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Interval-Valued Fuzzy Ideals of a Ring

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Abstract

We introduce the notions of interval-valued fuzzy prime ideals, interval-valued fuzzy completely prime ideals and interval-valued fuzzy weakly completely prime ideals. And we give a characterization of interval-valued fuzzy ideals and establish relationships between interval-valued fuzzy completely prime ideals and interval-valued fuzzy weakly completely prime ideals.

Key Words: interval-valued fuzzy set, interval-valued fuzzy subring. interval-valued fuzzy ideal, interval-valued fuzzy prime ideal, interval-valued fuzzy completely prime ideal, interval-valued fuzzy weakly completely prime ideal.

1. Introduction and Preliminaries

In 1975, Zadeh[11] introduced the concept of interval-valued fuzzy sets as a generalization of fuzzy sets introduced by himself[10]. After then, Biswas[1] applied the notion of interval-valued fuzzy sets to group theory. Moreover, Gorzalczany[4] applied it to a method of inference in approximate reasoning, and Montal and Samanta[8] applied it to topology. Recently, Hur et al.[5] introduced the concept of an interval-valued fuzzy relations and obtained some of it's properties. Also, Choi et al.[3] applied it to topology in the sense of Šostak, Kang and Hur[7], and Kang[6] applied it to algebra.

In this paper, we introduce the notions of interval-valued fuzzy prime ideals, interval-valued fuzzy completely prime ideals and interval-valued fuzzy weakly completely prime ideals. And we give a characterization of interval-valued fuzzy ideals and establish relationships between interval-valued fuzzy completely prime ideals and interval-valued fuzzy weakly completely prime ideals.

Now, we will list some basic concepts and well-known results which are needed in the later sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0, 0]$, $\mathbf{1} = [1, 1]$, and $\boldsymbol{a}$=$[a, a]$ for every $a \in (0, 1)$. We also note that

(i) $(\forall M, N \in D(I))\ (M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))\ (M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$(See [8]).

Definition 1.1 [8, 11]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set* (in short, *IVS*) in X and is denoted by $A = [A^L, A^U]$. Thus for each $x \in X$, $A(x) = [A^L(x), A^U(x)]$, where $A^L(x)$[resp. $A^U(x)$] is called the *lower*[resp. *upper*] *end point* of x to A. For any $[a, b] \in D(I)$, the interval-valued fuzzy set A in X defined by $A(x) = [a, b]$ for each $x \in X$ is denoted by $\widetilde{[a, b]}$ and if $a = b$, then the IVS $\widetilde{[a, b]}$ is denoted by simply $\widetilde{a}$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVSs in X as $D(I)^X$. It is clear that set $A = [A^L, A^U] \in D(I)^X$ for each $A \in I^X$.

Definition 1.2 [8]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(a) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.
(b) $A = B$ iff $A \subset B$ and $B \subset A$.
(c) $A^c = [1 - A^U, 1 - A^L]$.
(d) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.
(d)′ $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha]$.
(e) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

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(e)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

Result 1.A [8, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then

(a) $\tilde{0} \subset A \subset \tilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma}(A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma}(A \cup A_\alpha)$.

(g) $(\tilde{0})^c = \tilde{1}$, $(\tilde{1})^c = \tilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A^c_\alpha$, $(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A^c_\alpha$.

Definition 1.3 [7]. Let A be an IVS in a set X and let $[\lambda, \mu] \in D(I)$. Then the set $A^{[\lambda,\mu]} = \{x \in X : A^L(x) \geq \lambda \text{ and } A^U(x) \geq \mu\}$ is called a $[\lambda, \mu]$-*level subset* of A.

Definition 1.4 [8]. Let $[\lambda, \mu] \in D(I)$. Then an *interval-valued fuzzy point*(in short, *IVP*) $x_{[\lambda,\mu]}$ of X is the IVS in X defined as follows : For each $y \in X$,

$$x_{[\lambda,\mu]}(y) = \begin{cases} [\lambda, \mu], & \text{if } y = x; \\ \tilde{0}, & \text{otherwise.} \end{cases}$$

In this case, x is called the *support* of $x_{[\lambda,\mu]}$ and, λ and μ are called the *value* and *nonvalue* of $x_{[\lambda,\mu]}$, respectively. In particular, if $\lambda = \mu$, then it is also denoted by $x_{\boldsymbol{\lambda}}$. An IVP x_M is said to *belong to* an IVS A in X, denoted by $x_M \in A$ if $M^L \leq A^L(x)$ and $M^U \leq A^U(x)$.

It is clear that $A = \cup_{x_M\in A} x_M$ and $x_M \in A$ if and only if $x_{M^L} \in A^L$ and $x_{M^U} \in A^U$, for each $A \in P(I)^X$.

We will denote the set of all IVPs in X as IVP(X).

The following is the immediate result of Definition 1.2 and 1.4.

