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(L, e)-filters on complete residuated lattices

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Abstract

We introduce the notion of (L, e)-filters with fuzzy partially order e on complete residuated lattice L. We investigate (L, e)-filters induced by the family of (L, e)-filters and functions. In fact, we study the initial and final structures for the family of (L, e)-filters and functions. From this result, we define the product and co-product for the family of (L, e)-filters and functions.

Key Words: (L, e)-filters, (L, e)-filter (preserving) maps, the product and co-product of (L, e)-filters.

1. Introduction

Höhle *et al.* [5,6] introduced the notion of L-filter on a complete quasi-monoidal lattice (including GL-monoid [4]) L instead of a completely distributive lattice ([2-4]) as an extension of fuzzy filters [1,2]. The notion of L-filter facilitated to study L-fuzzy topologies [3,5,6], L-fuzzy uniform spaces [5,6] and topological structures [7]. Kim [9] introduced (L, e)-filters with fuzzy partially order e on complete residuated lattice L and investigate their properties.

In this paper, we investigate (L, e)-filters induced by the family of (L, e)-filters and functions. In fact, we investigate the initial and final structures for the family of (L, e)-filters and functions. From this result, we define the product and co-product for the family of (L, e)-filters and functions.

2. Preliminaries

Definition 2.1. [5,6,10] A triple $(X, \leq, *)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(X, \leq, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(X, *, 1)$ is a commutative monoid;

(L3) $*$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) * b = \bigvee_{i\in\Gamma} (a_i * b).$$

Let $(L, \leq, \odot)$ be a complete residuated lattice. An order reversing map ${}^c : L \to L$ defined by $a^c = a \to 0$ is called a *strong negation* if $a^{cc} = a$ for each $a \in L$.

In this paper, we assume $(L, \leq, \odot, {}^c)$ is a complete residuated lattice with a strong negation ^{c}.

Definition 2.2. [5,6,9,10] Let X be a set. A function $e_X : X \times X \to L$ is called a *fuzzy partially order* on X if it satisfies the following conditions:

(E1) $e_X(x, x) = 1$ for all $x \in X$,

(E2) $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

The pair (X, e_X) is a *fuzzy partially order set* (simply, fuzzy poset).

Let $(X, \leq, *)$ be a complete residuated lattice. A fuzzy poset (X, e_X) is a *p-fuzzy poset* if $e_X(x_1, y_1) \odot e_X(x_2, y_2) \leq e_X(x_1 * x_2, y_1 * y_2)$ for each $x_i, y_i \in X$ and $e_X(x, y) = 1$ if $x \leq y$.

Lemma 2.3. [5,6,9,10] For each $x, y, z, x_i, y_i \in L$, we define $x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$. Then the following properties hold.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$ and $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$

(4) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(5) $x \to (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \to y_i)$

(6) $(\bigwedge_{i\in\Gamma} x_i) \to y = \bigvee_{i\in\Gamma}(x_i \to y)$.

(7) $\bigwedge_{i\in\Gamma} y_i^c = (\bigvee_{i\in\Gamma} y_i)^c$ and $\bigvee_{i\in\Gamma} y_i^c = (\bigwedge_{i\in\Gamma} y_i)^c$.

(8) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

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(9) $1 \to x = x$.
(10) If $x \leq y$, then $x \to y = 1$.
(11) $(x \to y) \odot (y \to z) \leq x \to z$.
(12) $(x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \odot x_2 \to y_1 \odot y_2)$.

Definition 2.4. [9] Let $(X, \leq, *)$ be a complete residuated lattice and e_X a fuzzy poset. A mapping $\mathcal{F} : X \to L$ is called a *complete residuated valued* (L, e_X)*-filter* (for short, (L, e_X)-filter) on X if it satisfies the following conditions:
(F1) $\mathcal{F}(0) = 0$ and $\mathcal{F}(1) = 1$,
(F2) $\mathcal{F}(x * y) \geq \mathcal{F}(x) \odot \mathcal{F}(y)$, for each $x, y \in X$,
(F3) $\mathcal{F}(x) \odot e_X(x, y) \leq \mathcal{F}(y)$.
The pair $(X, \mathcal{F})$ is called an (L, e_X)*-filter space*.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be (L, e)-filters on X. We say $\mathcal{F}_1$ is *finer* than $\mathcal{F}_2$ (or $\mathcal{F}_2$ is *coarser* than $\mathcal{F}_1$) iff $\mathcal{F}_2 \leq \mathcal{F}_1$.

