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Interval-valued Fuzzy Normal Subgroups

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Abstract

We study some properties of interval-valued fuzzy normal subgroups of a group. In particular, we obtain two characterizations of interval-valued fuzzy normal subgroups. Moreover, we introduce the concept of an interval-valued fuzzy coset and obtain several results which are analogous of some basic theorems of group theory.

Key Words: interval-valued fuzzy normal subgroup, interval-valued fuzzy coset, interval-valued fuzzy quotient group.

1. Introduction and Preliminaries

In 1975, Zadeh[11] introduced the concept of interval-valued fuzzy sets as the generalization of fuzzy sets introduced by himself[10]. After that time, Biswas[1] applied the notion of interval-valued fuzzy set to group theory, and Samanta and Montal[9] to topology. Recently, Choi et al.[2] introduced the concept of interval-valued smooth topological spaces and studied some of it's properties. Hur et al.[3] investigated interval-valued fuzzy relations, Kang and Hur[6] applied the concept of interval-valued fuzzy sets to algebra. In particular, Kang[7] studied interval-valued fuzzy subgroups preserved by homomorphisms. In this paper, we investigate some properties of interval-valued fuzzy normal subgroups of a group. In particular, we obtain two characterizations of interval-valued fuzzy normal subgroups. introduce the concept of interval-valued fuzzy subgroups. Moreover, we introduce the concept of an interval-valued fuzzy coset and obtain several results which are analogous of some basic theorems of group theory.

Now, we will list some concepts and results related to interval-valued fuzzy set theory and needed in next sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0,1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\boldsymbol{a}$=$[a,a]$ for every $a \in (0,1)$.

We also note that

(i) $(\forall M, N \in D(I))$ $(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,
(ii) $(\forall M, N \in D(I))$ $(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$ (See [9]).

Definition 1.1 [9, 11]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set* (in short, *IVS*) in X and is denoted by $A = [A^L, A^U]$. Thus $A(x) = [A^L(x), A^U(x)]$, where $A^L(x)$[resp. $A^U(x)$] is called the *lower*[resp. *upper*] *end point* of x to A. For any $[a,b] \in D(I)$, the interval-valued fuzzy set A in X defined by $A(x) = [A^L(x), A^U(x)] = [a,b]$ for each $x \in X$ is denoted by $\widetilde{[a,b]}$ and if $a = b$, then the IVS $\widetilde{[a,b]}$ is denoted by simply $\tilde{a}$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVSs in X as $D(I)^X$. It is clear that set $A = [A^L, A^U] \in D(I)^X$ for each $A \in I^X$.

Definition 1.2 [9]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(a) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.
(b) $A = B$ iff $A \subset B$ and $B \subset A$.
(c) $A^C = [1 - A^U, 1 - A^L]$.
(d) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.
(d)′ $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A_\alpha^L, \bigvee_{\alpha \in \Gamma} A_\alpha^U]$.
(e) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.
(e)′ $\bigcap_{\alpha \in \Gamma} A_\alpha = [\bigwedge_{\alpha \in \Gamma} A_\alpha^L, \bigwedge_{\alpha \in \Gamma} A_\alpha^U]$.

Result 1.A [9, Theorem 1]. Let $A, B, C \in D(I)^X$ and let

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$\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then
(a) $\tilde{0} \subset A \subset \tilde{1}$.
(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.
(c) $A \cup (B \cup C) = (A \cup B) \cup C$,
$A \cap (B \cap C) = (A \cap B) \cap C$.
(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.
(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma}(A \cap A_\alpha)$.
(f) $A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma}(A \cup A_\alpha)$.
(g) $(\tilde{0})^c = \tilde{1}$, $(\tilde{1})^c = \tilde{0}$.
(h) $(A^c)^c = A$.
(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A_\alpha^c$, $(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A_\alpha^c$.

Definition 1.3 [6]. An interval-valued fuzzy set A in G is called an *interval-valued fuzzy subgroupoid*(in short, IVGP) in G if for any $x, y \in G$,
$A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.

We will denote IVGPs in G as IVGP(G). Then it is clear that $\widetilde{0}$ and $\widetilde{1} \in$ IVGP(G).

Definition 1.4 [7]. Let A be an IVS in a group G. Then A is called an *interval-valued fuzzy subgroup*(in short, *IVG*) in G if it satisfies the conditions: For any $x, y \in G$,
(a) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.
(b) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$.
We will denote the set of all IVGs of G as IVG(G).

Result 1.B [1, Proposition 3.1]. Let A be an IVG in a group G.
(a) $A(x^{-1}) = A(x), \forall x \in G$.
(b) $A^L(e) \geq A^L(x)$ *and* $A^U(e) \geq A^U(x), \forall x \in G.$, where e is the identity of G.

Result 1.C [6, Proposition 4.7]. Let $A \in$ IVG(G). If $A(xy^{-1}) = A(e)$, for any $x, y \in G$, then $A(x) = A(y)$.

Definition 1.5 [6]. Let A be an IVS in a set X and let $[\lambda, \mu] \in D(I)$. Then the set $A^{[\lambda,\mu]} = \{x \in X : A^L(x) \geq \lambda$ and $A^U(x) \geq \mu\}$ is called a $[\lambda, \mu]$-*level subset* of A.

Result 1.D [6, Propositions 4.16 and 4.17]. Let A be an IVS in a group G. Then $A \in$ IVG(G) if and only if for each $[\lambda, \mu] \in$ Im A with $\lambda \leq A^L(e)$ and $\mu \leq A^U(e)$, $A^{[\lambda,\mu]}$ is a subgroup of G.

Result 1.E [7, Proposition 3.2]. Let A be an IVFS in a set X and let $[\lambda_1, \mu_1], [\lambda_2, \mu_2] \in ImA$. If $\lambda_1 < \lambda_2$ and $\lambda_2 < \mu_2$, then $A^{[\lambda_2,\mu_2]} \subset A^{[\lambda_1,\mu_1]}$.

Let A be an IVG of a group G. Then for each $[\lambda, \mu] \in D(I)$ with $A(e) \geq [t, s]$, i.e., $A^L(e) \geq t$ and $A^U(e) \geq s$, the level subset $A^{[\lambda,\mu]}$ is a subgroup of G. If Im A = $\{[t_0, s_0], [t_1, s_1], \cdots, [t_n, s_n]\}$, the family of level subgroups $\{A^{[t_i,s_i]} : 0 \leq i \leq n\}$ constitutes the complete list of level subgroups of A. If the image set of the IVG A of a finite group G consists of $\{[t_0, s_0], [t_1, s_1], \cdots, [t_n, s_n]\}$, where $t_0 > t_1 > \cdots > t_n$ and $s_0 > s_1 > \cdots > s_n$, then, by Results 1.D and 1.E, the level subgroups of A form a chain:

$$A^{[t_0,s_0]} \subset A^{[t_1,s_1]} \subset \cdots \subset A^{[t_n,s_n]} = G,$$

where $A(e) = [t_0, s_0]$.