Theorem 1.5. Let $A, B \in D(I)^X$. Then $A \subset B$ if and only if for each $x_M \in$ IVP(X), $x_M \in A$ implies $x_M \in B$.

Definition 1.6 [7]. Let $(X, \cdot)$ be a groupoid and let $A, B \in D(I)^X$. Then the *interval-valued fuzzy product* of A and B, $A \circ B$ is defined as follows : For each $x \in X$,

$$A \circ B(x) = \begin{cases} [\bigvee_{x=yz} (A^L(y) \wedge B^L(z)), \bigvee_{x=yz} (A^U(y) \wedge B^U(z))], & \text{if } x = yz; \\ \tilde{0}, & \text{orherwise.} \end{cases}$$

Result 1.B [7, Proposition 3.2]. Let $(X, \cdot)$ be a groupoid, let "$\circ$" be the same as above, let $x_M, y_N \in$ IVP(X) and let $A, B \in D(I)^X$. Then

(a) $x_M \circ y_N = (xy)_{M\cap N}$.

(b) $A \circ B = \bigcup_{x_M\in A, y_N\in B} x_M \circ y_N$.

Definition 1.7 [1]. Let G be a group and let $A \in D(I)^G$. Then A is called an *interval-valued fuzzy subgroup*(in short, IVG) of G if it satisfies the following conditions :

(a) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$ for any $x, y \in G$.

(b) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$ for each $x \in G$.

We will denote the set of all IVGs as IVG(G).

Result 1.C [1, Proposition 3.1]. Let A be an IVG of a group G with identity e. Then $A(x^{-1}) = A(x)$ and $A^L(x) \geq A^L(e), A^U(x) \geq A^U(e)$ for each $x \in G$.

Definition 1.8 [7]. Let $(R, +, \cdot)$ be a ring and let $\widetilde{0} \neq A \in D(I)^R$. Then A is called an *interval-valued fuzzy subring*(in short, *IVR*) of R if it satisfies following conditions :

(a) A is an IVG with respect to the operation " + ".

(b) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$ for any $x, y \in R$.

We will denote the set of all IVRs as IVR(R).

2. Interval-valued fuzzy ideals

Definition 2.1 [7]. Let A be a non-empty IVR of a ring R. Then A is called an :

(i) *interval-valued fuzzy left ideal* (in short, *IVLI*) of R if $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for any $x, y \in R$.

(ii) *interval-valued fuzzy right ideal* (in short, *IVRI*) of R if $A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$ for any $x, y \in R$.

(iii) *interval-valued fuzzy ideal* (in short, *IVRI*) of R if it is an IVLI and an IVRI of R .

We will denote the set of all IVRIs [resp. IVLIs and IVIs] of R as IVRI(R) [resp. IVLI(R) and IVI(R)].

Result 2.A [7, Proposition 6.6]. Let R be a ring. Then A is an ideal [resp. a left ideal and a right ideal] of R if and any of $[\chi_A, \chi_A] \in$ IVI(R) [resp. IVLI(R) and IVRI(R)].

Result 2.B [7, Proposition 6.5]. Let R be a ring and let $\tilde{0} \neq A \in D(I)^R$. Then $A \in$ IVI(R)[resp. IVLI(R) and IVRI(R)] if and only if for any $x, y \in R$,

(i) $A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y)$.

(ii) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$ [resp. $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$, $A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$].

Lemma 2.2. Let R be a ring and let $A, B \in D(I)^R$.

(a) If $A, B \in$ IVLI(R) [resp. IVRI(R) and IVI(R)], then $A \cap B \in$ IVLI(R) [resp. IVRI(R) and IVI(R)].

(b) If $A \in$ IVRI(R) and $B \in$ IVLI(R), then $A \circ B \subset A \cap B$.

Proof. (a) Suppose $A, B \in$ IVLI(R) and let $x, y \in R$. Then

$$\begin{aligned}&(A\cap B)^L(x-y)\\ =&A^L(x-y)\wedge B^L(x-y)\\ \geq&(A^L(x)\wedge A^L(y))\wedge(B^L(x)\wedge B^L(y))\\ =&(A\cap B)^L(x)\wedge(A\cap B)^L(y).\end{aligned}$$

Similarly, we have $(A\cap B)^U(x-y) \geq (A\cap B)^U(x) \wedge (A\cap B)^U(y)$. Also

$$\begin{aligned}&(A\cap B)^L(xy)\\ =&A^L(xy)\wedge B^L(xy)\\ \geq&A^L(y)\wedge B^L(y)\ \ (\text{Since }\ A,B\in\ \text{IVLI(R)})\\ =&(A\cap B)^L(y).\end{aligned}$$

Similarly, we have $(A\cap B)^U(xy) \geq (A\cap B)^U(y)$. Hence, by Result 2.B, $A \cap B \in$ IVLI(R). Similarly, we can easily see the rest.

