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THE LATTICE OF INTERVAL-VALUED FUZZY IDEALS OF A RING

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Abstract. We investigate the lattice structure of various sublattices of the lattice of interval-valued fuzzy subrings of a given ring. We prove that a special class of interval-valued fuzzy ideals of a ring. Finally, we show that the lattice of interval-valued fuzzy ideals of R is not complemented[resp. has no atoms(dual atoms)].

1. Introduction and Preliminaries

The concept of a fuzzy set was introduced by Zadeh[7], and then he introduced the notion of interval-valued fuzzy sets as a generalization of fuzzy sets in 1975[8]. After that time, Biswas[2] applied it to group theory, and Montal and Samanta[6] to topology. Recently, Cheong and Hur[3] investigated interval-valued ideals and bi-ideals of a subgroup, Kang and Hur[5] applied the concept of interval-valued fuzzy sets to algebra. Moreover, Choi et.al[4] introduced the notion of interval-valued smooth topological spaces and studied some of it's properties.

In this paper, we investigate the lattice of various sublattices of the lattice of interval-valued fuzzy subgroups of a given ring. We prove that a special class of interval-valued fuzzy ideals forms a modular sublattice of the lattice of interval-valued fuzzy ideals of a ring. Finally, we show that the lattice of interval-valued fuzzy ideals of R is not complemented[resp. has no atoms(dual atoms)].

Now, we will list some basic concepts and two results which are needed in the later sections.

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Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0, 0]$, $\mathbf{1} = [1, 1]$, and $\boldsymbol{a}=[a, a]$ for every $a \in (0, 1)$. We also note that

(i) $(\forall M, N \in D(I))$ $(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))$ $(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$(See [6]).

Definition 1.1 [6, 8]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set* (in short, *IVS*) in X and is denoted by $A = [A^L, A^U]$.

Thus for each $x \in X$, $A(x) = [A^L(x), A^U(x)]$, where $A^L(x)$[resp. $A^U(x)$] is called the *lower*[resp. *upper*] *end point* of x to A. For any $[a, b] \in D(I)$, the interval-valued fuzzy set A in X defined by $A(x) = [a, b]$ for each $x \in X$ is denoted by $\widetilde{[a, b]}$ and if $a = b$, then the IVS $\widetilde{[a, b]}$ is denoted by simply $\widetilde{a}$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVSs in X as $D(I)^X$. It is clear that $[A, A] \in D(I)^X$ for each $A \in I^X$.

Definition 1.2 [6]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then

(a) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(b) $A = B$ iff $A \subset B$ and $B \subset A$.

(c) $A^c = [1 - A^U, 1 - A^L]$.

(d) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(d)′ $\bigcup_{\alpha\in\Gamma} A_\alpha = [\bigvee_{\alpha\in\Gamma} A^L_\alpha, \bigvee_{\alpha\in\Gamma} A^U_\alpha]$.

(e) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(e)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

Result 1.A [6, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then

(a) $\tilde{0} \subset A \subset \tilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} (A \cap A_\alpha)$.
(f) $A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma} (A \cup A_\alpha)$.
(g) $(\tilde{0})^c = \tilde{1}$, $(\tilde{1})^c = \tilde{0}$.
(h) $(A^c)^c = A$.
(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A_\alpha^c$, $(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A_\alpha^c$.

It is obvious that $(D(I)^X, \cup, \cap)$ is complete lattice satisfying the De-Morgan's Laws.

Definition 1.3 [2]. Let A be an IVS in a group G. Then A is called an *interval-valued fuzzy subgroup* (in short, *IVG*) in G if it satisfies the conditions : For any $x, y \in G$,
(a) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.
(b) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$.
We will denote the set of all IVGs of G as IVG(G).

Result 1.B [2, Proposition 3.1]. Let A be an IVG in a group G.
(a) $A(x^{-1}) = A(x), \forall x \in G$.
(b) $A^L(e) \geq A^L(x)$ and $A^U(e) \geq A^U(x), \forall x \in G$, where e is the identity of G.

Throughout this paper, $L = (L, +, \cdot)$ denotes a lattice, where "+" and "·" denote the sup and inf, respectively. For a general background of lattice theory, we refer to [1]. Moreover, we will denote by R a ring having the zero "0", with respect to binary operations "+" and "·".

2. Lattice of interval-valued fuzzy subrings

Definition 2.1 [5]. Let R be a ring and let $A \in D(I)^R$. Then A is called an *interval-valued fuzzy subring*(in short, *IVR*) of R if it satisfies the following conditions: For any $x, y \in R$,
(i) $A^L(x + y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x + y) \geq A^U(x) \wedge A^U(y)$.
(ii) $A^L(-x) \geq A^L(x)$ and $A^U(-x) \geq A^U(x)$.
(iii) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.

We will denote the set of all IVRs of R as IVR(R).

From Result 1.B, it can be easily verified that if $A \in$ IVR(R), $A^L(x) \leq A^L(0), A^U(x) \leq A^U(0)$ and $A(x) = A(-x)$ for each $x \in R$. We shall call $A(0)$ as the *tip* of the interval-valued fuzzy subring A.

Result 2.A [5, Proposition 6.2]. Let $A \in D(I)^R$. Then $A \in$ IVR(R) if and only if for any $x, y \in R$,

(a) $A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y)$.

(b) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.

Proposition 2.2. let $\{A_\alpha\}_{\alpha\in\Gamma} \subset$ IVR(R). Then $\cap_{\alpha\in\Gamma} A_\alpha \in$ IVR(R).

Proof. let $A = \cap_{\alpha\in\Gamma} A_\alpha$ and let $x, y \in R$. Then

$$\begin{aligned} A^L(x-y) &= \bigwedge_{\alpha\in\Gamma} A^L_\alpha(x-y) \geq \bigwedge_{\alpha\in\Gamma}(A^L_\alpha(x) \wedge A^L_\alpha(y)) \text{ (Since } A^\alpha \in \text{IVR(R))} \\ &= (\bigwedge_{\alpha\in\Gamma} A^L_\alpha(x)) \wedge (\bigwedge_{\alpha\in\Gamma} A^L_\alpha(y)) = (\bigcap_{\alpha\in\Gamma} A_\alpha)^L(x) \wedge (\bigcap_{\alpha\in\Gamma} A_\alpha)^L(y) \\ &= A^L(x) \wedge A^L(y). \end{aligned}$$

By the similar arguments, we have that $A^U(x-y) \geq A^U(x) \wedge A^U(y)$. Similarly, we have $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$. Hence, by Result 2.A, $A = \bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVR(R). □

Definition 2.3. Let $A \in$ IVR(R). Then an interval-valued fuzzy subring *generated* by A is the least interval-valued fuzzy subring of R containing A and denoted by (A).