Notation. $N \lhd G$ denotes that N is a normal subgroup of a group G.

2. Interval-valued fuzzy normal subgroups and interval-valued fuzzy cosets

Lemma 2.1. If A is an IVGP of a finite group G, then A is an IVG of G.

Proof. Let $x \in G$. Since G is finite, x has finite order, say n. Then $x^n = e$, where e is the identity of G. Thus $x^{-1} = x^{n-1}$. Since A is an IVGP of G,
$A^L(x^{-1}) = A^L(x^{n-1}) = A^L(x^{n-2}x) \geq A^L(x)$
and
$A^U(x^{-1}) = A^U(x^{n-1}) = A^U(x^{n-2}x) \geq A^U(x)$.
Hence A is an IVG of G. □

Lemma 2.2. Let A be an IVG of a group G and let $x \in G$. Then $A(xy) = A(y)$, for each $y \in G$ if and only if $A(x) = A(e)$.

Proof. (⇒): Suppose $A(xy) = A(y)$ for each $y \in G$. Then clearly $A(x) = A(e)$.
(⇐): Suppose $A(x) = A(e)$. Then, by Result 1.B(b), $A^L(y) \leq A^L(x)$ and $A^U(y) \leq A^U(x)$ for each $y \in G$. Since A is an IVG of G, Then $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$. Thus $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for each $y \in G$.
On the other hand, by the hypothesis and Result 1.B(b),
$A^L(y) = A^L(x^{-1}xy) \geq A^L(x) \wedge A^L(xy)$ and $A^U(y) = A^U(x^{-1}xy) \geq A^U(x) \wedge A^U(xy)$.
Since $A^L(x) \geq A^L(y)$ for each $y \in G$, $A^L(x) \wedge A^L(xy) = A^L(xy)$ and $A^U(x) \wedge A^L(xy) = A^U(xy)$. So $A^L(y) \geq A^L(xy)$ and $A^U(y) \geq A^U(xy)$ for each $y \in G$. Hence $A(xy) = A(y)$ for each $y \in G$. □

Remark 2.3. It is easy to see that if $A(x) = A(e)$, then $A(xy) = A(yx)$ for each $y \in G$.

Definition 2.4. Let A be an IVS of a group G and let $x \in G$. We define two mappings $Ax, xA : G \to D(I)$ as follows, respectively : For each $g \in G$, $Ax(g) = A(gx^{-1})$ and $xA(g) = A(x^{-1}g)$. Then Ax[resp, xA] is called the *interval-valued fuzzy right* [resp.*left*] *coset* of G determined by x and A.

Remark 2.5. Definition 2.4 extends in a natural way to usual definition of a "coset" of a group. This is seen as follows: Let H be a subgroup of a group G and let $A = [\chi_H, \chi_H]$, where χ_H is the characteristic function of H. Let $x, y \in G$. Then $Ax = [\chi_H, \chi_H]$.
Suppose $g \in H$. Then

$$\begin{aligned} Ax(gx) &= [\chi_{H_x}(gx), \chi_{H_x}(gx)] \\ &= [\chi_H(gxx^{-1}), \chi_H(gxx^{-1})] \\ &= [\chi_H(g), \chi_H(g)] \\ &= [1, 1]. \end{aligned}$$

Suppose $g \notin H$. Then

$$\begin{aligned} Ax(gx) &= [\chi_{H_x}(gx), \chi_{H_x}(gx)] \\ &= [\chi_H(gxx^{-1}), \chi_H(gxx^{-1})] \\ &= [\chi_H(g), \chi_H(g)] \\ &= [0, 0]. \end{aligned}$$

So $Ax = [\chi_{H_x}, \chi_{H_x}]$.

The following is the immediate result of Definition 2.4.

Proposition 2.6. Let A be an IVG of a group G. Then
(a) $(xy)A = x(yA)$.
(b) $A(xy) = (Ax)y$.
(c) $xA = A$ if $A(x) = [1, 1]$.

We know that any two left[resp. right] cosets of a subgroup H of a group G are equal or disjoint. However this fact is not valid in the interval-valued fuzzy case as shown in the following example.

Example 2.7. Let $G = \{e, a, b, c, d\}$ be the Klein's four group and let A be the IVG of G defined by: $A(a) = [1, 1], A(b) = [t_1, t_1], A(c) = A(d) = [t_2, t_2]$, where $0 < t_2 \leq t_1 < 1$. Then $bA \neq cA$.

Definition 2.8 [6]. Let $A \in$ IVG(G). Then A is called an *interval-valued fuzzy normal subgroup*(in short, *IVNG*) of G if $A(xy) = A(yx)$, for any $x, y \in G$.

We will denote the set of all IVNGs of a group G as IVNG(G).

The following is the immediate result of Definitions 2.4 and 2.8.

Theorem 2.9. Let A be an IVG of a group G. Then the followings are equivalent:

(a) $A^L(xyx^{-1}) \geq A^L(y)$ and $A^U(xyx^{-1}) \geq A^U(y)$ for any $x, y \in G$.

(b) $A(xyx^{-1}) = A(y)$ for any $x, y \in G$.

(c) $A \in$ IVNG(G).

(d) $xA = Ax, \forall x \in G$.

(e) $xAx^{-1} = A, \forall x \in G$.

Remark 2.10. Let G be a group.

(a) If A is a fuzzy normal subgroup of G, then $[A, A] \in$ IVNG(G).

(b) If $A = [A^L, A^U] \in$ IVNG(G), then A^L and A^U are fuzzy normal subgroups of G.

Let G be a group and $a, b \in G$. We say that a is *conjugate* to b if there exists $x \in G$ such that $b = x^{-1}ax$. It is well-known that conjugacy is an equivalence relation on G. The equivalence classes in G under the relation of conjugacy are called *conjugate classes*[4].

Theorem 2.11. Let A be an IVG of a group G. Then $A \in$ IVNG(G) if and only if A is constant on the conjugate classes of G.

Proof. $(\Rightarrow)$: Suppose $A \in$ IVNG(G) and let $x, y \in G$. Then $A(y^{-1}xy) = A(xyy^{-1}) = A(x)$. Hence A is constant on the conjugate classes.