(b) Let $x \in G$ and suppose $A \circ B(x) = [0,0]$. Then there is nothing to show. Suppose $A \circ B(x) \neq [0,0]$. Then $A \circ B(x) = [\bigvee_{x=yz}(A^L(y) \wedge B^L(z)), \bigvee_{x=yz}(A^U(y) \wedge B^U(z))]$. Since $A \in$ IVRI(R) and $B \in$ IVLI(R),

$$A^L(y) \leq A^L(yz) = A^L(x), A^U(y) \leq A^U(yz) = A^U(x)$$

and

$$B^L(z) \leq B^L(yz) = B^L(x), B^U(z) \leq B^U(yz) = B^U(x).$$

Thus

$$\begin{aligned}(A\circ B)^L(x) &= \bigvee_{x=yz}(A^L(y)\wedge B^L(z))\\ &\leq A^L(x)\wedge B^L(x) = (A\cap B)^L(x).\end{aligned}$$

Similarly, we have $(A\circ B)^U(x) \leq (A\cap B)^U(x)$. Hence $A \circ B \subset A \cap B$. This completes the proof. □

A ring R is said to be *regular* if for each $a \in R$ there exists an $x \in R$ such that $a = axa$.

Result 2.C [2, Theorem 9.4]. A ring R is regular if and only if $JM = J \cap M$ for each right ideal J and left ideal M of R.

Theorem 2.3. A ring R is regular if and only if for each $A \in$ IVRI(R) and each $B \in$ IVLI(R), $A \circ B = A \cap B$.

Proof. $(\Rightarrow)$: Suppose R is regular. From Lemma 2.2(b), it is clear that $A \circ B \subset A \cap B$. Thus it is sufficient to show that $A \cap B \subset A \circ B$. Let $a \in R$. Then, by the hypothesis, there exists an $x \in R$ such that $a = axa$. Thus $A^L(a) = A^L(axa) \geq A^L(ax) \geq A^L(a)$ and $A^U(a) = A^U(axa) \geq A^U(ax) \geq A^U(a)$. So $A(ax) = A(a)$.
On the other hand,

$$\begin{aligned}(A\circ B)^L(a) &= \bigvee_{a=yz}(A^L(y)\wedge B^L(z))\\ &\geq A^L(ax)\wedge B^L(a) \quad (\text{Since }\ a=axa)\\ &= A^L(a)\wedge B^L(a) = (A\cap B)^L(a).\end{aligned}$$

Similarly, we have $(A\circ B)^U(a) \geq (A\cap B)^U(a)$. Thus $A \cap B \subset A \circ B$. Hence $A \circ B = A \cap B$.

$(\Leftarrow)$: Suppose the necessary condition holds. Let J and M be right and left ideals of R, respectively. Then, by Result 2.A, $[\chi_J, \chi_J] \in$ IVRI(R) and $[\chi_M, \chi_M] \in$ IVLI(R). Let $a \in J \cap M$ and let $A = [\chi_J, \chi_J], B = [\chi_M, \chi_M]$. Then, by the hypothesis, $(A\circ B)(a) = (A\cap B)(a) = [1,1]$. Thus

$$\begin{aligned}(A\circ B)^L(a) &= \bigvee_{a=a_1a_2}(A^L(a_1)\wedge B^L(a_2))\\ &= \bigvee_{a=a_1a_2}(\chi_J(a_1)\wedge\chi_M(a_2))\\ &= 1.\end{aligned}$$

Similarly, we have $(A\circ B)^U(a) = 1$. So there exist $b_1, b_2 \in R$ such that $\chi_J(b_1) = 1$ and $\chi_M(b_2) = 1$ with $a = b_1b_2$. Thus $a \in JM$, i.e., $J \cap M \subset JM$. Since $JM \subset J \cap M$, $JM = J \cap M$. Hence, by Result 2.C, R is regular. This completes the proof. □

3. Interval-valued fuzzy prime ideals

Definition 3.1. Let P be an IVI of a ring R. Then P is said to be *prime* if P is not a constant mapping and for any $A, B \in$ IVI(R), $A \circ B \subset P$ implies either $A \subset P$ or $B \subset P$.

We will denote the set of all interval-valued fuzzy prime ideals of R as IVPI(R).

Theorem 3.2. Let J be an ideal of a ring R such that $J \neq R$. Then J is a prime ideal of R if and only if

$[\chi_J, \chi_J] \in$ IVPI(R).