Here, we construct the lattice of interval-valued fuzzy subrings such as interval-valued fuzzy (left, right) ideals. The common feature of all these constructions is that the intersection of an arbitrary family of interval-valued fuzzy subrings is always an interval-valued fuzzy subring(See proposition 2.2). Therefore, we consider the inf of a family of interval-valued fuzzy subrings to be their intersection, whereas the union of two interval-valued fuzzy subrings may not be an interval-valued fuzzy subring. Hence, we shall always be talking the sup of a family of interval-valued fuzzy subrings to be the interval-valued fuzzy subring generated by the union of that family. The outcome of the above discussion can be described can be described by the following propositions.

Proposition 2.4. IVR(R) forms a complete lattice under the ordering of interval-valued fuzzy set inclusion $\subset$.

Definition 2.5 [5]. Let $A \in$ IVR(R). Then A is called an:

(1) *interval-valued fuzzy left ideal* (in short, IVLI) in R if $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for any $x, y \in R$.

(2) *interval-valued fuzzy right ideal* (in short, IVRI) in X if $A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$ for any $x, y \in R$.

(3) *interval-valued fuzzy ideal* (in short, IVI) in X if it both an IVLI and an IVRI in R.

We will denote the set of all IVIs [resp. IVLIs and IVRIs] of R as IVI(R) [resp. IVLI(R) and IVRI(R)]. In particular, $\mathrm{IVI}_{[\lambda_0,\mu_0]}(R)$ denotes the set of all IVIs with the same tip "$[\lambda_0, \mu_0]$". It is clear that IVI(R)=IVLI(R) $\cap$ IVRI(R).

Result 2.B [5, Proposition 6.5]. Let $A \in D(I)^R$. Then $A \in$ IVI(R)[resp. IVLI(R) and IVRI(R)] if and only if for any $x, y \in R$,

(a) $A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y)$.

(b) $A^L(xy) \geq A^L(x) \vee A^L(y)$ and $A^U(xy) \geq A^U(x) \vee A^U(y)$ [resp. $A^L(xy) \geq A^L(y), A^U(xy) \geq A^U(y)$ and $A^L(xy) \geq A^L(x), A^U(xy) \geq A^U(x)$].

The proof of the following result is similar to Proposition 2.2.

Proposition 2.6. Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset$ IVI(R) [resp. IVLI(R) and IVRI(R)]. Then $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVI(R) [resp. IVLI(R) and IVRI(R)].

Definition 2.7. Let $A \in D(I)^R$. Then the IVI[resp. IVLI and IVRI] *generated* by A is the least IVI[resp. IVLI and IVRI] of R containing A and denoted by (A).

The following is easily verified.

Proposition 2.8. (a) IVI(R)[resp. IVLI(R) and IVRI(R)] is a meet complete sublattice of IVR(R).

(b) $\mathrm{IVI}_{[\lambda,\mu]}(R)$ is a complete sublattice of IVR(R).

Definition 2.9[2,5]. Let X be a set and let $A \in D(I)^X$. Then A is said to have the *sup-property* if for each $Y \in P(X)$, there exists $y_0 \in Y$ such that $A(y_0) = [\bigvee_{x\in Y} A^L(x), \bigvee_{x\in Y} A^U(x)]$, where $P(X)$ denotes the power set of X.

Definition 2.10. Let $A, B \in D(I)^R$. Then the *sum* $A+B$ and the *product* $A \circ B$ of A and B are defined as follows, respectively: For each $z \in R$,

(i) $(A+B)(z) = [\bigvee_{z=x+y}(A^L(x) \wedge B^L(y)), \bigvee_{z=x+y}(A^L(x) \wedge B^U(y))]$,

(ii)

$$(A \circ B)(z) = \begin{cases} [\bigvee_{z=xy}(A^L(x) \wedge B^L(y)), \bigvee_{z=xy}(A^U(x) \wedge B^U(y))], & \text{if } z = xy; \\ [0,0], & \text{otherwise.} \end{cases}$$

Definition 2.11. Let X be a set, let $A \in D(I)^X$ and let $[\lambda, \mu] \in D(I)$.

(i) [5] The set $A_{[\lambda,\mu]} = \{x \in X : A^L(x) \geq \lambda \text{ and } A^U(x) \geq \mu\}$ is called a $[\lambda, \mu]$-*level-subset* of A.

(ii) The set $A^*_{[\lambda,\mu]} = \{x \in X : A^L(x) > \lambda \text{ and } A^U(x) > \mu\}$ is called a strong $[\lambda, \mu]$-*level-subset* of A.

The following is the immediate result of Propositions 4.16 and 4.17 in [5].

Theorem 2.12. Let $A \in D(I)^R$. Then $A \in$ IVI(R) if and only if $A_{[\lambda,\mu]}$ is an ideal for each $[\lambda, \mu] \in D(I)$ with $\lambda \leq A^L(0)$ and $\mu \leq A^U(0)$.

Lemma 2.13. Let $A, B \in$ IVI(R). If A and B have the sup-property, then $(A+B)_{[\lambda,\mu]} = A_{[\lambda,\mu]} + B_{[\lambda,\mu]}$ for each $[\lambda, \mu] \in D(I)$.

Proof. Let $z \in (A+B)_{[\lambda,\mu]}$. Then

$$(A+B)^L(z) = \bigvee_{z=x+y}(A^L(x) \wedge B^L(y))$$

and (2.1)

$$(A+B)^U(z) = \bigvee_{z=x+y}(A^U(x) \wedge B^U(y)).$$

For each decomposition $z = x + y$, we have either $A^L(x) \leq B^L(y)$ and $A^U(x) \leq B^U(y)$ or $A^L(x) \geq B^L(y)$ and $A^U(x) \geq B^U(y)$. This contradiction leads as to define the following subsets of R:

$X(z) = \{x \in R : z = x + y$ for some $y \in R$ such that $A^L(x) \leq B^L(y)$ and $A^U(x) \leq B^U(y)\}$,

$Y(z) = \{x \in R : z = x + y$ for some $y \in R$ such that $A^L(x) \geq B^L(y)$ and $A^U(x) \geq B^U(y)\}$,

$X^*(z) = \{x \in R : z = x + y$ for some $y \in R$ such that $A^L(x) \geq B^L(y)$ and $A^U(x) \geq B^U(y)\}$.

Then clearly $R = X(z) \cup X^*(z)$. Since A and B have the sup-property, there exist $x_0 \in X(z)$ and $y_0 \in Y(z)$ such that

$$A^L(x_0) = \bigvee_{x \in X(z)} A^L(x),\ A^U(x_0) = \bigvee_{x \in X(z)} A^U(x)$$

and (2.2)

$$B^L(y_0) = \bigvee_{y \in Y(z)} B^L(y),\ B^U(y_0) = \bigvee_{y \in Y(z)} B^U(y).$$

Since $x_0 \in X(z)$, there exists $y_0' \in R$ with $z = x_0 + y_0'$ such that $A^L(x_0) \leq B^L(y_0')$ and $A^U(x_0) \leq B^U(y_0')$.

Since $y_0 \in Y(z)$, there exists $x_0' \in R$ with $z = x_0' + y_0$ such that $A^L(x_0') \geq B^L(y_0)$ and $A^U(x_0') \geq B^U(y_0)$.