$(\Leftarrow)$: Suppose the necessary condition holds and let $x, y \in G$. Then $A(xy) = A(xyxx^{-1})$
$= A(x(yx)x^{-1}) = A(yx)$. Hence $A \in$ IVNG(G). □

Let G be a group and $x, y \in G$. Then the element $x^{-1}y^{-1}xy$ is usually denoted by $x, y]$ and called the *commutator* of x and y. It is clear that if x and y commute with each other, then clearly $[x, y] = e$. Let H and K be two subgroups of a group G. Then the subgroup $[H, K]$ is defined as the subgroup generated by the elements $\{[x, y] : x \in H, y \in K\}$. It is well-known that $N \lhd G$ if and only if $[N, G] \leq N$.

The following is the generalization of the above result using interval-valued fuzzy sets.

Theorem 2.12. Let A be an IVG of a group G. Then $A \in$ IVNG(G) if and only if $A^L([x, y]) \geq A^L(x)$ and $A^U([x, y]) \geq A^U(x)$ for any $x, y \in G$.

Proof. $(\Rightarrow)$: Suppose $A \in$ IVNG(G) and let $x, y \in G$.

Then

$$\begin{aligned}A^L([x,y]) =& A^L(x^{-1}y^{-1}xy)\\ =& A^L(y^{-1}xyx^{-1}) \text{ (By the hypothesis)}\\ \geq& A^L(y^{-1}xy) \wedge A^L(x^{-1})\\ & \text{(Since } A \in \text{IVG(G))}\\ =& A^L(x) \wedge A^L(x)\\ & \text{(By Theorem 2.9 and Result 1.B(a))}\\ =& A^L(x).\end{aligned}$$

By the similar arguments, we have that $A^U([x,y]) \geq A^U(x)$. Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold and let $x, z \in G$. Then

$$\begin{aligned}A^L(x^{-1}zx) &= A^L(zz^{-1}x^{-1}zx)\\ &\geq A^L(z) \wedge A^L([z,x]) \text{ (Since } A \in \text{IVG(G))}\\ &\geq A^L(z) \wedge A^L(z) \text{ (By the hypothesis)}\\ &= A^L(z).\end{aligned}$$

By the similar arguments, we have that $A^U(x^{-1}zx) \geq A^U(z)$. On the other hand,

$$\begin{aligned}A^L(z) &= A^L(xx^{-1}zxx^{-1})\\ &\geq A^L(x) \wedge A^L(x^{-1}zx) \wedge A^L(x^{-1})\\ & \quad \text{(Since } A \in \text{IVG(G))}\\ &= A^L(x) \wedge A^L(x^{-1}zx). \text{ (By Result 1.B(a))}\end{aligned}$$

By the similar arguments, we have that $A^U(z) \geq A^U(x) \wedge A^U(x^{-1}zx)$.

Case(i): Suppose $A^L(x) \wedge A^L(x^{-1}zx) = A^L(x)$ and $A^U(x) \wedge A^U(x^1zx) = A^U(x)$. Then $A^L(z) \geq A^L(x)$ and $A^U(z) \geq A^U(x)$ for any $x, z \in G$. Thus A is a constant mapping. So $A(xy) = A(yx)$ for any $x, z \in G$, i.e., $A \in$ IVNG(G).

Case(ii): Suppose $A^L(x) \wedge A^L(x^{-1}zx) = A^L(x^{-1}zx)$ and $A^U(x) \wedge A^U(x^{-1}zx) = A^U(x^{-1}zx)$. Then $A^L(z) \geq A^L(x^{-1}zx)$ and $A^U(z) \geq A^U(x^{-1}zx)$ for any $x, z \in G$, i.e., $A(x^{-1}zx) = A(z)$ for any $x, z \in G$. So A is constant on the conjugate classes. By Theorem 2.11, $A \in$ IVNG(G). Hence, in either cases, $A \in$ IVNG(G). This completes the proof. □

Proposition 2.13. Let A be an IVNG of a group G and let $[\lambda, \mu] \in D(I)$ such that $\lambda \leq A^L(e), \mu \leq A^U(e)$, where e denotes the identity of G. Then $A^{[\lambda,\mu]} \lhd G$.

Proof. By Result 1.D, $A^{[\lambda,\mu]}$ is a subgroup of G. Let $x \in A^{[\lambda,\mu]}$ and let $z \in G$. Since $A \in$ IVNG(G), by Proposition 2.9(b), $A(z^{-1}xz) = A(x)$. Since $x \in A^{[\lambda,\mu]}$, $A^L(x) \geq \lambda$ and $A^U(x) \geq \mu$. Thus $A^L(z^{-1}xz) \geq \lambda$ and $A^U(z^{-1}xz) \geq \mu$. So $z^{-1}xz \in A^{[\lambda,\mu]}$. Hence $A^{[\lambda,\mu]} \lhd G$. □

Let A be an IVNG of a finite group G with ImA= $\{[t_0, s_0], [t_1, s_1], \cdots, [t_r, s_r]\}$, where $t_0 > t_1 > \cdots > t_r$ and $s_0 > s_1 > \cdots > s_r$. Then it follows from Theorem 2.7 that the level subgroups of A form a chain of normal subgroups:

$$A^{[t_0,s_0]} \subset A^{[t_1,s_1]} \subset \cdots, A^{[t_r,s_r]} = G. \qquad (2.1)$$

The following is the immediate result of Proposition 2.13.

Corollary 2.13 [6, Proposition 5.4]. Let A be an IVNG of a group G with identity e. Then $G_A \lhd G$, where $G_A = \{x \in G : A(x) = A(e)\}$.

The following is the converse of Proposition 2.13.

Proposition 2.14. If A is an IVG of a finite group G such that all the level subgroups of A are normal in G, then $A \in$ IVNG(G).

Proof. Let Im A = $\{[t_0, s_0], [t_1, s_1], \cdots, [t_r, s_r]\}$, where $t_0 > t_1 > \cdots > t_r$ and $s_0 > s_1 > \cdots > s_r$. Then the family $\{A^{[t_i,s_i]} : 0 \leq i \leq r\}$ is the complete set of level subgroups of G. By the hypothesis, $A^{[t_i,s_i]} \lhd G$ for each $0 \leq i \leq r$. From the definition of the level subgroup, it is clear that $A^{[t_i,s_i]} \setminus A^{[t_{i-1},s_{i-1}]} = \{x \in G : A(x) = [t_i, s_i]\}$. Since a normal subgroup of a group is a complete union of conjugate classes, it follows that in the given chain (2.1) of normal subgroups, each $A^{[t_i,s_i]} \setminus A^{[t_{i-1},s_{i-1}]}$ is a union of some conjugate classes. Since A is constant on $A^{[t_i,s_i]} \setminus A^{[t_{i-1},s_{i-1}]}$, it follows that A must be constant on each conjugate class of G. Hence, by Theorem 2.11, $A \in$ IVNG(G). □

Example 2.15. Let G be the group of all symmetries of a square. Then G is a group of order 8 generated by a rotation through $\pi/2$ and a reflection along a diagonal of the square. Let us denote the elements of G by $\{1, 2, 3, 4, 5, 6, 7, 8\}$, where 1 is the identity, 2 is rotation through $\pi/2$ and 5 is a reflection along a diagonal: the multiplication table of G is as shown in Table 1.