Proof. $(\Rightarrow)$: Suppose J is a prime ideal of R and let $P = [\chi_J, \chi_J]$. Since $J \neq R$, P is not a constant mapping on R. Assume that there exist $A, B \in$ IVI(R) such that $A \circ B \subset P$ and $A \not\subset P$ and $B \not\subset P$. Then there exist $x, y \in R$ such that

$$A^L(x) > P^L(x) = \chi_J(x),\ A^U(x) > P^U(x) = \chi_J(x)$$

and

$$B^L(y) > P^L(y) = \chi_J(y),\ B^U(y) > P^U(y) = \chi_J(y).$$

Thus $A^L(x) \neq 0$, $A^U(x) \neq 0$ and $B^L(y) \neq 0$, $B^U(y) \neq 0$. But $\chi_J(x) = 0$ and $\chi_J(y) = 0$. So $x \notin J$ and $y \notin J$. Since J is a prime ideal of R, by the process of the proof of Theorem 2 in [9], there exist an $r \in R$ such that $xry \notin J$. Let $a = xry$. Then clearly, $\chi_J(a) = 0$. Thus

$$A \circ B(a) = [0, 0]. \quad (3.1)$$

On the other hand,

$$\begin{aligned}(A \circ B)^L(a) &= \bigvee_{a=cd} (A^L(c) \wedge B^L(d)) \\ &\geq A^L(x) \wedge B^L(ry) \ \text{(Since } a = xry) \\ &= A^L(x) \wedge B^L(y) \ \text{(Since } B \in \text{IVI(R))} \\ &> 0. \ \text{(Since } A^L(x) \neq 0 \text{ and } B^L(y) \neq 0)\end{aligned}$$

Similarly, we have $(A \circ B)^U(a) > 0$. Then $A \circ B(a) \neq \widetilde{0}$. This contradicts (3.1). So P satisfies the second condition of Definition 3.1. Hence $P = [\chi_J, \chi_J] \in$ IVPI(R).

$(\Leftarrow)$: Suppose $P = [\chi_J, \chi_J] \in$ IVPI(R). Since P is not a constant mapping on R, $J \neq R$. Let A and B be two ideals of R such that $AB \subset J$. Let $\widetilde{A}, \widetilde{B} \in$ IVI(R). Consider the product $\widetilde{A} \circ \widetilde{B}$. Let $x \in R$.

Suppose $\widetilde{A} \circ \widetilde{B}(x) = [0, 0]$. Then clearly $\widetilde{A} \circ \widetilde{B} \subset P$.

Suppose $\widetilde{A} \circ \widetilde{B}(x) \neq [0, 0]$. Then $(\widetilde{A} \circ \widetilde{B})^L(x) = \bigvee_{x=yz}(\chi_A(y) \wedge \chi_B(z)) \neq 0$. Similarly, we have $(\widetilde{A} \circ \widetilde{B})^U(x) \neq 0$. Thus there exist $y, z \in R$ with $x = yz$ such that $\chi_A(y) \neq 0$ and $\chi_B(z) \neq 0$. So $\chi_A(y) = 1$ and $\chi_B(z) = 1$. This implies $y \in A$ and $z \in B$. Thus $x = yz \in AB \subset J$. So $\chi_J(x) = 1$. It follows that $\widetilde{A} \circ \widetilde{B} \subset P$. Since $P \in$ IVPI(R), either $\widetilde{A} \subset P$ or $\widetilde{B} \subset P$. Thus either $A \subset J$ or $B \subset J$. Hence J is a prime ideal of R. This completes the proof. □

Proposition 3.3. Let P be an interval-valued fuzzy prime ideals of a ring R and let $R_P = \{x \in R : P(x) = P(0)\}$. Then R_P is a prime ideal of R.

Proof. Let $x, y \in R_P$. Then $P(x) = P(0)$ and $P(y) = P(0)$. Thus $P^L(x - y) \geq P^L(x) \wedge P^L(y) = P^L(0)$. Similarly, we have $P^U(x-y) \geq P^U(0)$. Since $P \in$ IVI(R),

$$P^L(0) = P^L(0(x - y)) \geq P^L(x - y).$$

Similarly, we have $P^U(0) \geq P^U(x - y)$. So $x - y \in R_P$.

Now let $r \in R$ and let $x \in R_P$. Then

$$P^L(rx) \geq P^L(x) = P^L(0) \text{ and } P^U(rx) \geq P^U(x) = P^U(0).$$

By Result 1.C, $P(rx) = P(0)$. So $rx \in R_P$. Similarly we have $xr \in R_P$. Hence R_P is an ideal of R.

Let J and M be two ideals of R such that $JM \subset R_P$. We define two mappings $A, B : R \to D(I)$ by $A = P(0)[\chi_J, \chi_J]$ and $B = P(0)[\chi_M, \chi_M]$, respectively, where $P(0)[\chi_J, \chi_J] = [P^L(0)\chi_J, P^U(0)\chi_J]$. Then we can easily prove that $A, B \in$ IVI(R). Let $x \in R$.

Suppose $A \circ B(x) = [0, 0]$. Then $A \circ B \subset P$.