But for $A(x_0)$ and $B(y_0)$, we have either $A^L(x_0) \geq B^L(y_0)$ and $A^U(x_0) \geq B^U(y_0)$ or $A^L(x_0) \leq B^L(y_0)$ and $A^U(x_0) \leq B^U(y_0)$.

Case (i): Suppose $A^L(x_0) \geq B^L(y_0)$ and $A^U(x_0) \geq B^U(y_0)$. Then

$$\begin{aligned}
\bigvee_{z=x+y} (A^L(x) \wedge B^L(y)) &= \bigvee_{x \in R} (A^L(x) \wedge B^L(z-x)) \text{ (Since } y = z - x) \\
&= (\bigvee_{x \in X(z)} (A^L(x) \wedge B^L(z-x))) \vee (\bigvee_{x \in X^*(z)} (A^L(x) \\
&\quad \wedge B^L(z-x))) \text{ (Since } R = X(z) \cup X^*(z)) \\
&= (\bigvee_{x \in X(z)} (A^L(x) \wedge B^L(y))) \vee (\bigvee_{x \in Y(z)} (A^L(x) \wedge B^L(z-x))) \\
&= (\bigvee_{x \in X(z)} A^L(x) \vee (\bigvee_{x \in Y(z)} B^L(y))) \\
&= A^L(x_0) \wedge B^L(y_0) \text{ (By (2.2)} \\
&= A^L(x_0). \text{ (By the hypothesis)}
\end{aligned}$$

Similarly, we have that $\bigvee_{z=x+y}(A^U(x) \wedge B^U(y)) = A^U(x_0)$. Thus, by (2.1), $A^L(x_0) = (A+B)^L(z) \geq \lambda$ and $A^U(x_0) = (A+B)^U(z) \geq \mu$. So $x_0 \in A_{[\lambda,\mu]}$. Since $B^L(y_0') \geq A^L(x_0)$ and $B^U(y_0') \geq A^U(x_0)$, $B^L(y_0' \geq \lambda$ and $B^U(y_0' \geq \mu$. Then $y_0' \in B_{[\lambda,\mu]}$. Thus $z = x_0 + y_0 \in A_{[\lambda,\mu]} + B_{[\lambda,\mu]}$.

Case (ii): Suppose $A^L(x_0) \leq B^L(y_0)$ and $A^U(x_0) \leq B^U(y_0)$. Then as in Case(i), it follows that $x_0 \in A_{[\lambda,\mu]}$ and $y_0 \in B_{[\lambda,\mu]}$. Thus $z = x_0' + y_0 \in A_{[\lambda,\mu]} + B_{[\lambda,\mu]}$. So, in either case, $z \in A_{[\lambda,\mu]} + B_{[\lambda,\mu]}$. Hence $(A+B)_{[\lambda,\mu]} \subset A_{[\lambda,\mu]} + B_{[\lambda,\mu]}$. Now let $z \in A_{[\lambda,\mu]} + B_{[\lambda,\mu]}$. Then there exist $x_0 \in A_{[\lambda,\mu]}$ and $y_0 \in B_{[\lambda,\mu]}$ such that $z = x_0 + y_0$. Thus $A^L(x_0) \geq \lambda, A^U(x_0) \geq \mu$ and $B^L(x_0) \geq \lambda, B^U(x_0) \geq \mu$. So

$$(A+B)^L(z) = \bigvee_{z=x+y} (A^L(x) \wedge B^L(y)) \geq \lambda$$

and

$$(A+B)^U(z) = \bigvee_{z=x+y} (A^U(x) \wedge B^U(y)) \geq \mu.$$

Thus $z \in (A+B)_{[\lambda,\mu]}$. Hence $A_{[\lambda,\mu]} + B_{[\lambda,\mu]} \subset (A+B)_{[\lambda,\mu]}$. Therefore $(A+B)_{[\lambda,\mu]} = A_{[\lambda,\mu]} + B_{[\lambda,\mu]}$. This completes the proof. □

Lemma 2.14. Let $A, B \in D(I)^R$. and let $[\lambda,\mu] \in D(I)$. Then $(A+B)^*_{[\lambda,\mu]} = A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]}$.

Proof. Suppose $(A+B)^*_{[\lambda,\mu]} = \emptyset$. Then clearly $(A+B)^*_{[\lambda,\mu]} \subset A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]}$. Suppose $(A+B)^*_{[\lambda,\mu]} \neq \emptyset$ and let $z \in (A+B)^*_{[\lambda,\mu]}$. Then

$$(A+B)^L(z) = \bigvee_{z=x+y} (A^L(x) \wedge B^L(y)) > \lambda$$

and

$$(A+B)^U(z) = \bigvee_{z=x+y} (A^U(x) \wedge B^U(y)) > \mu.$$

Thus there exist $x_0, y_0 \in R$ with $z = x_0 + y_0$ such that $A^L(x_0) \wedge B^L(y_0) > \lambda$ and $A^U(x_0) \wedge B^U(y_0) > \mu$. So $A^L(x_0) > \lambda, A^U(x_0) > \mu$ and $B^L(y_0) > \lambda, B^U(y_0) > \mu$. Then $x_0 \in A^*_{[\lambda,\mu]}$ and $y_0 \in B^*_{[\lambda,\mu]}$. Thus $z = x_0 + y_0 \in A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]}$. So $(A+B)^*_{[\lambda,\mu]} \subset A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]}$.

Now for each $[\lambda,\mu] \in D(I_1)$, suppose

$$(\bigvee_{x \in R} A^L(x)) \wedge (\bigvee_{y \in R} B^L(y)) \leq \lambda$$

and

$$(\bigvee_{x \in R} A^U(x)) \wedge (\bigvee_{y \in R} B^U(y)) \leq \mu.$$

Then one of $A^*_{[\lambda,\mu]}$ and $B^*_{[\lambda,\mu]}$ is $\emptyset$. Thus $A^*_{[\lambda,\mu]}+B^*_{[\lambda,\mu]}=\emptyset\subset(A+B)^*_{[\lambda,\mu]}$. Otherwise, $A^*_{[\lambda,\mu]}\neq\emptyset$ and $B^*_{[\lambda,\mu]}\neq\emptyset$. Then $A^*_{[\lambda,\mu]}+B^*_{[\lambda,\mu]}\neq\emptyset$. Let $z\in A^*_{[\lambda,\mu]}+B^*_{[\lambda,\mu]}$. Then it exist $x_0\in A^*_{[\lambda,\mu]}$ and $y_0\in B^*_{[\lambda,\mu]}$ such that $z=x_0+y_0$ and

$$(A+B)^L(z)=\bigvee_{z=x+y}(A^L(x)\wedge B^L(y))\geq A^L(x_0)\wedge B^L(y_0)>\lambda$$

and

$$(A+B)^U(z)=\bigvee_{z=x+y}(A^U(x)\wedge B^U(y))\geq A^U(x_0)\wedge B^U(y_0)>\mu.$$

Thus $z\in(A+B)^*_{[\lambda,\mu]}$. So $A^*_{[\lambda,\mu]}+B^*_{[\lambda,\mu]}\subset(A+B)^*_{[\lambda,\mu]}$. Hence $(A+B)^*_{[\lambda,\mu]}=A^*_{[\lambda,\mu]}+B^*_{[\lambda,\mu]}$. This completes the proof. □

Theorem 2.15. Let $A\in D(I)^R$. Then $A\in$ IVI(R)[resp. IVLI(R) and IVRI(R)] if and only if $A^*_{[\lambda,\mu]}=\emptyset$ or $A^*_{[\lambda,\mu]}\in$ I(R)[resp. LI(R) and RI(R)] for each $[\lambda,\mu]\in D(I)$, where I(R)[resp. LI(R) and RI(R)] denotes the set of all ideals[resp. left ideals and right ideals] of R.

