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CONVERGENCE OF DOUBLE SERIES OF RANDOM ELEMENTS IN BANACH SPACES

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Abstract. For a double array of random elements $\{X_{mn}; m \geq 1, n \geq 1\}$ in a p-uniformly smooth Banach space, $\{b_{mn}; m \geq 1, n \geq 1\}$ is an array of positive numbers, convergence of double random series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_{mn}$, $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}^{-1} X_{mn}$ and strong law of large numbers

$$b_{mn}^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} \to 0 \text{ as } m \wedge n \to \infty$$

are established.

1. Introduction

Consider a double array $\{X_{mn}; m \geq 1, n \geq 1\}$ of random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$, $\{b_{mn}; m \geq 1, n \geq 1\}$ is an array of positive numbers. In the current work, we establish convergence a.s of double random series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_{mn}$ and $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}^{-1} X_{mn}$, and since the convergence of double random series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}^{-1} X_{mn}$ we obtain strong laws of large numbers $b_{mn}^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} \to 0$ as $m \wedge n \to \infty$.

Strong law of larger number for double array of random element in Banach spaces have studied by many authors. For example, Dung et al. [1], Dung and Tien [2], Quang et al. [8], Roralsky and Thanh [9], Stadtmuller and Thanh [11]. The three-series theorem for martingale in Banach spaces in case of single series was established by Tien [13]. However, convergence of double random series has not been studied. In this paper we not only extend some results of Su and Tong [12] and Hong and Tsay [4] but also establish the convergence of double random series.

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2. Preliminaries

Technical definitions relevant to the current work will be discussed in this section.

For $a, b \in \mathbb{R}$, $\min\{a, b\}$ and $\max\{a, b\}$ will be denoted, respectively, by $a \wedge b$ and $a \vee b$. Denote $\mathbb{N}$ be the set of all positive integers, for (i, j) and $(m, n) \in \mathbb{N}^2$, $(i, j) \prec (m, n)$ means that $i \leq m$ and $j \leq n$. Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

Scalora [10] introduced the idea of the conditional expectation of a random element in a Banach space. For a random element V and sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the conditional expectation $E(V|\mathcal{G})$ is defined analogously to that in the random variable case and enjoys similar properties.

A real separable Banach space $\mathcal{X}$ is said to be *p-uniformly smooth* ($1 \leq p \leq 2$) if there exists a finite positive constant C such that such that for any L^p integrable $\mathcal{X}$-valued martingale difference sequence $\{X_n, n \geq 1\}$,

$$E\|\sum_{i=1}^{n} X_n\|^p \leq C \sum_{i=1}^{n} E\|X_i\|^p.$$

Clearly every real separable Banach space is of 1-uniformly smooth and the real line (the same as any Hilbert space) is of 2-uniformly smooth. If a real separable Banach space of p-uniformly smooth for some $1 < p \leq 2$, then it is of r-uniformly smooth for all $r \in [1, p)$. For more details, the reader may refer to Pisier [7].

To prove the main result we need the following lemmas.

Lemma 2.1. *Let $\{S_{mn}; m \geq 1, n \geq 1\}$ be an array of random elements taking values in Banach space $\mathcal{X}$. Then, S_{mn} converges a.s. as $m \wedge n \to \infty$ if only if for all $\varepsilon > 0$,*

$$\lim_{N\to\infty} P\left(\sup_{\substack{N\leq m\leq p\\ N\leq n\leq q}} \|S_{pq} - S_{mn}\| > \varepsilon\right) = 0. \tag{2.1}$$

Proof. Omitted. □

Remark 2.2. Since inequalities

$$\sup_{\substack{m\leq p\\ n\leq q}} \|S_{pq} - S_{mn}\| \leq \sup_{\substack{m\wedge n\leq p'\leq p\\ m\wedge n\leq q'\leq q}} \|S_{p'q'} - S_{pq}\| \leq 2 \sup_{\substack{m\leq p\\ n\leq q}} \|S_{pq} - S_{mn}\|,$$

we have that the condition (2.1) is equivalent with

$$\lim_{m\wedge n\to\infty} P\left(\sup_{\substack{m\leq p\\ n\leq q}} \|S_{pq} - S_{mn}\| > \varepsilon\right) = 0.$$

Lemma 2.3. *Let $\{a_{mnij}; 1 \le i \le m, 1 \le j \le n\}$ be an array of positive constants such that*

$$\sup_{m\ge1,n\ge1} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{mnij} \le C < \infty \text{ and } \lim_{m\wedge n\to\infty} a_{mnij} = 0 \text{ for fixed } i, j.$$