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7
5	5	8	7	6	1	4	3	2
6	6	5	8	7	2	1	4	3
7	7	6	5	8	3	2	1	4
8	8	7	6	5	4	3	2	1

Table 1.

We can easily see that the conjugate classes of G are $\{1\}, \{3\}, \{5,7\}, \{6,8\}, \{2,4\}$.
Let $H = \{1,3\}$ and let $K = \{1,2,3,4\}$. Then clearly, $H \lhd G$ and $K \lhd G$(in fact, H is the center of G). Thus we have a chain of normal subgroups given by

$$\{1\} \subset H \subset K \subset G. \tag{2.2}$$

Now we will construct an IVG of G whose level subgroups are precisely the members of the chain (2.2). Let $[t_i, s_i] \in D(I), 0 \leq i \leq 3$ such that $t_0 > t_1 > t_2 > t_3$ and $s_0 > s_1 > s_2 > s_3$. Define a mapping $A : G \to D(I)$ as follows:
$A(1) = [t_0, s_0], A(H \setminus \{1\}) = [t_1, s_1], A(K \setminus H) = [t_2, s_2], A(G \setminus K) = [t_3, s_3]$. From the definition of A, it is clear that $A(x) = A(x^{-1})$ for each $x \in G$. Also, we can easily check that for any $x, y \in G$,

$$A^L(xy) \geq A^L(x) \wedge A^L(y) \text{ and } A^U(xy) \geq A^U(x) \wedge A^U(y).$$

Furthermore, it is clear that A is constant on the conjugate classes. Hence, by Theorem 2.11, $A \in$ IVNG(G). □

The following can be easily proved and the proof is omitted.

Lemma 2.16. Let A be an IVG of a group and let $x \in G$. Then $A(x) = [\lambda, \mu]$ if and only if $x \in A^{[\lambda,\mu]}$ and $x \notin A^{[t,s]}$ for each $[t, s] \in D(I)$ such that $t > \lambda$ and $s > \mu$.

It is well-known that if N is a normal subgroup of a group G, then $xy \in N$ if and only if $yx \in N$ for any $x, y \in G$.

The following result is the generalization of Proposition 2.14.

Proposition 2.17. Let A be an IVG of a group G. If $A^{[\lambda,\mu]}, [\lambda, \mu] \in$ Im A, is a normal subgroup of G, then $A \in$ IVNG(G).

Proof. For any $x, y \in G$, let $A(x, y) = [\lambda, \mu]$ and let $A(xy) = [t, s]$ be such that $t > \lambda$ and $s > \mu$. Then, by Lemma 2.16, $xy \in A^{[\lambda,\mu]}$ and $xy \notin A^{[t,s]}$. Thus $yx \in A^{[\lambda,\mu]}$ and $yx \notin A^{[t,s]}$. So $A(yx) = [\lambda, \mu]$, i.e., $A(xy) = A(yx)$. Hence $A \in$ IVNG(G). □

3. Homomorphisms

Definition 3.1 [9]. Let $f : X \to Y$ be a mapping, let $A = [A^L, A^U] \in D(I)^X$ and let $B = [B^L, B^U] \in D(I)^Y$. Then

(a) the *image* of A under f, denoted by $f(A)$, is an IVS in Y defined as follows: For each $y \in Y$,

$$f(A^L)(y) = \begin{cases} \bigvee_{y=f(x)} A^L(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$f(A^U)(y) = \begin{cases} \bigvee_{y=f(x)} A^U(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

(b) the *preimage* of B under f, denoted by $f^{-1}(B)$, is an IVS in Y defined as follows: For each $y \in Y$,

$$f^{-1}(B^L)(y) = (B^L \circ f)(x) = B^L(f(x))$$

and

$$f^{-1}(B^U)(y) = (B^U \circ f)(x) = B^U(f(x)).$$

It can be easily seen that $f(A) = [f(A^L), f(A^U)]$ and $f^{-1}(B) = [f^{-1}(B^L), f^{-1}(B^U)]$.

Result 3.A [9, Theorem 2]. Let $f : X \to Y$ be a mapping and $g : Y \to Z$ be a mapping. Then
(a) $f^{-1}(B^c) = [f^{-1}(B)]^c$, $\forall B \in D(I)^Y$.
(b) $[f(A)]^c \subset f(A^c)$, $\forall A \in D(I)^Y$.
(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in D(I)^Y$.
(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where $A_1, A_2 \in D(I)^X$.
(e) $f(f^{-1}(B)) \subset B, \forall B \in D(I)^Y$.
(f) $A \subset f(f^{-1}(A)), \forall A \in D(I)^Y$.
(g) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)), \forall C \in D(I)^Z$.
(h) $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.
(h)′ $f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha) = \bigcap_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.

Proposition 3.2. Let $f : X \to Y$ be a groupoid homomorphism. If $A \in$ IVGP(X), then $f(A) \in$ IVGP(Y).

Proof. For each $y \in Y$, let $X_y = f^{-1}(y)$. Since f is a homomorphism, it is clear that
$X_y X_{y'} \subset X_{yy'}$ for any $y, y' \in Y$. (*)
Let $y, y' \in Y$.
Case (i): Suppose $yy' \notin f(A)$. Then clearly $f(A)(yy') = [0, 0]$. Since $yy' \notin f(X), X_{yy'} = \emptyset$. By (*), $X_y = \emptyset$ or $X_{y'} = \emptyset$. Thus $f(A)(y) = [0, 0]$ or $f(A)(y') = [0, 0]$. So

$$\begin{aligned} f(A)(yy') &= [0, 0] \\ &= [f(A)^L(y) \wedge f(A)^L(y'), \\ &\quad f(A)^U(y) \wedge f(A)^U(y')]. \end{aligned}$$