Proof. We prove this lemma for left ideal, since other cases are similar. It is clear that $A=\mathbf{0}$ if and only if $A^*_{[\lambda,\mu]}=\emptyset$ for each $[\lambda,\mu]\in D(I)$. Now we assume that $A\neq\mathbf{0}$.

($\Rightarrow$): Suppose $A\in$ IVLI(R) and let $[\lambda,\mu]\in D(I)$. Let $x,y\in A^*_{[\lambda,\mu]}$ and let $z\in R$. Then

$$\begin{aligned}A^L(x-y)&\geq A^L(x)\wedge A^L(y)\text{ (Since }A\in\text{IVLI(R))}\\&>\lambda\text{ (Since }x,y\in A^*_{[\lambda,\mu]})\end{aligned}$$

and

$$A^L(x-y)\geq A^L(x)\wedge A^L(y)>\mu.$$

Also,

$$\begin{aligned}A^L(zx)&\geq A^L(x)\text{ (Since }A\in\text{IVLI(R))}\\&>\lambda\text{ (Since }x\in A^*_{[\lambda,\mu]})\end{aligned}$$

and

$$A^U(zx)\geq A^U(x)>\mu.$$

Thus $x-y\in A^*_{[\lambda,\mu]}$ and $zx\in A^*_{[\lambda,\mu]}$. Hence $A^*_{[\lambda,\mu]}\in$ LI(R).

($\Leftarrow$): Suppose the necessary condition holds. For any $x,y\in R$, let $A(x)=[\lambda,\mu]$ and let $A(y)=[s,t]$ such that $\lambda\leq s$ and $\mu\leq t$.

Case (i): Suppose $[\lambda,\mu]=[0,0]$. Then

$$A^L(x-y)\geq\lambda=A^L(x)\wedge A^L(y),A^U(x-y)\geq\mu=A^U(x)\wedge A^U(y)$$

and

$$A^L(zx)\geq\lambda=A^L(x)\text{ and }A^U(zx)\geq\mu=A^U(x),\text{ foreach }z\in R.$$

Thus $A\in$ IVLI(R).

Case (ii): Suppose $[\lambda,\mu]\neq[0,0]$. For each $\epsilon>0$, let $\epsilon<\lambda$. Then we have

$$A^L(y)>s-\epsilon\geq\lambda-\epsilon,\ A^U(y)>t-\epsilon\geq\mu-\epsilon.$$

and

$$A^L(x)>\lambda-\epsilon,\ A^U(x)>\mu-\epsilon.$$

Thus $x,y\in A^*_{[\lambda-\epsilon,\mu-\epsilon]}$. By the hypothesis, $A^*_{[\lambda-\epsilon,\mu-\epsilon]}\in$ I(R). So $x-y\in A^*_{[\lambda-\epsilon,\mu-\epsilon]}$ and $zx\in A^*_{[\lambda-\epsilon,\mu-\epsilon]}$ for each $z\in R$. Then $A^L(x-y)>\lambda-\epsilon$, $A^U(x-y)>\mu-\epsilon$ and $A^L(zx)>\lambda-\epsilon$, $A^U(zx)>\mu-\epsilon$ for each $z\in R$. Since ϵ is an arbitrary, $A^L(x-y)\geq\lambda=A^L(x)\wedge A^L(y))$, $A^U(x-y)\geq\mu=A^U(x)\wedge A^U(y)$ and $A^L(zx)\geq\lambda=A^L(x)$, $A^U(zx)\geq\mu=A^U(x)$. Hence $A\in$ IVLI(R). This completes the proof. □

Proposition 2.16. $\text{IVLI}_{[\lambda_0,\mu_0]}(R)$, $\text{IVRI}_{[\lambda_0,\mu_0]}(R)$ and $\text{IVI}_{[\lambda_0,\mu_0]}(R)$ are sublattices of IVR(R) and for any $A,B\in\text{IVLI}_{[\lambda_0,\mu_0]}(R)$[resp. $\text{IVRI}_{[\lambda_0,\mu_0]}(R)$ and $\text{IVI}_{[\lambda_0,\mu_0]}(R)$], $A\vee B=A+B$.

Proof. It is easy to see that $\text{IVLI}_{[\lambda_0,\mu_0]}(R)$, $\text{IVRI}_{[\lambda_0,\mu_0]}(R)$ and $\text{IVI}_{[\lambda_0,\mu_0]}$ (R) are sublattices of IVR(R). We do only prove that $A\vee B=A+B$ for any $A,B\in\text{IVLI}_{[\lambda_0,\mu_0]}(R)$(For $\text{IVRI}_{[\lambda_0,\mu_0]}(R)$ and $\text{IVI}_{[\lambda_0,\mu_0]}(R)$, the proofs are similar). Let $z\in R$. Then

$$(A+B)^L(z)=\bigvee_{z=x+y}(A^L(x)\wedge B^L(y))\leq A^L(0)\wedge B^L(0)=\lambda_0$$

and

$$(A+B)^U(z)=\bigvee_{z=x+y}(A^U(x)\wedge B^U(y))\leq A^U(0)\wedge B^U(0)=\mu_0.$$

Thus $\bigvee_{z=x+y}(A+B)^L(z)\leq\lambda_0$ and $\bigvee_{z=x+y}(A+B)^U(z)\leq\mu_0$. On the other hand, $\bigvee_{z\in R}(A+B)^L(z)\geq(A+B)^L(0)=\bigvee_{0=x+y}(A^L(x)\wedge B^L(y))\geq A^L(0)\wedge B^L(0)=\lambda_0$. By the similar arguments, we have that $\bigvee_{z\in R}(A+B)^U(z)\geq\mu_0$. So

$$[\bigvee_{z\in R}(A+B)^L(z), \bigvee_{z\in R}(A+B)^U(z)] = (A+B)(0) = [\lambda_0, \mu_0]. \tag{2.3}$$