Theorem 2.5. [9] Let $(X, \leq, *)$ be a complete residuated lattice and (X, e_X) a p-fuzzy poset. If $\mathcal{H} : X \to L$ is a function satisfying the following condition:
(C) $\mathcal{H}(1) = 1$ and for every finite index set K,

$$\bigvee_K \odot_{i \in K} \mathcal{H}(x_i) \odot e_X(*_{i \in K} x_i, 0) = 0.$$

We define a function $\mathcal{F}_\mathcal{H} : L^X \to L$ as

$$\mathcal{F}_\mathcal{H}(x) = \bigvee (\odot_{i \in K} \mathcal{H}(x_i)) \odot e_X(*_{i \in K} x_i, x)$$

where the $\bigvee$ is taken for every finite set K.

Then:
(1) $\mathcal{F}_\mathcal{H}$ is an (L, e_X)-filter on X,
(2) if $\mathcal{H} \leq \mathcal{F}$ and $\mathcal{F}$ is an (L, e_X)-filter on X, then $\mathcal{F}_\mathcal{H} \leq \mathcal{F}$.

Definition 2.6. [9]Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two (L, e_X) and (L, e_Y)-filter spaces. Then a function $\phi : X \to Y$ is said to be:
(1) a *filter map* iff $\mathcal{G}(y) \leq \bigvee_{x \in \phi^{-1}(\{y\})} \mathcal{F}(x)$, for all $y \in Y$,
(2) a *filter preserving map* iff $\mathcal{F}(x) \leq \mathcal{G}(\phi(x))$ for all $x \in X$.
(3) an *ordered preserving map* iff $e_X(x, y) \leq e_Y(\phi(x), \phi(x))$ for all $x, y \in X$.
(4) $\phi^{-1} : Y \to X$ is an *ordered preserving relation* iff for all $x, y \in Y$,

$$e_Y(x, y) \leq \bigwedge_{a \in \phi^{-1}(\{x\}), b \in \phi^{-1}(\{y\})} e_X(a, b).$$

Naturally, the composition of filter maps (resp. filter preserving maps) is a filter map (resp. filter preserving map).

Definition 2.7. [9]Let $\phi : X \to Y$ be a function, $\mathcal{F}$ an (L, e_X)-filter on X and $\mathcal{G}$ an (L, e_Y)-filter Y.

(1) The *image* of $\mathcal{F}$ is a function $\phi_L^{\to}(\mathcal{F}) : Y \to L$ defined by

$$\phi_L^{\to}(\mathcal{F})(y) = \bigvee \{\mathcal{F}(x) \mid x = \phi^{-1}(y)\}.$$

(2) The *preimage* of $\mathcal{G}$ is a function $\phi_L^{\leftarrow}(\mathcal{G}) : X \to L$ defined by

$$\phi_L^{\leftarrow}(\mathcal{G})(x) = \mathcal{G}(\phi(x)).$$

(3) Let $\mathcal{H} : X \to L$ be a function and $x \in X$. We denote

$$[\mathcal{H}](x) = \bigvee_{y \in X} \mathcal{H}(y) \odot e_X(y, x).$$

Theorem 2.8. [9] Let $(X, \leq, *)$ and $(Y, \leq, \star)$ be complete residuated lattices. Let $\phi : X \to Y$ be an order preserving function with $\phi(x * y) \geq \phi(x) \star \phi(y)$, $\phi(0) = 0$ and $\phi(1) = 1$, e_X, e_Y p-fuzzy posets and $\mathcal{G}$ an (L, e_Y)-filter on Y. Then:
(1) $[\phi_L^{\leftarrow}(\mathcal{G})]$ is the coarsest (L, e_X)-filter for which $\phi : (X, [\phi_L^{\leftarrow}(\mathcal{G})]) \to (Y, \mathcal{G})$ is a filter map.
(2) If $e_X(x, y) = e_Y(\phi(x), \phi(y))$ for $x, y \in X$, then $[\phi_L^{\leftarrow}(\mathcal{G})] = \phi_L^{\leftarrow}(\mathcal{G})$.

Theorem 2.9. [9] Let $(X, \leq, *)$ and $(Y, \leq, \star)$ be complete residuated lattices. Let $\phi : X \to Y$ be a function with $\phi(x*y) \leq \phi(x) \star \phi(y)$ with $\phi(1) = 1$ and $\phi(0) = 0$, e_X, e_Y p-fuzzy posets. Let $\mathcal{F}$ and $\mathcal{G}$ be (L, e_X) and (L, e_Y)-filters, respectively. Then we have the following properties.
(1) If $\mathcal{F}(x) \odot e_Y(\phi(x), 0) = 0$, then $[\phi_L^{\to}(\mathcal{F})]$ is the coarsest (L, e_Y)-filter for which $\phi : (X, \mathcal{F}) \to (Y, [\phi_L^{\to}(\mathcal{F})])$ is a filter preserving map.
(2) If ϕ is injective and ϕ^{-1} is an order-preserving relation, $[\phi_L^{\to}(\mathcal{F})]$ is an (L, e_X)-filter.
(3) If ϕ is surjective, ϕ^{-1} is an order-preserving relation and $\mathcal{F}$ is an (L, e_X)-filter with $\mathcal{F}(x) \odot e_Y(\phi(x), 0) = 0$, then $\phi_L^{\to}(\mathcal{F})$ is an (L, e_X)-filter.
(4) If $\phi : X \to Y$ is an order preserving map with $\phi(x * y) = \phi(x) \star \phi(y)$, then $[\phi_L^{\to}([\phi_L^{\leftarrow}(\mathcal{G})])]$ is an (L, e_Y)-filter on Y with $[\phi_L^{\to}([\phi_L^{\leftarrow}(\mathcal{G})])] \leq \mathcal{G}$.

