuous solutions $f : \mathbb{C} \to \mathbb{C}$ of the equation.

1. Introduction

In [3], Marcin E. Kuczma introduced the functional equation

$$x + g(y + f(x)) = y + g(x + f(y)) \tag{1.1}$$

which arises while studying a problem of compatibility of means. In particular, he obtained analytic solutions $f, g : \mathbb{R} \to \mathbb{R}$ of the equation (1.1). In [2], Nicole Brillouët-Belluot refined the result and find the differentiable solution of the equation (1.1). In the paper, it is also proved that the functional equation

$$x + f(y + f(x)) = y + f(x + f(y)), \;\; f(0) = 0 \tag{1.2}$$

has no differentiable solution $f : \mathbb{R} \to \mathbb{R}$. In the present paper we prove, with a different approach from those in [2], that there exist no measurable solutions $f : \mathbb{R} \to \mathbb{R}$ of the equation (1.2). As a matter of fact we find a general solution of the equation (1.2) for the function

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$f : G \to G$ when G is a 2-divisible abelian group. As a consequence we find a continuous solution $f : \mathbb{C} \to \mathbb{C}$ of the equation (1.2).

2. Main theorems

We denote by G a 2-divisible abelian group with identity 0 and $f : G \to G$. We first consider the functional equation

$$x + f(y + f(x)) = y + f(x + f(y)), \quad f(0) = 0 \tag{2.1}$$

for all $x \in G$. We say that f is additive provided that

$$f(x + y) = f(x) + f(y) \tag{2.2}$$

for all $x, y \in G$.

Theorem 2.1. *Let f satisfy* (2.1). *Then f is an additive function satisfying*

$$f(f(x)) = f(x) - x \tag{2.3}$$

for all $x, y \in G$. Conversely, if an additive function f satisfies (2.3), *then f satisfies* (2.1).

Proof. Letting $y = 0$ in (2.1) we get (2.3). Replacing x by $f(x)$ in (2.3) we have

$$f(f(f(x))) = f(f(x)) - f(x) = -x \tag{2.4}$$

Replacing x by $f(f(x))$ in (2.1) and using (2.4) and (2.3) we have

$$f(f(y)+f(f(x))) = f(f(x))+f(y-x)-y = f(x)+f(y-x)-x-y \tag{2.5}$$

Replacing x by $f(x)$ and y by $f(y)$ in (2.1) we have

$$f(x) + f(f(y) + f(f(x))) = f(y) + f(f(x) + f(f(y))). \tag{2.6}$$

From (2.5) and (2.6) we have

$$2f(x) + f(y - x) = 2f(y) + f(x - y). \tag{2.7}$$

Letting $y = 0$ in (2.7) we have $f(-x) = -f(x)$. Thus, replacing $x - y$ by u and y by v we have

$$2f(u+v) = 2f(u) + 2f(v), \tag{2.8}$$

which implies that f is additive since G is 2-divisible. Conversely, assume that f satisfies (2.2) and (2.3). Then we have

$$\begin{aligned} x + f(y + f(x)) &= x + f(y) + f(f(x)) = f(y) + f(x), \\ y + f(x + f(y)) &= y + f(x) + f(f(y)) = f(x) + f(y). \end{aligned}$$

Also, from (2.2) we have $f(0) = 0$. This completes the proof. □

From the above result it is easy to see that there exist no regular solutions $f : \mathbb{R} \to \mathbb{R}$ of (2.1).

Corollary 2.2. *The functional equation* (2.1) *has no solution* $f : \mathbb{R} \to \mathbb{R}$ *which is bounded in a set of positive measure.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (2.1). Then by Theorem 2.1, f satisfies (2.2) and (2.3). It is well known that every solution $f : \mathbb{R} \to \mathbb{R}$ of (2.2), which is bounded in a set of positive measure, has the form $f(x) = ax$ for some $a \in \mathbb{R}$([1]). By (2.3) we have $a^2 = a - 1$. Thus a is not a real number. This completes the proof. □

Remark. The Corollary 2.2 implies there are no measurable (continuous, increasing, and so on) function $f : \mathbb{R} \to \mathbb{R}$ satisfying (2.1).

Now we find a continuous solution $f : \mathbb{C} \to \mathbb{C}$ of the equation (2.1).

Corollary 2.3. *The continuous solution* $f : \mathbb{C} \to \mathbb{C}$ *of the equation* (2.1) *has the form*

$$f(z) = \left(\frac{1}{2} \pm i\sqrt{\frac{3}{4} + |\beta|^2}\right) z + \beta\bar{z} \tag{2.9}$$

for some constant $\beta \in \mathbb{C}$.

Proof. Let $f(z) = g(z) + ih(z)$, $g(z), h(z) \in \mathbb{R}$ satisfy (2.1). Then by Theorem 2.1, both g and h are additive since f is additive. Let $z = x + iy$, $x, y \in \mathbb{R}$ and $g_1(x) = g(x)$, $g_2(y) = g(iy)$, $h_1(x) = h(x)$, $h_2(y) = h(iy)$. Then g_1, g_2, h_1, h_2 are all continuous additive function. By the well known fact([1]), we obtain

$$g_1(x) = a_1x,\ g_2(y) = a_2y,\ h_1(x) = b_1x,\ h_2(y) = b_2y$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Now we have

$$\begin{aligned} f(z) = g(z) + ih(z) &= g(x) + g(iy) + ih(x) + ih(iy) \\ &= (a_1 + ib_1)x + (a_2 + ib_2)y \\ &= \alpha z + \beta \bar{z} \end{aligned}$$

for some $\alpha, \beta \in \mathbb{C}$. Now from the equation

$$f(f(z)) = f(z) - z$$

we have for all $z \in \mathbb{C}$,

$$(\alpha^2 - \alpha + 1 + |\beta|^2)z = -\beta(\alpha + \bar{\alpha} - 1)\bar{z},$$

which implies

$$\alpha^2 - \alpha + 1 + |\beta|^2 = -\beta(\alpha + \bar{\alpha} - 1) = 0.$$

Thus we have

$$\alpha = \frac{1}{2} \pm i\sqrt{\frac{3}{4} + |\beta|^2}.$$

This completes the proof. □

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