For each $[\lambda_0, \mu_0] \in D(I_1)$ with $\lambda < \lambda_0$ and $\mu < \mu_0, (A+B)^*_{[\lambda,\mu]} \neq \emptyset$. By Lemma 2.14, $(A+B)^*_{[\lambda,\mu]} = A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]}$. Since $A, B \in$ IVLI(R), by Theorem 2.15, $A^*_{[\lambda,\mu]}, B^*_{[\lambda,\mu]} \in$ LI(R). Thus $(A+B)^*_{[\lambda,\mu]} \in$ LI(R). So, by Theorem 2.15, $A+B \in$ IVLI(R). Moreover,

$$A+B \in \text{ IVLI}_{[\lambda_0,\mu_0]}(R). \tag{2.4}$$

Let $z \in R$. Then

$$(A+B)^L(z) = \bigvee_{z=x+y}(A^L(x) \wedge B^L(y)) \geq A^L(z) \wedge B^L(0) = A^L(z)$$

and

$$(A+B)^U(z) = \bigvee_{z=x+y}(A^U(x) \wedge B^U(y)) \geq A^U(z) \wedge B^U(0) = B^U(z).$$

Thus $A \subset A+B$. By the similar arguments, we have $B \subset A+B$. So

$$A \subset A+B \text{ and } B \subset A+B. \tag{2.5}$$

Now let $C \in$ IVLI(R) such that $A \subset C$ and $B \subset C$ and let $z \in R$. Then

$$\begin{aligned}(A+B)^L(z) &= \bigvee_{z=x+y}(A^L(x) \wedge B^L(y)) \leq \bigvee_{z=x+y}(C^L(x) \wedge C^L(y)) \\ &\leq \bigvee_{z=x+y} C^L(z) \text{ (Since } C^L(z) = C^L(x+y) \geq C^L(x) \wedge C^L(y)) \\ &= C^L(z).\end{aligned}$$

Similarly, we have that $(A+B)^U(z) \geq C^U(z)$. Thus

$$A+B \subset C. \tag{2.6}$$

Hence, by (2.3),(2.4), (2.5) and (2.6), $A+B = A \vee B$. This completes the proof. □

Remark 2.16. (a) $A \vee B = A+B$ is not true in IVR(R), IVLI(R), IVRI(R) and IVI(R) (See Example 2.17).

(b) As well-known, $S+T$ is not subring in general, where S and T are subrings of R. Hence $A \vee B = A+B$ is not true in $\text{IVR}_{[\lambda_0,\mu_0]}(R)$(See Example 2. 18).

Example 2.17. We define two mappings $A : R \to D(I)$ and $B : R \to D(I)$ as follows, respectively: For each $x \in R$,

$$A(x) = [0.4, 0.5] \text{ and } B(x) = [0.3, 0.6].$$

Then clearly $A, B \in$ IVR(R)[resp. IVLI(R), IVRI(R) and IVI(R)]. Moreover, it is easy to see that $(A + B)(0) = [0.3, 0.6]$ and $(A \vee B)(0) = [0.4, 0.5]$. $\square$

Example 2.18. Let $R = \{(a, b) : a, b \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the ring of integers. We define the additive operation and the multiplicative operation on R as follows, respectively: For any $(a, b), (c, d) \in R$,

$$(a, b) + (c, d) = (a + c, b + d) \text{ and } (a, b) \cdot (c, d) = (0, 0).$$

Then $(R, +, \cdot)$ forms a ring with zero $(0, 0)$. Now we define three mappings $A, B, C : R \to D(I)$ as follows, respectively: For each $(x, y) \in R$,

$$A(x, y) = \begin{cases} [\frac{1}{3}, \frac{3}{5}], & \text{if } y = 0; \\ [0, 0], & \text{if } y \neq 0. \end{cases}$$

$$B(x, y) = \begin{cases} [\frac{1}{3}, \frac{3}{5}], & \text{if } x = 0; \\ [0, 0], & \text{if } x \neq 0. \end{cases}$$

$$C(x, y) = \begin{cases} [\frac{1}{3}, \frac{3}{5}], & \text{if } x = y; \\ [0, 0], & \text{if } x \neq y. \end{cases}$$

Then it is easy to see that $A, B, C \in$ IVI(R). Let $(x, y) \in R$. Then

$$\begin{aligned} (A + B)^L(x, y) &= \bigvee_{(x,y)=(x_1,y_1)+(x_2,y_2)} (A^L(x_1, y_1) \wedge B^L(x_2, y_2)) \\ &\leq A^L(x, 0) \wedge B^L(0, y) = \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} (A + B)^U(x, y) &= \bigvee_{(x,y)=(x_1,y_1)+(x_2,y_2)} (A^U(x_1, y_1) \wedge B^U(x_2, y_2)) \\ &\leq A^U(x, 0) \wedge B^U(0, y) = \frac{3}{5}. \end{aligned}$$

Thus $(C \wedge (A + B))^L(x, y) = C^L(x, y) \wedge (A + B)^L(x, y) = C^L(x, y)$ and $(C \wedge (A + B))^U(x, y) = C^U(x, y) \wedge (A + B)^U(x, y) = C^U(x, y)$. So $C \wedge (A + B) = C$. On the other hand,

$$(C \wedge A)^L(x,y) = C^L(x,y) \wedge A^L(x,y) = \begin{cases} \dfrac{1}{3}, & \text{if } (x,y) = (0,0); \\ 0, & \text{if } (x,y) \neq (0,0). \end{cases}$$

and

$$(C \wedge A)^U(x,y) = C^U(x,y) \wedge A^U(x,y) = \begin{cases} \dfrac{3}{5}, & \text{if } (x,y) = (0,0); \\ 0, & \text{if } (x,y) \neq (0,0). \end{cases}$$

Also,

$$(C \wedge B)^L(x,y) = C^L(x,y) \wedge B^L(x,y) = \begin{cases} \dfrac{1}{3}, & \text{if } (x,y) = (0,0); \\ 0, & \text{if } (x,y) \neq (0,0). \end{cases}$$

and

$$(C \wedge B)^U(x,y) = C^U(x,y) \wedge B^U(x,y) = \begin{cases} \dfrac{3}{5}, & \text{if } (x,y) = (0,0); \\ 0, & \text{if } (x,y) \neq (0,0). \end{cases}$$

Thus

$$((C \wedge A) + (C \wedge B))^L(x,y) = \begin{cases} \dfrac{1}{3}, & \text{if } (x,y) = (0,0); \\ 0, & \text{if } (x,y) \neq (0,0). \end{cases}$$

and

$$((C \wedge A) + (C \wedge B))^U(x,y) = \begin{cases} \dfrac{3}{5}, & \text{if } (x,y) = (0,0); \\ 0, & \text{if } (x,y) \neq (0,0). \end{cases}$$

So $C \wedge (A+B) \neq (C \wedge A) + (C \wedge B)$. Hence IVI(R) is not distributive. □

Lemma 2.19. Let $A, B \in$ IVI(R). If A and B have the sup-property, then the following holds:

(a) $A+B$ has the sup-property.

(b) $A \cap B$ has the sup-property.