3. The preimages and images of (L, e)-filters

Theorem 3.1. Let $(X, \leq, *)$ and $(X_i, \leq, \star_i)$ be complete residuated lattices. Let $\phi_i : (X, e_X) \to (X_i, e_{X_i})$ be order preserving functions with $\phi_i(x * y) \geq \phi_i(x) \star_i \phi_i(y)$, $\phi_i(1) = 1$, $\phi_i(0) = 0$, e_X, e_{X_i} p-fuzzy posets for all $i \in \Gamma$. Let $\{\mathcal{G}_i\}_{i \in \Gamma}$ be a family of (L, e_{X_i})-filters on X_i satisfying the following condition:
(C) For every finite subset K of Γ, $\odot_{i \in K} \phi_i^{\leftarrow}(\mathcal{G}_i)(x_i) \odot e_X(*_{i \in K} x_i, 0) = 0$.

We define a function $[\bigotimes_{i \in \Gamma} \phi_i^{\leftarrow}(\mathcal{G}_i)] : X \to L$ as

$$[\bigotimes_{i \in \Gamma} \phi_i^{\leftarrow}(\mathcal{G}_i)](x) = \bigvee_K (\odot_{i \in K} \phi_i^{\leftarrow}(\mathcal{G}_i)(x_i) \odot e_X(*_{i \in K} x_i, x)$$

where the $\bigvee$ is taken for every finite subset K of Γ. Put $\mathcal{F} = [\bigotimes_{i\in\Gamma} \phi_i^{\leftarrow}(\mathcal{G}_i)]$. Then the following properties hold.

(1) $\mathcal{F}$ is the coarsest (L, e_X)-filter for which $\phi_i : (X, \mathcal{F}) \to (X_i, \mathcal{G}_i)$ is a filter map.

(2) If for each $i \in \Gamma$, $\phi_i \circ \phi : (Y, \mathcal{F}^*) \to (X_i, \mathcal{G}_i)$ is a filter map and $e_Y(x, y) \geq e_X(\phi(x), \phi(y))$ for all $x, y \in Y$, then a map $\phi : (Y, \mathcal{F}^*) \to (X, \mathcal{F})$ is a filter map.

Proof. (1) (F1) By the condition (C),

$$\mathcal{F}(0) = \bigvee_K (\odot_{k\in K}\phi_k^{\leftarrow}(\mathcal{G}_k)(x_k) \odot e_X(*_{k\in K}x_k, 0)) = 0.$$

$$\mathcal{F}(1) \geq \bigvee_K (\odot_{k\in K}\phi_k^{\leftarrow}(\mathcal{G}_k)(1) \odot e_X(1, 1)) = 1.$$

(F2) For each two finite subsets K and J of Γ,

$$\begin{aligned}
&\mathcal{F}(x_1) \odot \mathcal{F}(x_2) \\
&= \bigvee_K(\odot_{k\in K}\phi_k^{\leftarrow}(\mathcal{G}_k)(u_k) \odot e_X(*_{k\in K}u_k, x_1) \\
&\odot \bigvee_J(\odot_{j\in J}\phi_j^{\leftarrow}(\mathcal{G}_j)(w_j) \odot e_X(*_{j\in J}w_j, x_2) \\
&\leq \bigvee_{K\cup J} \Big(\odot_{m\in(K\cup J)-(K\cap J)} \mathcal{G}_m(p_m)\Big) \odot \\
&\Big(\odot_{m\in(K\cap J)} \mathcal{G}_m(\phi_m(u_m * w_m))\Big) \\
&\odot e_X((*_{k\in K}u_k) * (*_{j\in J}w_j), x_1 * x_2) \\
&= \bigvee_{K\cup J} \odot_{m\in(K\cup J)} \Big(\mathcal{G}_m(p_m)\odot \\
&e_X((*_{k\in K}u_k) * (*_{j\in J}w_j), x_1 * x_2)\Big) \\
&\leq \mathcal{F}(x_1 * x_2)
\end{aligned}$$

where for $m \in K \cup J$,

$$p_m = \begin{cases} \phi_m(u_m) \text{ if } m \in K - (K \cap J), \\ \phi_m(w_m) \text{ if } m \in J - (K \cap J), \\ \phi_m(u_m * w_m) \text{ if } m \in K \cap J. \end{cases}$$

because, for each $m \in K \cap J$,

$$\begin{aligned}
\mathcal{G}_m(\phi_m(u_m * w_m)) &\geq \mathcal{G}_m(\phi_m(u_m) \star_m \phi_m(w_m)) \\
&\geq \mathcal{G}_m(\phi_m(u_m)) \odot \mathcal{G}_m(\phi_m(w_m)).
\end{aligned}$$