Case (ii): Suppose $yy' \in f(X)$. Then $X_{yy'} \neq \emptyset$. If $X_y = \emptyset$ and $X_{y'} = \emptyset$, then $f(A)(y) = [0,0]$ and $f(A)(y') = [0,0]$. Thus

$$f(A)^L(yy') \geq f(A)^L(y) \wedge f(A)^L(y')$$

and

$$f(A)^U(yy') \geq f(A)^U(y) \wedge f(A)^U(y').$$

If $X_y \neq \emptyset$ or $X'_y \neq \emptyset$, then, by (*),

$$\begin{aligned} f(A)^L(yy') &= \bigvee_{z \in X_{yy'}} A^L(z) \geq \bigvee_{z \in X_y X_{y'}} A^L(z) \\ &= \bigvee_{x \in X_y, x' \in X_{y'}} A^L(xx') \\ &\geq \bigvee_{x \in X_y, x' \in X_{y'}} (A^L(x) \wedge A^L(x')) \\ &\quad \text{(Since } A \in \text{IVGP(X))} \\ &= (\bigvee_{x \in X_y} A^L(x)) \wedge (\bigvee_{x' \in X_{y'}} A^L(x')) \\ &= f(A)^L(y) \wedge f(A)^L(y'). \end{aligned}$$

By the similar arguments, we have that $f(A)^U(yy') \geq f(A)^U(y) \wedge f(A)^U(y')$. Consequently, $f(A)^L(yy') \geq f(A)^L(y) \wedge f(A)^L(y')$ and $f(A)^U(yy') \geq f(A)^U(y) \wedge f(A)^U(y')$. Hence $f(A) \in$ IVGP(Y). □

Definition 3.3 [1, 6]. Let A be an IVS in a groupoid G. Then A is said to have the *sup-property* if for any $T \in P(G)$, there exists a $t_0 \in T$ such that $A(t_0) = \cup_{t \in T} A(t)$, i.e., $A^L(t_0) = \bigvee_{t \in T} A^L(t)$ and $A^U(t_0) = \bigvee_{t \in T} A^U(t)$, where $P(G)$ denotes the power set of G.

Result 3.B [6, Proposition 4.11]. Let $f : G \to G'$ be a group homomorphism, let $A \in$ IVG(G) and let $B \in$ IVG(G'). Then the followings hold:
(a) If A has the sup property, then $f(A) \in$ IVG(G').
(b) $f^{-1}(B) \in$ IVG(G).

Proposition 3.4. Let $f : X \to Y$ be a group[resp. ring, algebra and field] homomorphism. If $A \in$ IVG(X)[resp. IVR(X), IVA(X) and IVF(X)], then $f(A) \in$ IVG(Y)[resp. IVR(Y), IVA(Y) and IVF(Y)], where IVG(X)[resp. IVR(X), IVA(X) and IVF(X)] denotes the set of all interval-valued fuzzy subgroups[resp. subrings, subalgebras and subfields] of a group[resp. ring, algebra and field] X.

Proof. Suppose $f : X \to Y$ is a group homomorphism and let $A \in$ IVG(X). Then, we need only to show that $f(A)^L(y^{-1}) \geq f(A)^L(y)$ and $f(A)^U(y^{-1}) \geq f(A)^U(y)$ for each $y \in Y$. Let $y \in Y$.

Case (i): Suppose $y^{-1} \notin f(X)$. Then $y \notin f(X)$. Thus $f(A)(y^{-1}) = [0,0] = f(A)(y)$.

Case (ii): Suppose $y^{-1} \in f(X)$. Then $y \in f(X)$. Thus

$$\begin{aligned} f(A)^L(y^{-1}) &= \bigvee_{t^{-1} \in f^{-1}(y^{-1})} A^L(t^{-1}) \\ &\geq \bigvee_{t \in f^{-1}(y)} A^L(t) = f(A)^L(y) \end{aligned}$$

and

$$\begin{aligned} f(A)^U(y^{-1}) &= \bigvee_{t^{-1} \in f^{-1}(y^{-1})} A^U(t^{-1}) \\ &\geq \bigvee_{t \in f^{-1}(y)} A^U(t) = f(A)^U(y). \end{aligned}$$

Hence $f(A) \in$ IVG(Y). The proofs of the rest are omitted. This completes the proof. □

Another Proof : Let $[\lambda, \mu] \in \text{Im } f(A)$. Then there exists a $y \in Y$ such that

$$f(A)(y) = [\bigvee_{x \in f^{-1}(y)} A^L(x), \bigvee_{x \in f^{-1}(y)} A^L(x)] = [\lambda, \mu].$$

Since $A \in$ IVG(X), by Result 1.B(b), $\lambda \leq A^L(e)$ and $\mu \leq A^U(e)$.

Case (i): Suppose $[\lambda, \mu] = [0,0]$. Then clearly $(f(A))^{[\lambda,\mu]} = Y$. So, by Result 1.D, $f(A) \in$ IVG(Y).

Case (ii): Suppose $\lambda > 0$. Then
$z \in (f(A))^{[\lambda,\mu]} \Leftrightarrow f(A)^L(z) \geq \lambda$ and $f(A)^U(z) \geq \mu$ $\Leftrightarrow \bigvee_{x \in f^{-1}(z)} A^L(x) \geq \lambda$ and $\bigvee_{x \in f^{-1}(z)} A^U(x) \geq \mu \Leftrightarrow$ there exists an $x \in X$ such that $f(x) = z, A^L(x) \geq \lambda$ and $A^U(x) \geq \mu \Leftrightarrow z \in (f(A^{[\lambda,\mu]}))$.
Thus $(f(A))^{[\lambda,\mu]} = f(A^{[\lambda,\mu]})$. Since f is a homomorphism and $A^{[\lambda,\mu]}$ is a subgroup of X, $f(A^{[\lambda,\mu]})$ is a subgroup of Y. So, by Result 1.D, $f(A) \in$ IVG(X). Hence, in all, $f(A) \in$ IVG(X). □

Remark 3.5. In Result 3.B, A has the sup property but in Proposition 3.4, there is no restriction on A.

Proposition 3.6. Let $f : G \to G'$ be a group homomorphism, let $A \in$ IVNG(G) and let $B \in$ IVNG (G'). Then the followings hold:
(a) If f is surjective, then $f(A) \in$ IVNG (G').
(b) $f^{-1}(B) \in$ IVNG(G).

Proof. (a) By Proposition 3.4, $f(A) \in$ IVG (G'). Let $[\lambda, \mu] \in \text{Im } f(A)$. From the process of the another proof of Proposition 3.4, it is clear that $\lambda \leq A^L(e), \mu \leq A^U(e)$ and $(f(A))^{[\lambda,\mu]} = f(A^{[\lambda,\mu]})$. Since $A \in$ IVNG(G), by Proposition 2.13, $A^{[\lambda,\mu]} \lhd G$. Since f is an epimorphism, $(f(A))^{[\lambda,\mu]} = f(A^{[\lambda,\mu]}) \lhd G'$. Hence, by Proposition 2.17, $f(A) \in$ IVNG (G').