Proof. (a) Let S be any subset of R. Then

$$\begin{aligned}\bigvee_{z\in S}(A+B)^L(z) &= \bigvee_{z\in S}(\bigvee_{z=x+y}(A^L(x)\wedge B^L(y)))\\ &= \bigvee_{z\in S, z=x+y}(A^L(x)\wedge B^L(y))\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\bigvee_{z\in S}(A+B)^U(z) &= \bigvee_{z\in S}(\bigvee_{z=x+y}(A^U(x)\wedge B^U(y)))\\ &= \bigvee_{z\in S, z=x+y}(A^U(x)\wedge B^U(y)).\end{aligned}$$

Let us define two subsets $X(S)$ and $Y(S)$ of R by

$X(S) = \{x \in R : z \in S, z = x + y$ for some $y \in R$ such that $A^L(x) \leq B^L(y)$ and $A^U(x) \leq B^U(y)\}$,

$Y(S) = \{y \in R : z \in S, z = x + y$ for some $x \in R$ such that $A^L(x) \geq B^L(y)$ and $A^U(x) \geq B^U(y)\}$.

Since A and B have the sup-property, there exist $x' \in X(S)$ and $y'' \in Y(S)$ such that

$$A^L(x') = \bigvee_{x\in X(S)}(A^L(x), A^U(x')) = \bigvee_{x\in X(S)} A^U(x)$$

and (2.7)

$$B^L(y'') = \bigvee_{y\in Y(S)}(B^L(y), B^U(y'')) = \bigvee_{y\in Y(S)} B^U(y).$$

Since $x' \in X(S)$, there exists $z_1 \in S$ such that $z_1 = x' + y'$ for some $y' \in R$ satisfying $A^L(x') \leq B^L(y')$ and $A^U(x') \leq B^U(y')$. Also, since $y'' \in Y(S)$, there exists $z_2 \in S$ such that $z_2 = x'' + y''$ for some $x'' \in R$ satisfying $A^L(x'') \geq B^L(y'')$ and $A^U(x'') \geq B^U(y'')$.

On the other hand, we have either $A^L(x') \geq B^L(y'')$, $A^U(x') \geq B^U(y'')$ or $A^L(x') \leq B^L(y'')$, $A^U(x') \leq B^U(y'')$.

Case (i): Suppose $A^L(x^{'}) \geq B^L(y^{''})$ and $A^U(x^{'}) \geq B^U(y^{''})$. Then

$$\begin{aligned}
\bigvee_{z\in S, z=x+y} (A^L(x) \wedge B^L(y)) &= \bigvee_{x\in X(S)} (A^L(x) \wedge B^L(y)) \vee (\bigvee_{y\in Y(S)} (A^L(x) \wedge B^L(y))) \\
&\quad \text{(As in Lemma 2.13)} \\
&= (\bigvee_{x\in X(S)} A^L(x)) \vee (\bigvee_{y\in Y(S)} B^L(y)) \\
&= A^L(x^{'}) \vee B^L(y^{''}) \text{ (By (2.7))} \\
&= A^L(x^{'}). \text{ (By the hypothesis)}
\end{aligned}$$

Similarly, we have that $\bigvee_{z\in S, z=x+y}(A^U(x) \wedge B^U(y)) = A^U(x')$. Thus

$$\bigvee_{z\in S} (A+B)^L(z) = A^L(x')$$

and (2.8)

$$\bigvee_{z\in S} (A+B)^U(z) = A^U(x').$$

Now we show that

$$\bigvee_{z\in S} (A+B)^L(z) = (A+B)^L(z_1) \text{ and } \bigvee_{z\in S} (A+B)^U(z) = (A+B)^U(z_1).$$

For decompositions $z_1 = x_i^{'} + y_i^{'}$, we have

$$(A+B)^L(z_1) = \bigvee_{z_1 = x_i^{'} + y_i^{'}} (A^L(x_i^{'}) \wedge B^L(y_i^{'}))$$

and

$$(A+B)^U(z_1) = \bigvee_{z_1 = x_i^{'} + y_i^{'}} (A^U(x_i^{'}) \wedge B^U(y_i^{'})).$$

Again, we construct subset $X(z_1)$ and $Y(z_1)$ of R as follows: $X(z_1) = \{x_i^{'} \in R : z_1 = x_i^{'} + y_i^{'}$ for some $y_i^{'} \in R$ such that $A^L(x_i^{'}) \leq B^L(y_i^{'})$ and $A^U(x_i^{'}) \leq B^U(y_i^{'})\}$,

$Y(z_1) = \{y_i^{'} \in R : z_1 = x_i^{'} + y_i^{'}$ for some $x_i^{'} \in R$ such that $A^L(x_i^{'}) \geq B^L(y_i^{'})$ and $A^U(x_i^{'}) \geq B^U(y_i^{'})\}$. Then

$$\begin{aligned}
\bigvee_{z_1 = x_i^{'} + y_i^{'}} (A^L(x_i^{'}) \wedge B^L(y_i^{'})) &= (\bigvee_{x_i^{'}\in X(z_1)} (A^L(x_i^{'}) \wedge B^L(y_i^{'}))) \vee (\bigvee_{x_i^{'}\in X(z_1)} (A^L(x_i^{'}) \wedge \\
&\quad B^L(y_i^{'}))) \text{ (As in Lemma 2.13)} \\
&= (\bigvee_{x_i^{'}\in X(z_1)} (A^L(x_i^{'}))) \vee (\bigvee_{y_i^{'}\in Y(z_1)} (B^L(y_i^{'}))).
\end{aligned}$$

By the similar arguments, we have that

$$\bigvee_{z_1=x_i'+y_i'} (A^U(x_i') \wedge B^U(y_i')) = (\bigvee_{x_i' \in X(z_1)} (A^U(x_i'))) \vee (\bigvee_{y_i' \in Y(z_1)} (B^U(y_i'))).$$

Since $X(z_1) \subset X(S)$ and $x_i' \in X(z_1)$,

$$A^L(x') \leq \bigvee_{x_i' \in X(z_1)} A^L(x_i') \leq \bigvee_{x \in X(S)} A^L(x) = A^L(x_i')$$

and

$$A^U(x') \geq \bigvee_{x_i' \in X(z_1)} A^U(x_i') \geq \bigvee_{x \in X(S)} A^U(x) = A^U(x_i').$$

Thus $\bigvee_{x_i' \in X(z_1)} A^L(x_i') = A^L(x_i')$ and $\bigvee_{x_i' \in X(z_1)} A^U(x_i') = A^U(x_i')$. Also, since $Y(z_1) \subset Y(S)$ and $y'' \in Y(z_1)$, we have

$$\bigvee_{y_i' \in Y(z_1)} B^L(y_i') = B^L(y_i'') \text{ and } \bigvee_{y_i' \in Y(z_1)} B^U(y_i') = B^U(x_i'').$$

By the hypothesis,

$$\bigvee_{x_i' \in X(z_1)} A^L(x_i') = A^L(x_i') \geq B^L(y_i'') = \bigvee_{y_i' \in Y(z_1)} B^L(y_i')$$

and

$$\bigvee_{x_i' \in X(z_1)} A^U(x_i') = A^U(x_i') \geq B^U(y_i'') = \bigvee_{y_i' \in Y(z_1)} B^U(y_i').$$

Thus

$$(A+B)^L = (\bigvee_{x_i' \in X(z_1)} A^L(x_i')) \vee (\bigvee_{y_i' \in Y(z_1)} B^L(y_i')) = A^L(x')$$

and (2.9)

$$(A+B)^U = (\bigvee_{x_i' \in X(z_1)} A^U(x_i')) \vee (\bigvee_{y_i' \in Y(z_1)} B^U(y_i')) = A^U(x').$$

So, by (2.8) and (2.9),

$$\bigvee_{z \in S} (A+B)^L(z) = (A+B)^L(z_1) \text{ and } \bigvee_{z \in S} (A+B)^U(z) = (A+B)^U(z_1).$$

Case (ii): Suppose $A^L(x') \leq B^L(y'')$ and $A^U(x') \leq B^U(y'')$. By proceeding in a similar way as in Case (i), we can verify that

$$\bigvee_{z\in S}(A+B)^L(z) = (A+B)^L(z_2) \text{ and } \bigvee_{z\in S}(A+B)^U(z) = (A+B)^U(z_2)$$

for some $z_2 \in S$. Hence, in all, $A+B$ has the sup-property.