(F3) For every finite subsets K,

$$\begin{aligned}
&\mathcal{F}(x) \odot e_X(x, z) \\
&= \bigvee_K(\odot_{k\in K}\phi_k^{\leftarrow}(\mathcal{G}_k)(x_k) \odot e_X(*_{k\in K}x_k, x)) \odot e_X(x, z) \\
&\leq \bigvee_K(\odot_{k\in K}\phi_k^{\leftarrow}(\mathcal{G}_k)(x_k) \odot e_X(*_{k\in K}x_k, z)) = \mathcal{F}(z).
\end{aligned}$$

Since $\bigvee_{x\in\phi_i^{-1}(\{x_i\})} \mathcal{F}(x) \geq \phi_i^{\leftarrow}(\mathcal{G}_i)(x) \odot e_X(x, x) = \mathcal{G}_i(x_i)$ for each $i \in \Gamma$, ϕ_i is a filter map.

Let $\bigvee_{x\in\phi_i^{-1}(\{x_i\})} \mathcal{G}(x) \geq \mathcal{G}_i(\phi_i(x)) = \mathcal{G}_i(x_i)$ be given for each $i \in \Gamma$. For each finite subset K of Γ, we have

$$\begin{aligned}
&\mathcal{G}(x) \\
&\geq \bigvee_{z_k\in\phi_k^{-1}(\{x_k\})} \mathcal{G}(*_{k\in K}z_k) \odot e_X(*_{k\in K}z_k, x) \\
&\geq \bigvee_{z_k\in\phi_k^{-1}(\{x_k\})} \odot_{k\in K}\mathcal{G}(z_k) \odot e_X(*_{k\in K}z_k, x) \\
&\geq \bigvee_{z_k\in\phi_k^{-1}(\{x_k\})} \odot_{k\in K}\mathcal{G}_k(x_k) \odot e_X(*_{k\in K}z_k, x) \\
&\geq \odot_{k\in K}\mathcal{G}_k(\phi_k(z_k)) \odot e_X(*_{k\in K}z_k, x).
\end{aligned}$$

Hence, by the definition of $\mathcal{F}$, $\mathcal{G} \geq \mathcal{F}$.

(2) Since for each $k \in K$, $\bigvee_{y\in(\phi_k\circ\phi)^{-1}(\{x_k\})} \mathcal{F}^*(y) \geq \mathcal{G}_k(x_k)$, $y = \odot_{k\in K}\phi^{-1}(\{z_k\}) = \phi^{-1}(*_{k\in K}z_k)$ and $\mathcal{F}^*(y) \odot e_X(y, z) \leq \mathcal{F}^*(z)$, for each finite index set K , we have

$$\begin{aligned}
&\bigvee_{z\in\phi^{-1}(\{x\})} \mathcal{F}^*(z) \\
&\geq \bigvee_{z\in\phi^{-1}(\{x\})} \Big(\Big(\bigvee_{y\in\odot_{k\in K}(\phi_k\circ\phi)^{-1}(\{x_k\})} \mathcal{F}^*(y)\Big) \\
&\odot e_Y(y, z)\Big) \\
&\geq \bigvee_{z\in\phi^{-1}(\{x\})} \Big(\Big(\bigvee_{z_k\in\phi_k^{-1}(\{x_k\})} \odot_{k\in K}\mathcal{G}_k(x_k)\Big) \\
&\odot e_X(\phi(y), \phi(z))\Big) \\
&\geq \odot_{k\in K}\mathcal{G}_k(\phi_k(z_k)) \odot e_X(*_k z_k, x) \\
&= \phi_k^{\leftarrow}(\mathcal{G}_k)(z_k) \odot e_X(*_k z_k, x)
\end{aligned}$$

By the definition of $\mathcal{F}$, $\mathcal{F}(x) \leq \bigvee_{z\in\phi^{-1}(\{x\})} \mathcal{F}^*(z)$. □

From Theorem 3.1, we can obtain the following corollaries.

Corollary 3.2. Let $(X, \leq, *)$ be a complete residuated lattice. Let $\{\mathcal{F}_i\}_{i\in\Gamma}$ be a family of (L, e_X)-filters on X and e_X a p-fuzzy poset, satisfying the following condition:

(C) For every finite subset K of Γ, $\odot_{i\in K}\mathcal{F}_i(x_i) \odot e_X(*_{i\in K}x_i, 0) = 0$.