(b) By Result 3.B(b), $f^{-1}(B) \in$ IVG(G). Let $x, y \in G$. Then

$$\begin{aligned} f^{-1}(B)(xy) &= [f^{-1}(B^L)(xy), f^{1}(B^U)(xy)] \\ &= [B^L(f(xy)), B^U(f(xy))] \\ &= [B^L(f(x)f(y)), B^U(f(x)f(y))] \\ &\quad \text{(Since f is a homomorphism)} \\ &= [B^L(f(y)f(x)), B^U(f(y)f(x))] \\ &\quad \text{(Since } B \in \text{IVNG}(f(G))\text{)} \\ &= [B^L(f(yx)), B^U(f(yx))] \\ &\quad \text{(Since f is a homomorphism)} \\ &= [f^{-1}(B^L)(yx), f^{-1}(B^U)(yx)] \\ &= f^{-1}(B)(yx). \end{aligned}$$

Hence $f^{-1}(B) \in$ IVNG(G). □

Result 3.C [6, Propositions 4.6 and 5.4]. Let G be a group.

(a) If $A \in$IVG(G), then G_A is a subgroup of G.

(b) If $A \in$ IVNG(G), then $G_A \lhd G$, where $G_A = x \in G : A(x) = A(e)$.

Theorem 3.7. Let A be an IVNG of a group G with identity e. We define a mapping $\hat{A} : G/G_A \to D(I)$ as follows: For each $x \in G$, $\hat{A}(G_A x) = A(x)$. Then $\hat{A} \in$ IVNG (G/G_A). Conversely, if $N \lhd G$ and $\hat{B} \in$ IVNG(G/N) such that $\hat{B}(N_g) = \hat{B}(N)$ only when $g \in N$, then there exists an $A \in$ IVNG(G) such that $G_A = N$ and $\hat{A} = \hat{B}$.

Proof. It is clear that $G_A \lhd G$ from Result 3.C(b). Moreover $\hat{A} \in D(I)^{G/G_A}$ from the definition of $\hat{A}$. Suppose $G_A x = G_A y$ for some $x, y \in G$. Then, by Corollary 2.13, $xy^{-1} \in G_A$. Thus $A(xy^{-1}) = A(e)$. By Result 1.C, $A(x) = A(y)$. So $\hat{A}(G_A x) = \hat{A}(G_A y)$. Hence $\hat{A}$ is well-defined. Furthermore, it is easy to see that $\hat{A} \in$ IVG(G/G_A). Let $x, y \in G$. Then

$$\begin{aligned} \hat{A}(G_A x G_A y) &= \hat{A}(G_A xy) \\ &= A(xy) \\ &= A(yx) \ \text{(Since } A \in \text{IVNG(G))} \\ &= \hat{A}(G_A y G_A x). \end{aligned}$$

Hence $\hat{A} \in$ IVNG(G/G_A).

Now let $N \lhd G$ and let $\hat{B} \in$ IVNG(G/G_A) such that $\hat{B}(N_g) = \hat{B}(N)$ only when $g \in N$. We define a mapping $A : G \to D(I)$ as follows: For each $x \in G, A(x) = \hat{B}(Nx)$. Then we can easily see that A is well-defined and $A \in$ IVG(G). Let $x, y \in G$. Then

$$\begin{aligned} A(y^{-1}xy)) &= \hat{B}(Ny^{-1}xy) \\ &= \hat{B}(Ny^{-1}NxNy) \\ &= \hat{B}(Nx) \ \text{(Since } \hat{B} \in \text{IVNG}(G/N)\text{)} \\ &= A(x). \end{aligned}$$

Thus A is constant on the conjugate classes of G. So, by Theorem 2.11, $A \in$ IVNG(G).

Now let $g \in N$. Then $A(g) = \hat{B}(N_g) = \hat{B}(N) = A(e)$. Thus $g \in G_A$. So $N \subset G_A$. Let $x \in G_A$. Then $A(x) = A(e)$. Thus $\hat{B}(Nx) = \hat{B}(N)$. So $x \in N$, i.e., $G_A \subset N$. Hence $N = G_A$. Furthermore, $\hat{A} = \hat{B}$. This completes the proof. □

4. Interval-valued fuzzy Lagrange's Theorem

Let A be an IVS in a group G and for each $x \in G$, ${}_xf : G \to G$[resp. $f_x : G \to G$] be a mapping defined as follows, respectively: For each $g \in G$,

${}_xf(g) = xg$ [resp. $f_x(g) = gx$].

Proposition 4.1. Let A be an IVG of a group G. Then ${}_xf(A) = xA$ [resp. $f_x(A) = Ax$] for each $x \in G$.

Proof. Let $g \in G$. Then

$$\begin{aligned} f_x(A)^L(g) &= \bigvee_{g' \in f_x^{-1}(g)} A^L(g') \\ &= \bigvee_{g'x=g} A^L(g') = A^L(gx^{-1}) \end{aligned}$$

and

$$\begin{aligned} f_x(A)^U(g) &= \bigvee_{g' \in f_x^{-1}(g)} A^U(g') \\ &= \bigvee_{g'x=g} A^U(g') = A^U(gx^{-1}). \end{aligned}$$

Hence, $f_x(A) = Ax$. Similarly, we can see that ${}_xf(A) = xA$. □

Theorem 4.2. Let A be an IVG of a group G and let $g_1, g_2 \in G$. Then $g_1A = g_2A$[resp. $Ag_1 = Ag_2$] if and only if $A(g_1^{-1}g_2) = A(g_2^{-1}g_1) = A(e)$[resp. $A(g_1g_2^{-1}) = A(g_2g_1^{-1}) = A(e)$].

Proof.(⇒): Suppose $g_1A = g_2A$. Then $g_1A(g_1) = g_2A((g_1)$ and $g_1A(g_2) = g_2A((g_2)$. $A(g_2^{-1}g_1) = A(e)$ and $A(g_1^{-1}g_2) = A(e)$. Hence $A(g_2^{-1}g_1) = A(g_1^{-1}g_2) = A(e)$.

(⇐): Suppose $A(g_1^{-1}g_2) = A(g_2^{-1}g_1) = A(e)$. let $x \in G$. Then $g_1A(x) = A(g_1^{-1}x) = A(g_1^{-1}g_2g_2^{-1}x)$. Since A is a IVG(G),

$$\begin{aligned} A^L(g_1^{-1}x) &= A^L(g_1^{-1}xg_2g_2^{-1}x) \\ &= A^L(g_1^{-1}g_2) \wedge A^L(g_2^{-1}x) \\ &= A^L(e) \wedge A^L(g_2^{-1}x) \\ &= A^L(g_2^{-1}x). \ \text{(By Result 1.B(b))} \end{aligned}$$

By the similar arguments, we have that $A^U(g_1^{-1}x) \geq A^U(g_2^{-1}x)$. Thus $g_2A \subset g_1A$. Similarly, we have that $g_1A \subset g_2A$. Hence $g_1A = g_2A$. This completes the proof. □

Proposition 4.3. Let A be an IVG of a group G. If $Ag_1 = Ag_2$ for any $g_1, g_2 \in G$, then $A(g_1) = A(g_2)$.