(b) The proof is left as an exercise for the reader. This completes the proof. □

Proposition 2.20. Let $\mathrm{IVIs}_{[\lambda_0,\mu_0]}(R)$ be the set of all IVIs with the sup-property and same tip "$[\lambda_0,\mu_0]$". Then $\mathrm{IVIs}_{[\lambda_0,\mu_0]}(R)$ forms a sublattice of $\mathrm{IVI}_{[\lambda_0,\mu_0]}(R)$ and hence of IVI(R).

Proof. Let $A, B \in \mathrm{IVIs}_{[\lambda_0,\mu_0]}(R)$. We show that $A \vee B = A + B$. Since $A, B \in$ IVI(R), by Lemma 2.13, $A_{[\lambda,\mu]}$ and $B_{[\lambda,\mu]}$ are ideals for each $[\lambda,\mu] \in D(I)$ with $\lambda \leq (A+B)^L(0) = \lambda_0$ $\mu \leq (A+B)^U(0) = \mu_0$. Then $A_{[\lambda,\mu]} + B_{[\lambda,\mu]}$ is an ideal of R. Since A and B have the sup-property, by Lemma 2.13, $A_{[\lambda,\mu]} + B_{[\lambda,\mu]} = (A+B)_{[\lambda,\mu]}$. Thus $(A+B)_{[\lambda,\mu]}$ is an ideal of R. So, by Theorem 2.12, $A+B \in$ IVI(R). Since A and B have the same tip "$[\lambda_0,\mu_0]$", we have

$$(A+B)^L(z) = (\bigvee_{z=x+y} A^L(x) \wedge B^L(y) \geq A^L(z) \wedge B^L(0) = A^L(z)$$

and

$$(A+B)^U(z) = (\bigvee_{z=x+y} A^U(x) \wedge B^U(y) \geq A^U(z) \wedge B^U(0) = A^U(z)$$

for each $z \in S$. Then $A \subset A+B$. By the similar arguments, we have $B \subset A+B$.

Now let $C \in$ IVI(R) contain A and B and let $z \in S$ such that $z = x+y$. Then

$$C^L(z) = C^L(x+y) \geq C^L(x) \wedge C^L(y) \text{ and } C^U(z) = C^U(x+y) \geq C^U(x) \wedge C^U(y).$$

Thus

$$\begin{aligned}(A+B)^L(z) &= \bigvee_{z=x+y}(A^L(x) \wedge B^L(y)) \\ &\leq \bigvee_{z=x+y}(C^L(x) \wedge C^L(y)) \text{ (Since } A \subset C \text{ and } B \subset C) \\ &= C^L(z).\end{aligned}$$

Similarly, we have that $(A+B)^U(z) \leq C^U(z)$. So $A+B \subset C$. Hence $A+B$ is the least interval-valued fuzzy ideal containing A and B. Therefore $A+B = A \vee B$. On the other hand, by Lemma 2.19(a), $A \vee B$ has the sup-property. Thus $A \vee B \in \mathrm{IVIs}_{[\lambda_0,\mu_0]}(R)$. So $\mathrm{IVIs}_{[\lambda_0,\mu_0]}(R)$ forms a sublattice of $\mathrm{IVI}_{[\lambda_0,\mu_0]}(R)$ and hence of IVI(R). This completes the proof. □

The following lattice diagram is the interrelationship of different sublattices of the lattice IVR(R):

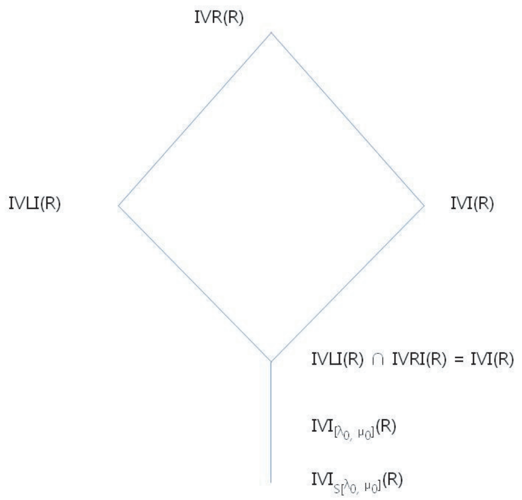

FIGURE 1

Now we obtain an interval-valued fuzzy analog of a well-known result that the set of ideals of a ring forms a modular lattice.

3. Interval-valued fuzzy ideals and modularity

In the previous section, we discussed various sublattices of the lattice of interval-valued fuzzy ideals of a ring. Hence, we obtain an interval-valued fuzzy analog of a well known result that the set of ideals of a ring forms a modular lattice.

Lemma 3.1. Let $A \in$ IVR(R). If $A^L(x) < A^L(y)$ and $A^U(x) < A^U(y)$ for some $x, y \in R$, then $A(x+y) = A(x)$.

Proof. Since $A \in$ IVR(R),

$$A^L(x+y) \geq A^L(x) \wedge A^L(y) = A^L(x)$$

and

$$A^U(x+y) \geq A^U(x) \wedge A^U(y) = A^U(x).$$

Assume that $A^L(x+y) > A^L(x)$ and $A^U(x+y) > A^U(x)$. Then

$$A^L(x) = A^L(y+x-y) \geq A^L(x+y) \wedge A^L(y) > A^L(x)$$

and

$$A^U(x) = A^U(y+x-y) \geq A^U(x+y) \wedge A^U(y) > A^U(x).$$

This contradicts the fact that $A(x) = A(x)$. Hence $A(x+y) = A(x)$. $\square$

Proposition 3.2. The sublattice $\mathrm{IVIs}_{[\lambda_0,\mu_0]}(R)$ is modular.