Suppose $A \circ B(x) \neq [0, 0]$. Then $(A \circ B)^L(x) = \bigvee_{x=yz}(A^L(y) \wedge B^L(z)) = \bigvee_{x=yz}(P^L(0)\chi_J(y) \wedge P^L(0)\chi_M(z)) \neq 0$. Similarly, we have $(A \circ B)^U(x) \neq 0$. Thus there exist $y, z \in R$ with $x = yz$ such that

$$P^L(0)\chi_J(y) \wedge P^L(0)\chi_M(z) \neq 0$$

and

$$P^U(0)\chi_J(y) \wedge P^U(0)\chi_M(z) \neq 0.$$

So $\chi_J(y) = 1$ and $\chi_M(z) = 1$. Thus $y \in J$ and $z \in M$, i.e., $x = yz \in JM \subset R_P$. So $P(x) = P(0)$, i.e., $A \circ B \subset P$. Since $P \in$ IVPI(R) and $A, B \in$ IVI(R), either $A \subset P$ or $B \subset P$. Suppose $A \subset P$. Then $P(0)[\chi_J, \chi_J] \subset P$. Assume that $J \subset R_P$. Then there exists an $a \in J$ such that $a \notin R_P$. Thus $P(a) \neq P(0)$. By Result 1.C, $P^L(a) < P^L(0)$ and $P^U(a) < P^U(0)$. Then $A^L(a) = P^L(0)\chi_J(a) = P^L(0) > P^L(a)$. Similarly, we have $A^U(a) > P^U(a)$. This contradicts the assumption that $A \subset P$. So $J \subset R_P$. By the similar arguments, we can show that if $B \subset P$, then $M \subset R_P$. Hence R_P is a prime ideal of R. This completes the proof. □

Remark 3.4. Let $P \in$ IVI($\mathbb{Z}$). Then, by Proposition 3.3, R_P is an ideal of $\mathbb{Z}$. Hence there exists an integer $n \geq 0$ such that $R_P = n\mathbb{Z}$.

Proposition 3.5. Let $P \in$ IVI($\mathbb{Z}$) with $R_P = n\mathbb{Z} \neq (0)$. Then P can take at most r values, where r is the number of distinct positive divisors of n.

Proof . Let $a \in \mathbb{Z}$ and let $d = (a, n)$. Then there exist $r, s \in \mathbb{Z}$ such that $d = ar + ns$. Thus

$$P^L(d) = P^L(ar+ns) \geq P^L(ar) \wedge P^L(ns) \geq P^L(a) \wedge P^L(n).$$

Similarly, we have $P^U(d) \geq P^U(a) \wedge P^U(n)$. Since $n \in R_P = n\mathbb{Z}$, by Result 1.C,

$$P^L(n) = P^L(0) \geq P^L(a) \text{ and } P^U(n) = P^U(0) \geq P^U(a).$$

Thus $P^L(d) \geq P^L(a)$ and $P^U(d) \geq P^U(a)$. Since d is a divisor of a, there exists a $t \in \mathbb{Z}$ such that $a = dt$. Then

$P^L(a) = P^L(dt) \geq P^L(d)$ and $P^U(a) = P^U(dt) \geq P^U(d)$.
So $P(a) = P(d)$. Moreover, by Result 1.C, $P(x) = P(-x)$ for each $x \in R$. Hence for each $a \in \mathbb{Z}$ there exists a positive divisor d of n such that $P(a) = P(d)$. This completes the proof. □

The following result gives a complete characterization of interval-valued fuzzy prime ideals of $\mathbb{Z}$:

Theorem 3.6. Let $P \in$ IVPI($\mathbb{Z}$) with $\mathbb{Z}_P \neq (0)$. Then P has two distinct values. Conversely, if $P \in D(I)^{\mathbb{Z}}$ such that $P(n) = [\lambda_1, \mu_1]$ when $p \mid n$ and $P(n) = [\lambda_2, \mu_2]$ when $p \nmid n$, where p is a fixed prime, $\lambda_1 > \lambda_2$ and $\mu_1 > \mu_2$, then $P \in$ IVPI($\mathbb{Z}$) with $\mathbb{Z}_P \neq (0)$.

Proof. Suppose $P \in$ IVPI($\mathbb{Z}$) with $\mathbb{Z}_P = n\mathbb{Z} \neq (0)$. Then, by Proposition 3.3, $\mathbb{Z}_P$ is a prime ideal of $\mathbb{Z}$. Thus n is a prime integer. Since n has two distinct positive integers, by Proposition 3.5, P has at most two distinct values. On the other hand, an interval-valued fuzzy prime ideals cannot be a constant mapping. Hence P has two distinct values.

Conversely, let P be an IVS in $\mathbb{Z}$ satisfying the given conditions. Let $a, b \in \mathbb{Z}$.

Case(i): Suppose $p \mid (a-b)$. Then $P(a-b) = [\lambda_1, \mu_1]$. Thus $\lambda_1 = P^L(a-b) \geq P^L(a) \wedge P^L(b)$ (Since $\lambda_1 > \lambda_2$) and $\mu_1 = P^U(a-b) \geq P^U(a) \wedge P^U(b)$ (Since $\mu_1 > \mu_2$).