If $\{x_{mn}; m \ge 1, n \ge 1\}$ is a double array of positive real numbers satisfying

$$\lim_{m\vee n\to\infty} x_{mn} = 0,$$

then

$$\lim_{m\wedge n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{mnij}x_{ij} = 0.$$

Proof. For proof is similar that of Lemma 2.2 of Stadtmuller and Thanh [11]. □

Lemma 2.4 ([1])**.** *Let $1 \le p \le 2$. Let $\{X_{ij}; 1 \le i \le m, 1 \le j \le n\}$ be a collection of mn random elements in a real separable Banach space p-uniformly smooth $\mathcal{X}$. Set $\mathcal{F}_{ij}$ is a σ-algebra generated by the family of random elements $\{X_{kl}; k < i \text{ or } l < j\}$ and $\mathcal{F}_{1,1} = \{\emptyset; \Omega\}$. If $E(X_{ij}|\mathcal{F}_{ij}) = 0$ for all $(i, j) \prec (m, n)$, then*

$$E \max_{\substack{1\le k\le m\\ 1\le l\le n}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l} X_{ij} \right\|^p \le C \sum_{i=1}^{m}\sum_{j=1}^{n} E\|X_{ij}\|^p,$$

where the constant C is independent of m and n.

Let $\{b_{mn}; m \ge 1, n \ge 1\}$ be an array of positive numbers. We define

$$N(x) = card\{(m, n) : b_{mn} \le x\},$$

and suppose that $N(x) < \infty$, $\forall x > 0$.

Now we define two other functions $L(x)$ and $R_p(x)$ which are little different from that of Su and Tong [12]:

$$L(x) = \int_0^x \frac{N(t)\log^+ N(t)}{t^2}dt \text{ and } R_p(x) = \int_x^\infty \frac{N(t)\log^+ N(t)}{t^{p+1}}dt$$

for $x > 0$ and $p > 0$. We have following lemma.

Lemma 2.5. *Let $\{b_{mn}; m \ge 1, n \ge 1\}$ be an array of positive numbers satisfying for each $m \ge 1$ and $n \ge 1$, $b_{ij} \le b_{mn}$ for all $(i, j) \prec (m, n)$ and $b_{mn} \to \infty$ as $m \wedge n \to \infty$. Let X be a non-negative real-valued random variables.*

(i) *If $EXL(X) < \infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P(X > b_{mn}) < \infty, \tag{2.2}$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{b_{mn}}\int_{b_{mn}}^{\infty}P(X>s)ds<\infty. \tag{2.3}$$

(ii) *If* $EX^pR_p(X)<\infty$ *for some* $p>0$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{b_{mn}^p}\int_{0}^{b_{mn}}s^{p-1}P(X>s)ds<\infty. \tag{2.4}$$

Proof. First we prove (i). Suppose that $EXL(X)<\infty$, denote d_k be the number of divisors of k and noting that $N(x)$ is non-decreasing we have

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P(X>b_{mn}) &\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P(N(X)>N(b_{mn}))\\
&\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P(N(X)>mn)\\
&\le \sum_{k=1}^{\infty}d_kP(N(X)>k)\\
&\le C\sum_{k=1}^{\infty}\log(k)P(N(X)>k)\\
&\le C\sum_{k=1}^{\infty}[(k+1)\log(k+1)-k\log(k)]P(N(X)>k)\\
&= C\sum_{k=1}^{\infty}k\log(k)[P(N(X)\le k+1)-P(N(X)\le k)]\\
&= C\sum_{k=1}^{\infty}k\log(k)\int_k^{k+1}dP(N(X)\le x)\\
&\le C\sum_{k=1}^{\infty}\int_k^{k+1}x\log x dP(N(X)\le x)\\
&= C\int_1^{\infty}x\log x dP(N(X)\le x)\\
&= CEN(X)\log^+N(X)\le CEXL(X)<\infty.
\end{aligned}$$

Next we prove (2.3). Let $s=b_{mn}t$. Then we have

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{l}\frac{1}{b_{mn}}\int_{b_{mn}}^{\infty}P(X>s)ds &= \sum_{m=1}^{k}\sum_{n=1}^{l}\int_1^{\infty}P(\frac{X}{t}>b_{mn})dt\\
&= \int_1^{\infty}\sum_{m=1}^{k}\sum_{n=1}^{l}P(\frac{X}{t}>b_{mn})dt
\end{aligned}$$

$$\le \int_1^\infty \sum_{m=1}^\infty \sum_{n=1}^\infty P(\frac{X}{t} > b_{mn})dt$$
$$\le \int_1^\infty EN(\frac{X}{t}) \log^+ N(\frac{X}{t})dt$$
$$= \int_0^\infty \left(\int_1^x N(\frac{x}{t}) \log^+ N(\frac{x}{t})dt \right) dP(X \le x)$$
$$= \int_0^\infty x \left(\int_1^x \frac{N(y)\log^+ N(y)}{y^2} dy \right) dP(X \le x)$$
$$= EXL(X) < \infty.$$

Letting $k \wedge l \to \infty$ we obtain (2.3).