Proof. Suppose $Ag_1 = Ag_2$ for any $g_1, g_2 \in G$. Then $Ag_1(g_2) = Ag_2(g_2)$. Thus $A(g_2g_1^{-1}) = A(e)$. Hence, by Result 1.C, $A(g_1) = A(g_2)$. □

Proposition 4.4. Let A be an IVG of a group G. If $A^{[\lambda,\mu]}x = A^{[\lambda,\mu]}y$ for any $x, y \in G \setminus A^{[\lambda,\mu]}$ and each $[\lambda, \mu] \in D(I)$, then $A(x) = A(y)$.

Proof. Suppose $A^{[\lambda,\mu]}x = A^{]\lambda,\mu]}y$ for any $x, y \in G \setminus A^{[\lambda,\mu]}$ and each $[\lambda, \mu] \in D(I)$. Then $yx^{-1} \in A^{[\lambda,\mu]}$. Thus $A^L(yx^{-1}) \geq \lambda$ and $A^U(yx^{-1}) \geq \mu$. Since $x \in G \setminus A^{[\lambda,\mu]}$, $A^L(x) < \lambda$ and $A^U(x) < \mu$. On the other hand,

$$A^L(y) = A^L(yx^{-1}x) \geq A^L(yx^{-1}) \wedge A^L(x)$$

and

$$A^U(y) = A^U(yx^{-1}x) \geq A^U(yx^{-1}) \wedge A^U(x).$$

Thus $A^L(y) \geq A^L(x)$ and $A^U(y) \geq A^U(x)$. By the similar arguments, we have that $A^L(y) \leq A^L(x)$ and $A^U(y) \leq A^U(x)$. Hence $A(x) = A(y)$. □

Proposition 4.5. Let A be an IVNG of a group G and let $x \in G$. Then $Ax(xg) = Ax(gx) = A(g)$ for each $g \in G$.

Proof. Let $g \in G$. Then

$$\begin{aligned} Ax(xg) &= [A_x^L(xg), A_x^U(xg)] \\ &= [A_x^L(xgx^{-1}x), A_x^U(xgx^{-1}x)] \\ &= [A_x^L(xgx^{-1}xx^{-1}), A_x^U(xgx^{-1}xx^{-1})] \\ &\quad \text{(By the definition of } Ax) \\ &= [A_x^L(xgx^{-1}), A_x^U(xgx^{-1})] \\ &= [A_x^L(g), A_x^U(g)] \ \text{(By Theorem 2.11)} \\ &= A(g). \end{aligned}$$

Similarly, we have that $Ax(gx) = A(g)$. This completes the proof. □

Remark 4.6. Proposition 4.5 is analogous to the result in group theory that if $N \lhd G$, then $Nx = xN$ for each $x \in G$.

If N is a normal subgroup of a group G, then the cosets of G with respect to N form a group(called the *quotient group* G/N). For an IVNG, we have the analogous result:

Proposition 4.7. Let A be an IVNG of a group G and let G/A be the set of all the interval-valued fuzzy cosets of A. We define an operation$*$ on G/A as follows: For any $x, y \in G$, $Ax * Ay = Axy$. Then $(G/A, *)$ is a group. In this case, G/A is called the *interval-valued fuzzy quotient group induced* by A.

Proof. Let $x, y, x_0, y_0 \in G$ such that $Ax = Ax_0$ and $Ay = Ay_0$, and let $g \in G$. Then $Axy(g) = A(gy^{-1}x^{-1})$ and $Ax_0y_0(g) = A(gy_0^{-1}x_0^{-1})$. On the other hand,

$$\begin{aligned} A^L(gy^{-1}x^{-1}) &= A^L(gy_0^{-1}y_0y^{-1}x^{-1}) \\ &= A^L(gy_0^{-1}x_0^{-1}x_0y_0y^{-1}x^{-1}) \\ &\geq A^L(gy_0^{-1}x_0^{-1}) \wedge A^L(x_0y_0y^{-1}x^{-1}). \\ &\quad \text{(Since } A \in \text{IVG(G))} \qquad (4.1) \end{aligned}$$

By the similar arguments, we have that
$A^U(gy^{-1}x^{-1}) \geq A^U(gy_0^{-1}x_0^{-1}) \wedge A^U(x_0y_0y^{-1}x^{-1})$.(4.2)
Since $Ax - Ax_0$ and $Ay = Ay_0$, $A(gx^{-1}) = A(gx_0^{-1})$ and $A(gy^{-1}) = A(gy_0^{-1})$. In Particular,

$$\begin{aligned} A(x_0y_0y^{-1}x^{-1}) &= A(x_0y_0y^{-1}x_0^{-1}) \\ &= A(y_0y^{-1}) \ \text{(Since } A \in \text{IVNG(G))} \\ &= A(e). \end{aligned}$$

So $[A^L(x_0y_0y^{-1}x^{-1}), A^U(x_0y_0y^{-1}x^{-1})] = [A^L(e), A^U(e)]$. By Result 1.B(b), $A^L(e) \geq A^L(gy_0^{-1}x_0^{-1})$ and $A^U(e) \geq A^U(gy_0^{-1}x_0^{-1})$. Thus, by (4.1) and (4.2),
$A^L(gy^{-1}x^{-1}) \geq A^L(gy_0^{-1}x_0^{-1})$ and $A^U(gy^{-1}x^{-1}) \geq A^U(gy_0^{-1}x_0^{-1})$. By the similar arguments, we have that

$$A^L(gy_0^{-1}x_0^{-1}) \geq A^L(gy^{-1}x^{-1})$$

and

$$A^U(gy_0^{-1}x_0^{-1}) \geq A^L(gy^{-1}x^{-1}).$$

So $A(gy_0^{-1}x_0^{-1}) = A(gy^{-1}x^{-1})$, i.e., $Ax_0y_0(g) = Axy(g)$. Hence $*$ is well-defined. Furthermore, we can easily check that the followings are true:

(i) $*$ is associative.

(ii) Ax^{-1} is the inverse of Ax for each $x \in G$.

(iii) $Ae = A$ is the identity of G/A. Therefore $(G/A, *)$ is a group. This completes the proof □

Proposition 4.8. Let A be an IVNG of a group G. We define a mapping $\bar{A} : G/A \to D(I)$ as follows: For each $x \in G$, $\bar{A}(Ax) = Ax$. Then $\bar{A}$ is an IVG of G/A. In this case, $\bar{A}$ is called the *interval- valued fuzzy subquotient group determined by* A.