Case(ii): Suppose $p \nmid (a-b)$. Then $p \nmid a$ or $p \nmid b$. Thus either $P(a) = [\lambda_2, \mu_2]$ or $P(b) = [\lambda_2, \mu_2]$. So $\lambda_2 = P^L(a-b) \geq P^L(a) \wedge P^L(b)$ and $\mu_2 = P^U(a-b) \geq P^U(a) \wedge P^U(b)$.

Case(iii): Suppose $p \mid ab$. Then clearly $P^L(ab) \geq P^L(b)$ and $P^U(ab) \geq P^U(b)$.

Case(iv): Suppose $p \nmid ab$. Then $p \nmid a$ and $p \nmid b$. Thus $P^L(ab) \geq P^L(b)$ and $P^U(ab) \geq P^U(b)$. Consequently, by Result 1.C, $P \in$ IVI($\mathbb{Z}$) with $\mathbb{Z}_P = p\mathbb{Z} \neq (0)$. Moreover, by the similar arguments of the proof of Proposition 3.2, we can see that $P \in$ IVPI($\mathbb{Z}$). This completes the proof. □

Proposition 3.7. Let R be a ring with 1. If every IVI of R has finite values, then R is a Noetherian ring.

Proof . Let $\{J_i\}_{i\in\mathbb{Z}^+}$ be a sequence of ideals of R such that $J_1 \subset J_2 \subset J_3 \subset \cdots$ and let $J = \bigcup_{i\in\mathbb{Z}^+} J_i$. Then clearly J is an ideal of R. We define a mapping $P : R \to D(I)$ as follows : For each $x \in R$,

$$P(x) = \begin{cases} \mathbf{0}, & \text{if } x \notin J; \\ [\frac{1}{i_1}, \frac{1}{i_1}], & \text{if } x \in J. \end{cases}$$

where $i_1 =$ minimum of i such that $x \in J_i$. Then it is clear that $P \in$ IVI(R) from the definition of P. Moreover, we can easily see that $P \in$ IVI(R). If the chain does not terminate, then P takes infinitely many values. This contradicts the hypothesis. Thus the chain terminates. Hence R is a Noetherian ring. This completes the proof. □

Proposition 3.8. Let $A : \mathbb{Z} \to D(I)$ be the mapping such that

(a) $A(x) = A(-x)$ for each $x \in \mathbb{Z}$.

(b) $A^L(x+y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x+y) \geq A^U(x) \wedge A^U(y)$ for any $x, y \in \mathbb{Z}$.

If there exists a non-zero integer m such that $A(m) = A(0)$, then A can take at most finitely many values.

Proof. It is clear that $A \in D(I)^{\mathbb{Z}}$ from the definition of A. Moreover, we can easily show that $A \in$ IVI($\mathbb{Z}$) such that $\mathbb{Z}_A \neq (0)$. Hence, by Proposition 3.5, A can take at most finitely many values. □

4. Interval-valued fuzzy completely prime ideals

Definition 4.1. Let P be an IVI of a ring R. Then P is called an *interval-valued fuzzy completely prime ideals*(in short, *IVCPI*) of R if it satisfies the following conditions :

(a) P is not a constant mapping.

(b) For any $x_M, y_N \in$ IVP(R), $x_M \circ y_N \in P$ implies either $x_M \in P$ or $y_N \in P$.

We will denote the set of all IVCPIs of R as IVCPI(G).

Proposition 4.2. (a) Let R be a ring. Then IVCPI(R) $\subset$ IVPI(R).

(b) Let R be a commutative ring. Then IVPI(R) $\subset$ IVCPI(R). Hence IVCPI(R) $=$ IVPI(R).

Proof . (a) Let $P \in$ IVCPI(R) and let $A, B \in$ IVI(R) such that $A \circ B \subset P$. Suppose $A \not\subset P$. Then, by Theorem 1.5, there exists an $x_{[\lambda,\mu]} \in$ IVP(R) such that $x_{[\lambda,\mu]} \in P$ but $x_{[\lambda,\mu]} \notin P$. Let $y_{[t,s]} \in B$. Then, by Result 1.B(a), $x_{[\lambda,\mu]} \circ y_{[t,s]} = (xy)_{[\lambda\wedge t, \mu\wedge s]}$. On the other hand,

$$P^L(xy) \geq (A \circ B)^L(xy) \geq A^L(x) \wedge B^L(y)$$
$$= \lambda \wedge t = (x_{[\lambda,\mu]} \circ y_{[t,s]})^L(xy).$$

Similarly, we have $P^U(xy) \geq (x_{[\lambda,\mu]} \circ y_{[t,s]})^U(xy)$.