We define a function $[\bigotimes_{i\in\Gamma} \mathcal{F}_i)] : X \to L$ as

$$[\bigotimes_{i\in\Gamma} \mathcal{F}_i](x) = \bigvee_K (\odot_{i\in K}\mathcal{F}_i(x_i) \odot e_X(*_{i\in K}x_i, x)$$

where the $\bigvee$ is taken for every finite subset K of Γ. Then $[\bigotimes_{i\in\Gamma} \mathcal{F}_i]$ is the coarsest (L, e_X)-filter finer than $\mathcal{F}_i$ for each $i \in \Gamma$.

Corollary 3.3. Let $X = \Pi_{i\in\Gamma}X_i$ be a product set and $\pi_i : X \to X_i$ projection maps for all $i \in \Gamma$. Let $(X, \leq, *)$ and $(X_i, \leq, \star_i)$ be complete residuated lattices. Let $\pi_i : (X, e_X) \to (X_i, e_{X_i})$ be order preserving functions with $\pi_i(x * y) \geq \pi_i(x) \star_i \pi_i(y)$ and e_X, e_{X_i} p-fuzzy posets for all $i \in \Gamma$. Let $\{\mathcal{F}_i\}_{i\in\Gamma}$ be a family of (L, e_{X_i})-filters on X_i satisfying the following condition:

(C) For every finite subset K of Γ, $\odot_{i\in K}\pi_i^{\leftarrow}(\mathcal{F}_i)(x_i) \odot e_X(*_{i\in K}x_i, 0) = 0$.

We define a function $[\bigotimes_{i\in\Gamma} \pi_i^{\leftarrow}(\mathcal{F}_i)] : X \to L$ as

$$[\bigotimes_{i\in\Gamma} \pi_i^{\leftarrow}(\mathcal{F}_i)](x) = \bigvee_K (\odot_{i\in K}\pi_i^{\leftarrow}(\mathcal{F}_i)(x_i) \odot e_X(*_{i\in K}x_i, x)$$

where the $\bigvee$ is taken for every finite subset K of Γ. Let $\mathcal{F} = [\bigotimes_{i\in\Gamma} \pi_i^{\leftarrow}(\mathcal{F}_i)]$ be given. Then:

(1) $\mathcal{F}$ is the coarsest (L, e_X)-filter for which $\pi_i : (X, \mathcal{F}) \to (X_i, \mathcal{F}_i)$ is a filter map,

(2) If for each $i \in \Gamma$, $\pi_i \circ \phi : (Y, \mathcal{F}^*) \to (X_i, \mathcal{F}_i)$ is a filter map and $e_Y(x, y) \geq e_X(\phi(x), \phi(y))$ for all $x, y \in Y$, then a map $\phi : (Y, \mathcal{F}^*) \to (X, \mathcal{F})$ is a filter map.

In Corollary 3.3, the structure $[\bigotimes_{i\in\Gamma} \pi_i^{\leftarrow}(\mathcal{F}_i)]$ is called a *product* (L, e_X)*-filter* on X.

Example 3.4. Let $(X = L = [0,1], \odot)$ be the complete residuated lattice with $x\odot y = (x+y-1)\vee 0$. We define p-fuzzy partially order $e_0 : [0,1]\times[0,1] \to [0,1]$ as follows:

$$e_0(x,y) = \begin{cases} 1 \text{ if } \ x \le y, \\ 0 \text{ otherwise,} \end{cases}$$

Define functions $\mathcal{F}_i : [0,1] \to [0,1]$ as follows: for $x \in [0,1]$,

$$\begin{aligned} \mathcal{F}_1(x) \quad &= (1\odot e_0(1,x)) \vee (0.6 \odot e_0(0.6,x)) \\ &\quad \vee (0.3 \odot e_0(0.2,x)) \\ \mathcal{F}_2(x) \quad &= (1\odot e_0(1,x)) \vee (0.5 \odot e_0(0.2,x)) \\ \mathcal{F}_3(x) \quad &= (1\odot e_0(1,x)) \vee (0.4 \odot e_0(0.6,x)) \\ &\quad \vee (0.3 \odot e_0(0.1,x)) \end{aligned}$$

Each $\mathcal{F}_i$ for $i = 1,2,3$ is a $([0,1], e_0)$-filter.