Proof. Since the modular inequality is valid for every lattice, for any $A, B, C \in \mathrm{IVIs}_{[\lambda_0,\mu_0]}(R)$ with $B \subset A$, we have that $B \vee (A \wedge C) \subset A \wedge (B \vee C)$.
Assume that $A \wedge (B \vee C) \neq B \vee (A \wedge C)$. Then there exists $z \in R$ such that

$$(A \wedge (B \vee C))^L(z) > (B \vee (A \wedge C))^L(z)$$

and

$$(A \wedge (B \vee C))^U(z) > (B \vee (A \wedge C))^U(z).$$

Thus, by Proposition 2.20,

$$A^L(z) \wedge (B+C)^L(z) > (B + (A \cap C))^L(z)$$

and

$$A^U(z) \wedge (B+C)^U(z) > (B + (A \cap C))^U(z).$$

So

$$A^L(z) > (B + (A \cap C))^L(z), A^U(z) > (B + (A \cap C))^U(z)$$

and (3.1)

$$(B+C)^L(z) > (B + (A \cap C))^L(z), (B+C)^U(z) > (B + (A \cap C))^U(z).$$

Then there exist $x_0, y_0 \in R$ with $z = x_0 + y_0$ such that

$$B^L(x_0) \wedge C^L(y_0) > (B + (A \cap C)^L(z)$$

and

$$B^U(x_0) \wedge C^U(y_0) > (B + (A \cap C)^U(z).$$

Thus

$$B^L(x_0) > (B+(A\cap C))^L(z), B^U(x_0) > (B+(A\cap C))^U(z) \tag{3.2}$$

and

$$C^L(y_0) > (B+(A\cap C))^L(z), C^U(y_0) > (B+(A\cap C))^U(z). \tag{3.3}$$

On the other hand,

$$\begin{aligned}(B+(A\cap C))^L(z) &= \bigvee_{z=x+y}(B^L(x)\wedge(A\cap C)^L(y)))\\ &\geq (B^L(x_0)\wedge(A\cap C)^L(y_0))\\ &= B^L(x_0)\wedge A^L(y_0)\wedge C^L(y_0).\end{aligned}$$

Similarly, we have that $(B+(A\cap C))^U(z)\geq B^U(x_0)\wedge A^U(y_0)\wedge C^U(y_0)$. Then, by (3.1),(3.2),(3.3),

$$A^L(z), B^L(x_0), C^L(y_0) > B^L(x_0)\wedge A^L(y_0)\wedge C^L(y_0)$$

and

$$A^U(z), B^U(x_0), C^U(y_0) > B^U(x_0)\wedge A^U(y_0)\wedge C^U(y_0).$$

Thus

$$B^L(x_0)\wedge A^L(y_0)\wedge C^L(y_0) = A^L(y_0)$$

and

$$B^U(x_0)\wedge A^U(y_0)\wedge C^U(y_0) = A^U(y_0).$$

So

$$A^L(-y_0) = A^L(y_0) < A^L(x_0+y_0) = A^L(z)$$

and

$$A^U(-y_0) = A^U(y_0) < A^U(x_0+y_0) = A^U(z).$$

By Lemma 3.1,

$$A^L(y_0) = A^L(x_0+y_0-y_0) = A^L(x_0)$$

and

$$A^U(y_0) = A^U(x_0+y_0-y_0) = A^U(x_0).$$

Then

$$B^L(x_0) > A^L(y_0) = A^L(x_0)$$

and

$$B^U(x_0) > A^U(y_0) = A^U(x_0).$$

This contradicts the fact that $B\subset A$. Hence $A\wedge(B\vee C) = B\vee(A\wedge C)$. Therefore $\mathrm{IVIs}_{[\lambda_0,\mu_0]}(R)$ is modular.This completes the proof. $\square$

Remark 3.3. As a special case, $\mathrm{IVI}_{[1,1]}(R)$ is a complete sublattice of IVI(R) and $\mathrm{IVIs}_{[1,1]}(R)$ is a modular sublattice of IVI(R).

Proposition 3.4.(The generalization of Proposition 3.2) $\mathrm{IVLI}_{[\lambda_0,\mu_0]}(R)$, $\mathrm{IVRI}_{[\lambda_0,\mu_0]}(R)$ and $\mathrm{IVI}_{[\lambda_0,\mu_0]}(R)$ are all modular.

Proof. The proofs are similar to Proposition 3.2. $\square$

Proposition 3.5. IVI(R) is bounded.

Proof. It is clear that $\mathbf{0} \in$ IVI(R) and $\mathbf{1} \in$ IVI(R). Moreover, $\mathbf{0} \subset A \subset \mathbf{1}$ for each $A \in$ IVI(R). Hence IVI(R) is bounded. $\square$

Proposition 3.6. (a) IVI(R) is not complemented.

(b) IVI(R) has no atoms.

(c) IVI(R) has no dual atoms.

Proof. (a) We define a mapping $A : R \to D(I)$ as follows: For each $x \in R, A(x) = [\frac{1}{2}, \frac{1}{2}]$. Then clearly $[A, A^c] \in$ IVI(R). But $A \cup A^c \neq \tilde{1}$ and $A \cap A^c \neq \tilde{0}$. Thus A has no complement in IVI(R). Hence IVI(R) is not complemented.

(b) Suppose $A \in$ IVI(R) with $A \neq \tilde{0}$. We define a mapping $B : R \to D(I)$ as follows: For each $x \in R, B^L(x) = \frac{1}{2}A^L(x)$ and $B^U(x) = \frac{1}{2}A^U(x)$. Then clearly $B \in$ IVI(R). Moreover, $\tilde{0} \subsetneqq B \subsetneqq A$. Hence IVI(R) has no atoms.

(c) Suppose $A \in$ IVI(R) with $A \neq \tilde{1}$. We define a mapping $B : R \to D(I)$ as follows: For each $x \in R$,

$$B^L(x) = \frac{1}{2} + \frac{1}{2}A^L(x) \text{ and } B^U(x) = \frac{1}{2} + \frac{1}{2}A^U(x).$$

Then clearly $A \subsetneqq B \subsetneqq \tilde{1}$.

Now let $x, y \in R$. Then

$$\begin{aligned} B^L(xy) &= \frac{1}{2} + \frac{1}{2}A^L(xy) \\ &\geq \frac{1}{2} + \frac{1}{2}(A^L(x) \vee A^L(y) \text{ (Since } A \in \text{IVI(R))} \\ &= (\frac{1}{2} + \frac{1}{2}A^L(x)) \vee (\frac{1}{2} + \frac{1}{2}A^L(y)). \\ &= B^L(x) \vee B^L(y). \end{aligned}$$

By the similar arguments, we have that $B^U(xy) \geq B^U(x) \vee B^U(y)$. Also,

$$\begin{aligned} B^L(x-y) &= \frac{1}{2} + \frac{1}{2}A^L(x-y) \\ &\geq \frac{1}{2} + \frac{1}{2}(A^L(x) \wedge A^L(y) \text{ (Since } A \in \text{IVI(R))} \\ &= (\frac{1}{2} + \frac{1}{2}A^L(x)) \wedge (\frac{1}{2} + \frac{1}{2}A^L(y)). \\ &= B^L(x) \wedge B^L(y). \end{aligned}$$

By the similar arguments, we have that $B^U(x-y) \geq B^U(x) \wedge B^U(y)$. So $B \in$ IVI(R). Hence IVI(R) has no dual atoms. □

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