Let $z \in R$ such that $x \neq xy$. Then clearly $[x_{[\lambda,\mu]} \circ y_{[t,s]}](z) = [0,0]$. Thus $x_{[\lambda,\mu]} \circ y_{[t,s]} \in P$. Since $P \in$ IVCPI(R), $x_{[\lambda,\mu]} \in P$ or $y_{[t,s]} \in P$. Since $x_{[\lambda,\mu]} \notin P$, $y_{[t,s]} \in P$. So, by Theorem 1.5, $B \subset P$. Hence $P \in$ IVPI(R).

(b) Let $P \in$ IVPI(R) and let $x_{[\lambda,\mu]}, y_{[t,s]} \in$ IVP(R) such that $x_{[\lambda,\mu]} \circ y_{[t,s]} \in P$. Then $(x_{[\lambda,\mu]} \circ y_{[t,s]})^L(xy) \leq P^L(xy)$ and $(x_{[\lambda,\mu]} \circ y_{[t,s]})^U(xy) \leq P^U(xy)$.

Thus, by Result 1.B(a),

$$\lambda \wedge t \leq P^L(xy) \text{ and } \mu \wedge s \leq P^U(xy). \tag{4.1}$$

We define two mappings $A, B : R \to D(I)$ as follows : For each $z \in R$,

$$A(z) = \begin{cases} [\lambda, \mu], & \text{if } z \in (x); \\ [0,0], & \text{otherwise.} \end{cases}$$

and

$$B(z) = \begin{cases} [t, s], & \text{if } z \in (y); \\ [0,0], & \text{otherwise,} \end{cases}$$

where (x) is the ideal generated by x. Then clearly $A, B \in D(I)^R$ from the definitions of A and B. It is easily seen that if z is not expressible in the form $z = uv$ for some $u \in (x)$ and $v \in (y)$, then $A \circ B(z) = [0,0]$. Suppose there exist $u \in (x)$ and $v \in (y)$ such that $z = uv$. Then

$$(A \circ B)^L(z) = \bigvee_{z=uv, u \in (x), v \in (y)} (A^L(u) \wedge B^L(v)) = \lambda \wedge t$$

and

$$(A \circ B)^U(z) = \bigvee_{z=uv, u \in (x), v \in (y)} (A^U(u) \wedge B^U(v)) = \mu \wedge s.$$

Since R is commutative and $u \in (x)$, there exist $n \in \mathbb{Z}$ and $b \in R$ such that $u = nx + xb$. Since $v \in (y)$, there exist $m \in \mathbb{Z}$ and $c \in R$ such that $v = my + yc$. Since R is commutative, $uv = (nx + xb)(my + yc) = xyd + mnxy$ for some $d \in R$. Then

$$\begin{aligned} P^L(uv) &\geq P^L(xy) && (\text{Since } P \in \text{IVI(R)}) \\ &\geq \lambda \wedge t. && (\text{By } (4.1)) \end{aligned}$$

Similarly, we have that $P^U(uv) \geq P^U(xy) \geq \mu \wedge s$. Thus $z_{[\lambda \wedge t, \mu \wedge s]} = u_{[\lambda, \mu]} \circ v_{[t,s]} \in P$. So, in all, $A \circ B \subset P$. On the other hand, from the definitions of A and B, we can easily prove that $A, B \in$ IVI(R). Since $P \in$ IVPI(R), either $A \subset P$ or $B \subset P$. Thus either $x_{[\lambda, \mu]} \in P$ or $y_{[t,s]} \in P$. Hence $P \in$ IVCPI(R). This completes the proof. □

Proposition 4.3. Let P be a non-constant IVI of a ring R.
(a) If P is an IVPI [resp. IVCPI] of R, then
(i)R_P is a prime [resp. completely prime] ideal of R.
(ii) ImP consists of exactly two points of $D(I)$.
(b) If $P(0) = [1, 1]$ and P satisfies the conditions (i) and (ii), then $P \in$ IVPI(R) [resp. IVCPI(R)].

Proof . (a) We shall confirm our proof to the case of interval-valued fuzzy prime ideals. An analogous proof can be given by for interval-valued fuzzy completely prime ideals. Suppose $P \in$ IVPI(R). Then, by Proposition 3.3, R_P is a prime ideal of R. Assume that ImP contains more than two values. Then there exist $x, y \in R \setminus R_P$ such that $P(x) \neq P(y)$. Suppose without loss of generality that $P^L(x) < P^L(y)$ and $P^U(x) < P^U(y)$. Since $P \in$ IVI(R) and $A(y) \neq A(0)$, by Result 1.C, $P^L(x) < P^L(y) < P^L(0)$ and $P^U(x) < P^U(y) < P^U(0)$. Let $[\lambda, \mu], [t, s] \in D(I)$ be chosen such that

$$P^L(x) < \lambda < P^L(y) < t < P^L(0) \quad \text{and} \quad P^U(x) < \mu < P^U(y) < s < P^U(0). \tag{4.2}$$