(1) $[\mathcal{F}_1 \otimes \mathcal{F}_2]$ does not exist from:

$$\mathcal{F}_1(0.6)\odot\mathcal{F}_2(0.2)\odot e_0(0.6\odot 0.2, 0) = 0.1 \neq 0.$$

(2) We can obtain $[\mathcal{F}_2 \otimes \mathcal{F}_3]$ as

$$\begin{aligned} [\mathcal{F}_2 \otimes \mathcal{F}_3](x) \quad &= (1\odot e_X(1,x)) \vee (0.5 \odot e_0(0.2,x)) \\ &\quad \vee (0.3 \odot e_0(0.1,x)). \end{aligned}$$

Example 3.5. We define $([0,1], e_0)$-filters $\mathcal{F}_1 : X \to [0,1]$ and $\mathcal{F}_2 : Y \to [0,1]$ as follows

$$\begin{aligned} \mathcal{F}_1(x) \quad &= (1\odot e_0(1,x)) \vee (0.5 \odot e_0(0.3,x)) \\ \mathcal{F}_2(y) \quad &= (1\odot e_0(1,y)) \vee (0.6 \odot e_0(0.6,y)) \\ &\quad \vee (0.2 \odot e_0(0.2,y)). \end{aligned}$$

Let $\pi_1 : X\times Y \to X$ and $\pi_2 : X\times Y \to Y$ be projection maps. We can obtain the product $([0,1], e_{X\times Y})$-filter $\mathcal{F} = [\pi_1^{-1}(\mathcal{F}_1)\otimes\pi_2^{-1}(\mathcal{F}_2)]$ as

$$\begin{aligned} \mathcal{F}(x,y) \quad &= (1\odot e_{X\times Y}(1,(x,y))) \\ &\quad \vee (0.3\odot e_{X\times Y}((0.3,1),(x,y))) \\ &\quad \vee (0.6\odot e_{X\times Y}((1,0.6),(x,y))) \\ &\quad \vee (0.2\odot e_{X\times Y}((1,0.2),(x,y))) \\ &\quad \vee (0.1\odot e_{X\times Y}((0.3,0.6),(x,y))). \end{aligned}$$

where $e_{X\times Y}((x_1,y_1),(x_2,y_2)) = e_0(x_1,x_2)\wedge e_0(y_1,y_2)$.

Theorem 3.6. Let $(X, \le, \star)$ and $(X_i, \le, *_i)$ be complete residuated lattices. Let $\phi_i : (X_i, *_i) \to (X, \star)$ be functions with $\phi_i(x_i *_i y_i) \le \phi_i(x_i)\star\phi_i(y_i)$ and e_X, e_{X_i} p-fuzzy posets for all $i\in\Gamma$. Let $\{\mathcal{F}_i\}_{i\in\Gamma}$ be a family of (L, e_{X_i})-filters on X_i satisfying the following condition:

(C) For every finite subset K, $\odot_{i\in K}(\mathcal{F}_i(x_i)\odot e_X(\star_{i\in K}\phi_i(x_i), 0) = 0$.

We define a function $[\bigoplus_{i\in K}\phi_i^{\rightarrow}(\mathcal{F}_i)] : X \to L$ as

$$[\bigoplus_{i\in K}\phi_i^{\rightarrow}(\mathcal{F}_i)](x) = \bigvee_K(\odot_{i\in K}\mathcal{F}_i(x_i)\odot e_X(\star_{i\in K}\phi_i(x_i), x))$$

where the $\bigvee$ is taken for every finite subset K of Γ.

Let $\mathcal{F} = [\bigoplus_{i\in K}\phi_i^{\rightarrow}(\mathcal{F}_i)]$ be given.

Then (1) $\mathcal{F}$ is the coarsest (L, e_X)-filter for which $\phi_i : (X_i, \mathcal{F}_i) \to (X, \mathcal{F})$ is a filter preserving map,

(2) If for each $i\in\Gamma$, $\phi\circ\phi_i : (X_i,\mathcal{F}_i)\to(Y,\mathcal{G})$ is a filter preserving map and ϕ is an order preserving map, then a map $\phi : (X,\mathcal{F}) \to (Y,\mathcal{G})$ is an (L, e_X)-filter preserving map.

Proof. (1) (F1) By the condition (C), $\mathcal{F}(0) = 0$. Since $e_X(\phi_i(1), 1) = 1$, $\mathcal{F}(1) = 1$.

(F2) For each two finite subsets K and J,

$$\begin{aligned} &\mathcal{F}(x)\odot\mathcal{F}(z) \\ &= \bigvee_K(\odot_{k\in K}\mathcal{F}_k(x_k)\odot e_X(\star_{k\in K}(\phi_k(x_k)), x) \\ &\odot \bigvee_J(\odot_{j\in J}\mathcal{F}_j(z_j)\odot e_X(\star_{j\in J}(\phi_j(z_j)), z) \\ &\le \bigvee_{K,J}\Big((\odot_{m\in(K\cup J)-(K\cap J)}\mathcal{F}_m(w_m)) \\ &\odot(\odot_{m\in(K\cap J)}\mathcal{F}_m(x_m *_m z_m))\Big) \\ &\odot e_X(\star_{k\in K}(\phi_k(x_k))\star(\star_{j\in J}(\phi_j(z_j))), x\star z) \\ &= \Big(\odot_{m\in(K\cup J)}\mathcal{F}_m(w_m)\odot e_X(\star_{m\in(K\cup J)}\phi_m(w_m), x\star z) \\ &\le \mathcal{F}(x\star z) \end{aligned}$$