Proof. From the definition of $\bar{A}$, it is clear that $\bar{A} \in$

$D(I)^{G/A}$. Let $x, y \in G$. Then

$$\begin{aligned}\bar{A}^L(Ax * Ay) &= \bar{A}^L(Axy)\\ &= \bar{A}^L(xy)\\ &\geq A^L(x) \wedge A^L(y)\\ &= \bar{A}^L(Ax) \wedge \bar{A}^L(Ay).\end{aligned}$$

By the similar arguments, we have that $\bar{A}^U(Ax * Ay) \geq \bar{A}^U(Ax) \wedge \bar{A}^U(Ay)$. On the other hand,

$$\begin{aligned}\bar{A}^L((Ax)^{-1}) = \bar{A}^L(Ax^{-1}) = \bar{A}^L(x)^{-1})\\ \geq A^L(x) = \bar{A}^L(Ax)\end{aligned}$$

and

$$\begin{aligned}\bar{A}^U((Ax)^{-1}) = \bar{A}^U(Ax^{-1}) = \bar{A}^U(x)^{-1})\\ \geq A^U(x) = \bar{A}^U(Ax).\end{aligned}$$

Hence $\bar{A} \in$ IVG(G/A). □

Proposition 4.9. Let A be an IVNG of a group G. We define a mapping $\pi : G \to G/A$ as follows: For each $x \in G, \pi(x) = Ax$. Then π is a homomorphism with $Ker(\pi) = G_A$. In this case, π is called the *natural homomorphism*.

Proof. Let $x, y \in G$. Then $\pi(xy) = Axy = Ax * Ay = \pi(x) * \pi(y)$. So π is a homomorphism. Furthermore,

$$\begin{aligned}Ker(\pi) &= \{x \in G : \pi(x) = Ae\}\\ &= \{x \in G : A(x) = Ae\}\\ &= \{x \in G : Ax(x) = Ae(x)\}\\ &= \{x \in G : A(e) = A(x)\}\\ &= G_A.\end{aligned}$$

This completes the proof. □

Now we obtain for interval-valued fuzzy subgroups an analogous result of the "Fundamental Theorem of Homomorphism of Groups".

Proposition 4.10. Let $A \in$ IVNG(G). Then each interval-valued fuzzy(normal) subgroup of G/A corresponds in a natural way to an interval-valued fuzzy(normal) subgroup of G.

Proof. Let A^* be an interval-valued fuzzy subgroup of G/A. Define a mapping $B : G \to D(I)$ as follows: For each $x \in G, B(x) = A^*(Ax)$. By the definition of B, it is clear that $B \in D(I)^G$. Let $x, y \in G$. Then

$$\begin{aligned}B^L(xy) &= {A^*}^L(Axy)\\ &= {A^*}^L(Ax * Ay)\\ &\geq {A^*}^L(Ax) \wedge {A^*}^L(Ay) \quad (\text{Since } A^* \in \text{IVG(G/A)})\\ &= B^L(x) \wedge B^L(y).\end{aligned}$$

By the similar arguments, we have that $B^U(xy) \geq B^U(x) \wedge B^U(y)$. Since $A^* \in$ IVG(G/A), $A^*(Ax^{-1}) = A^*(Ax)$. Thus

$$\begin{aligned}B(x^{-1}) &= [B^L(x^{-1}), B^U(x^{-1})]\\ &= [{A^*}^L(Ax^{-1}), {A^*}^U(Ax^{-1})]\\ &= [{A^*}^L(Ax), {A^*}^U(Ax)]\\ &= [B^L(x) \wedge B^U(y)] = B(x).\end{aligned}$$

Hence $B \in$ IVG(G). It is easy to see that if B is an IVNG of G/A, then B is an IVNG of G. This completes the proof. □

Now we will obtain an interval-valued fuzzy analog of the famous "Lagrange's Theorem" for finite groups which is a basic result in group theory. Let A be an IVG of a finite group G. Then it clear that G/A is finite.

Definition 4.11. Let A be an IVG of a finite group G. Then the cardinality $\mid G/A \mid$ of G/A is called the *index* of A.

Theorem 4.12 (Interval-valued Fuzzy Lagrange's Theorem). Let A be an IVG of a finite group G. Then the index of A divides the order of G.

Proof. By Proposition 4.9, there is the natural homomorphism $\pi : G \to G/A$. Let H be the subgroup of G defined by $H = \{h \in G : Ah = Ae\}$, where e is the identity of G. Let $h \in H$. Then $Ah(g) = Ae(g)$ or $A(gh^{-1}) = A(g)$ for each $g \in G$. In particular, $A(h^{-1}) = A(e)$. Since A is an IVG of G, by Result 1.B(a), $A(h) = A(e)$. Thus $h \in G_A$. So $H \subset G_A$. Now let $h \in G_A$. Then $A(h) = A(e)$. Thus, by Result 1.B(a), $A(h^{-1}) = A(e)$. By Lemma 2.2, $A(gh^{-1}) = A(g)$ or $Ah(g) = Ae(g)$ for each $g \in G$. Thus $Ah = Ae$, i.e., $h \in H$. So $G_A \subset H$. Hence $H = G_A$.
Now decompose G as a disjoint union of the cosets of G with respect to H:

$$G = Hx_1 \cup Hx_2 \cup \cdots \cup Hx_k \tag{4.3}$$

where $hx_1 = H$. We show that corresponding to each coset Hx_i given in (4.3), there is an interval-valued fuzzy coset belonging to G/A, and further that this correspondence is injective. Consider any coset Hx_i. Let $h \in H$. Then $\pi(hx_i) = Ahx_i = Ah * Ax_i = Ae * Ax_i = Ax_i$.
Thus π maps each element of Hx_i into the interval-valued fuzzy coset Ax_i. Now we define a mapping $\bar{\pi} : \{Hx_i : 1 \leq i \leq k\} \to G/A$ as follows: For each $i \in \{1, 2, \cdots, K\}$,

$$\bar{\pi}(Hx_i) = Ax_i.$$

Then clearly, $\bar{\pi}$ is well-defined. Suppose $Ax_i = Ax_j$. Then $Ax_ix_j^{-1} = Ae$. Thus $x_ix_j^{-1} \in H$. So $Hx_i = Hx_j$. Hence $\bar{\pi}$ is injective. From the above discussion, it is clear that

$\mid G/A \mid = k$. Since k divides the order of G, $\mid G/A \mid$ also divides the order of G. This completes the proof. □

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