Let (x) and (y) denote respectively the ideals generated by x and y. We define two mappings $A, B : R \to D(I)$ as follows: $A = [\lambda\chi_{(x)}, \mu\chi_{(x)}]$ and $B = [t\chi_{(y)}, s\chi_{(y)}]$. Then it is easily seen that $A, B \in$ IVI(R) from the definitions of A and B. Let $z \in R$ which cannot be expressed in the from $z = uv$ for $u \in (x)$ and $v \in (y)$. Then $A \circ B(z) = [0, 0]$. Thus $A \circ B \subset P$. Now let $z \in R$. Suppose there exist $u \in (x)$ and $v \in (y)$ such that $z = uv$ for some $u \in (x)$ and $v \in (y)$. Then

$$(A \circ B)^L(z) = \bigvee_{z=uv, u \in (x), v \in (y)} (A^L(u) \wedge B^L(v)) = \lambda \wedge t = \lambda.$$

Similarly, we have $(A \circ B)^U(z) = \mu$. Since $u \in (x)$, there exist $m \in \mathbb{Z}$ and $r_i \in R(i = 1, 2, 3, 4)$ such that $u = mx + r_1x + xr_2 + r_3xr_4$. Similarly, there exist $n \in \mathbb{Z}$ and $s_i \in R(i = 1, 2, 3, 4)$ such that $v = ny + s_1y + ys_2 + s_3ys_4$. Since $P \in$ IVI(R), by Result 1.C,

$$P^L(z) = P^L(uv) \geq P^L(x) \wedge P^L(y) > \lambda$$

and

$$P^U(z) = P^U(uv) \geq P^U(x) \wedge P^U(y) > \mu.$$

Thus $(A \circ B)^L(z) \leq P^L(z)$ and $(A \circ B)^U(z) \leq P^U(z)$. So $A \circ B \subset P$. Since $P \in$ IVPI(R), either $A \subset P$ or $B \subset P$. Then either $A^L(x) = \lambda \leq P^L(x)$, $A^U(x) = \mu \leq P^U(x)$ or $B^L(y) = t \leq P^L(y)$, $B^U(y) = s \leq P^U(y)$. This contradicts (4.2). Hence ImP consists of exactly two points of $D(I)$.

(b) Suppose $P(0) = [1, 1]$ and P satisfies the conditions (i) and (ii). Then, by the similar arguments of proof of Theorem 3.2, we can see that $P \in$ IVPI(R). This completes the proof. □

Corollary 4.3. Let P be an interval-valued fuzzy completely prime ideal of a ring R. Then for any $x, y \in R$, $P(xy) = [P^L(x) \wedge P^L(y), P^U(x) \wedge P^U(y)]$.

Remark 4.4. Proposition 4.3 generalizes Proposition 3.5.

Definition 4.5. Let A be a non-constant IVI of a ring R. Then A is called an *interval-valued fuzzy weakly completely prime ideal of* R if for any

$x, y \in R, A(xy) = [A^L(x) \wedge A^L(y), A^U(x) \wedge A^U(y)]$.

The following is the immediate result of Definitions 4.1 and 4.5.

Proposition 4.6. Let A be an interval-valued fuzzy weakly completely prime ideal of a ring R. Then for each $[\lambda, \mu] \in D(I)$, $x_{[\lambda,\mu]} \circ y_{[t,s]} \in A$ implies that either $x_{[\lambda,\mu]} \in A$ or $y_{[t,s]} \in A$. Furthermore, for each $[\lambda, \mu] \in D(I)$ such that $\lambda + \mu \leq 1, \lambda < A^L(0)$ and $\mu < A^U(0)$, $A^{[\lambda,\mu]}$ is a completely prime ideal of R. In particular, $A^{[0,0]}$ is a completely prime ideal of R. Conversely if for each $[\lambda, \mu] \in D(I)$, $A^{[\lambda,\mu]}$ is a completely prime ideal then A is an interval-valued fuzzy weakly completely prime ideal.

The following is the example that an interval-valued fuzzy weakly completely prime ideal need not be an interval-valued fuzzy completely prime ideal.

Example 4.7. Let $R = \mathbb{Z} \times \mathbb{Z}$, let $S = \{0\} \times \mathbb{Z}$ and let $T = (2) \times \mathbb{Z}$. We define a mapping $A : R \to D(I)$ as follows : For each $x \in R$,

$$A(x) = \begin{cases} [1,1], & \text{if } x \in S; \\ (\frac{1}{2}, \frac{1}{3}), & \text{if } x \in T \backslash S; \\ [0,0], & \text{if } x \in R \backslash T. \end{cases}$$

Then clearly $A \in D(I)^R$ from the definition of A. Moreover, we can easily show that A is an interval-valued fuzzy weakly completely prime ideal but, by Proposition 4.2, A is not an interval-valued fuzzy weakly completely prime ideal. □

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