where for $m\in K\cup J$,

$$w_m = \begin{cases} x_m \text{ if } m\in K-(K\cap J), \\ z_m \text{ if } m\in J-(K\cap J), \\ x_m *_m z_m \text{ if } m\in K\cap J. \end{cases}$$

because, for each $m \in K\cap J$, $\mathcal{F}_m(x_m *_m z_m) \ge \mathcal{F}_m(x_m)\odot\mathcal{F}_m(z_m)$ and

$$\begin{aligned} &e_X(\star_{k\in K\cap J}(\phi_k(x_k *_k y_k), \star_{k\in K\cap J}\phi_k(x_k)\star\phi_k(x_k)) \\ &\odot e_X(\star_{k\in K\cap J}(\star_{k\in K\cap J}\phi_k(x_k)\star\phi_k(x_k), x\star z) \\ &\le e_X(\star_{k\in K\cap J}(\phi_k(x_k *_k y_k), x\star z) \end{aligned}$$

Since $\mathcal{F}(\phi_i(x_i)) \ge \mathcal{F}_i(x_i)\odot e_X(\phi_i(x_i),\phi_i(x_i)) = \mathcal{F}_i(x_i)$ for each $i\in\Gamma$, ϕ_i is an (L, e_X)-filter preserving map.

Let $\mathcal{G}(\phi_i(x_i)) \ge \mathcal{F}_i(x_i)$ be given for each $i\in\Gamma$. For each finite subset K of Γ, since $\mathcal{G}(\phi_k(x_k)) \ge \mathcal{F}_k(x_k)$ for all $k\in K$, we have

$$\begin{aligned} \mathcal{G}(x) \quad &\ge \mathcal{G}(\star_{k\in K}\phi_k(x_k))\odot e_X(\star_{k\in K}\phi_k(x_k), x) \\ &\ge \odot_{k\in K}\mathcal{G}(\phi_k(x_k))\odot e_X(\star_{k\in K}\phi_k(x_k), x) \\ &\ge \odot_{k\in K}\mathcal{F}_k(x_k)\odot e_X(\star_{k\in K}\phi_k(x_k), x). \end{aligned}$$

Hence, by the definition of $\mathcal{F}$, $\mathcal{G}\ge\mathcal{F}$.

(2) Since for each $k\in K$, $\phi\circ\phi_k : (X_k,\mathcal{F}_k)\to(Y,\mathcal{F}^*)$ is an L-filter preserving map;i.e. $\mathcal{F}_k(x_k)\le\mathcal{F}^*(\phi\circ\phi_k(x_k))$ for each finite index set K , we have

$$\begin{aligned} \mathcal{F}^*(\phi(z)) \quad &\ge \odot_{k\in K}\mathcal{F}^*((\phi\circ\phi_k)(x_k)) \\ &\quad \odot e_Y(\star_{k\in K}(\phi\circ\phi_k)(x_k), \phi(z)) \quad \text{(by (F2))} \\ &\ge \odot_{k\in K}\mathcal{F}_k(x_k)\odot e_Y(\star_{k\in K}\phi_k(x_k), z). \end{aligned}$$

By the definition of $\mathcal{F}$, $\mathcal{F}(z)\le\mathcal{F}^*(\phi(z))$.

□

From Theorem 3.6, we can obtain the following corollary.

Corollary 3.7. Let $X = \oplus_{i\in\Gamma} X_i$ be a direct sum and $\mu_i : X_i \to X$ inclusion maps for all $i \in \Gamma$. Let $(X, *)$ and $(X_i, \star_i)$ be complete residuated lattices.

Let $\mu_i : (X_i, *_i) \to (X, \star)$ be functions with $\mu_i(x_i *_i y_i) \leq \mu_i(x_i) \star \mu_i(y_i)$ and e_X, e_{X_i} p-fuzzy posets for all $i \in \Gamma$. Let $\{\mathcal{F}_i\}_{i\in\Gamma}$ be a family of (L, e_{X_i})-filters on X_i satisfying the following condition:

(C) For every finite subset K, $\odot_{i\in K}(\mathcal{F}_i(x_i) \odot e_X(\star_{i\in K}\mu_i(x_i), 0) = 0$.

We define a function $[\bigoplus_{i\in K} \mu_i^{\rightarrow}(\mathcal{F}_i)] : X \to L$ as

$$[\bigoplus_{i\in K} \mu_i^{\rightarrow}(\mathcal{F}_i)](x) = \bigvee_K (\odot_{i\in K}\mathcal{F}_i(x_i) \odot e_X(\star_{i\in K}\mu_i(x_i), x))$$

where the $\bigvee$ is taken for every finite subset K of Γ.

Let $\mathcal{F} = [\bigoplus_{i\in K} \mu_i^{\rightarrow}(\mathcal{F}_i)]$ be given.

Then (1) $\mathcal{F}$ is the coarsest (L, e_X)-filter for which $\mu_i : (X_i, \mathcal{F}_i) \to (X, \mathcal{F})$ is a filter preserving map,

(2) If for each $i \in \Gamma$, $\mu \circ \mu_i : (X_i, \mathcal{F}_i) \to (Y, \mathcal{G})$ is a filter preserving map and μ is an order preserving map, then a map $\mu : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is an (L, e_X)-filter preserving map.

In Corollary 3.7, the structure $[\bigoplus_{i\in\Gamma} \mu_i^{\rightarrow}(\mathcal{F}_i)]$ is called a *co-product* (L, e_X)*-filter* on X.

Example 3.8. Let $(X = \{0, \frac{1}{2}, 1\}, \odot)$, $(Y = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}, \odot)$ and $(L = [0, 1], \odot)$ be complete residuated lattices with $x \odot y = (x + y - 1) \vee 0$ and $x \to y = (1 - x + y) \wedge 1$. Define functions $\phi_i : X \to Y$ as follows:

$$\phi_1(0) = 0, \phi_1(\frac{1}{2}) = \frac{3}{4}, \phi_1(1) = 1, \phi_2(x) = x$$

$$\phi_3(0) = 0, \phi_3(\frac{1}{2}) = \frac{1}{4}, \phi_3(1) = 1.$$

Define functions $\mathcal{F}_i : X \to [0, 1]$ as follows:

$$\mathcal{F}_1(x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{1}{2} \text{ if } & x = \frac{1}{2}, \\ 0 \text{ if } & x = 0, \end{cases} \quad \mathcal{F}_2(x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{3}{4} \text{ if } & x = \frac{1}{2} \\ 0 \text{ if } & x = 0. \end{cases}$$

$e_0, e_1 : X \times X \to [0, 1]$ as follows:

$$e_0(x, y) = \begin{cases} 1 \text{ if } & x \leq y, \\ 0 \text{ otherwise}, \end{cases}$$

and $e_1(x, y) = x \to y$.

(1) Since $(\mathcal{F}_1(x_1) \odot \mathcal{F}_2(x_2) \odot e_0(\phi_1(x_1) \odot \phi_2(x_2), 0) = 0$, we obtain $([0, 1], e_0)$-filter $[\phi_1^{\rightarrow}(\mathcal{F}_1) \oplus \phi_2^{\rightarrow}(\mathcal{F}_2)] : Y \to [0, 1]$ as follows:

$$[\phi_1^{\rightarrow}(\mathcal{F}_1) \oplus \phi_2^{\rightarrow}(\mathcal{F}_2)](x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{1}{4} \text{ if } & x = \frac{1}{4}, \\ \frac{3}{4} \text{ if } & x = \frac{1}{2}, \\ \frac{3}{4} \text{ if } & x = \frac{3}{4} \\ 0 \text{ if } & x = 0. \end{cases}$$

(2) Since $(\mathcal{F}_1(x_1) \odot \mathcal{F}_2(x_2) \odot e_1(\phi_1(x_1) \odot \phi_2(x_2), 0) = 0$, we obtain $([0, 1], e_1)$-filter $[\phi_1^{\rightarrow}(\mathcal{F}_1) \oplus \phi_2^{\rightarrow}(\mathcal{F}_2)] : Y \to [0, 1]$ as follows:

$$[\phi_1^{\rightarrow}(\mathcal{F}_1) \oplus \phi_2^{\rightarrow}(\mathcal{F}_2)](x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{1}{4} \text{ if } & x = \frac{1}{4}, \\ \frac{3}{4} \text{ if } & x = \frac{1}{2}, \\ \frac{3}{4} \text{ if } & x = \frac{3}{4} \\ 0 \text{ if } & x = 0. \end{cases}$$

(3) Since $(\mathcal{F}_1(\frac{1}{2}) \odot \mathcal{F}_2(\frac{1}{2}) \odot e_0(\phi_1(\frac{1}{2}) \odot \phi_3(\frac{1}{2}), 0) = \frac{1}{2} \odot \frac{3}{4} \odot e_0(\frac{1}{2} \odot \frac{1}{4}, 0) = \frac{1}{2} \neq 0$, we cannot obtain $([0, 1], e_0)$-filter $[\phi_1^{\rightarrow}(\mathcal{F}_1) \oplus \phi_2^{\rightarrow}(\mathcal{F}_2)]$. By a similarly, we cannot obtain $([0, 1], e_1)$-filter $[\phi_1^{\rightarrow}(\mathcal{F}_1) \oplus \phi_2^{\rightarrow}(\mathcal{F}_2)]$. Moreover,

$$\frac{1}{4} = \phi_3(\frac{1}{2} \odot 1) \not\leq \phi_3(\frac{1}{2}) \odot \phi_3(1) = \frac{1}{4}.$$

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