Finally, we easily prove (ii) by using method of the proof is similar to that of (2.3). □

The array of random elements $\{X_{mn}; m \ge 1, n \ge 1\}$ is said to be *weakly mean dominated* by the random element X if, for some $0 < C < \infty$,

$$P\{\|X_{mn}\| \ge x\} \le CP\{\|X\| \ge x\}$$

for all $m \ge 1$, $n \ge 1$ and $x > 0$.

3. Main results

With the preliminaries accounted for, the main results may now be established. In the following we let $\{X_{mn}; m \ge 1, n \ge 1\}$ be an array of random elements defined on a probability $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$, $\mathcal{F}_{kl}$ be a σ-algebra generated by $\{X_{ij}; i < k \text{ or } j < l\}$, $\mathcal{F}_{1,1} = \{\emptyset; \Omega\}$. Suppose that $E(X_{mn}|\mathcal{F}_{mn}) = 0$ for all $m \ge 1, n \ge 1$.

Theorem 3.1. *Let $\mathcal{X}$ be a p-uniformly smooth Banach space for some $1 \le p \le 2$. If*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty E\|X_{mn}\|^p < \infty, \tag{3.1}$$

then

$$\sum_{m=1}^\infty \sum_{n=1}^\infty X_{mn} \text{ converges a.s.,} \tag{3.2}$$

$$\sum_{n=1}^\infty X_{mn} \text{ converges a.s. for every } m \ge 1 \text{ and} \tag{3.3}$$

$$\sum_{m=1}^\infty X_{mn} \text{ converges a.s. and for every } n \ge 1. \tag{3.4}$$

Proof. Set $S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$.

For an arbitrary $\varepsilon > 0$,

$$P\left(\max_{\substack{m\le p\le k\\ n\le q\le l}} \|S_{pq} - S_{mn}\| > \varepsilon\right) \le P\left(\max_{\substack{1\le m\le k\\ n\le q\le l}} \|\sum_{i=1}^{m}\sum_{j=n}^{q} X_{ij}\| > \varepsilon/2\right) + P\left(\max_{\substack{m\le p\le k\\ 1\le n\le l}} \|\sum_{i=1}^{m}\sum_{j=n}^{q} X_{ij}\| > \varepsilon/2\right). \tag{3.5}$$

If $\mathcal{G}_{mq}$ is the σ-algebra generated by the family of random elements $\{X_{ij}; (1 \le i \le k \text{ and } n \le j < q) \text{ or } (1 \le i < m \text{ and } n \le j \le k)\}$ for $1 \le m \le k$ and $n \le q \le l$, $\mathcal{G}_{1n} = \{\emptyset; \Omega\}$, then $\mathcal{G}_{mq} \subset \mathcal{F}_{mq}$ for all $1 \le m \le k$, $n \le q \le l$, which imply that $E(X_{mq}|\mathcal{G}_{mq}) = 0$ for all $1 \le m \le k$, $n \le q \le l$.

Applying Markov inequality and Lemma 2.3 we obtain

$$P\left(\max_{\substack{1\le m\le k\\ n\le q\le l}} \|\sum_{i=1}^{m}\sum_{j=n}^{q} X_{ij}\| > \varepsilon/2\right) \le \frac{2^p}{\varepsilon^p} E\left(\max_{\substack{1\le m\le k\\ n\le q\le l}} \|\sum_{i=1}^{m}\sum_{j=n}^{q} X_{ij}\|^p\right) \le \frac{C}{\varepsilon^p}\sum_{i=1}^{k}\sum_{j=n}^{l} E\|X_{ij}\|^p. \tag{3.6}$$

It is the same (3.6) we also have

$$P\left(\max_{\substack{m\le p\le k\\ 1\le q\le l}} \|\sum_{i=1}^{m}\sum_{j=n}^{q} X_{ij}\| > \varepsilon/2\right) \le \frac{C}{\varepsilon^p}\sum_{i=m}^{k}\sum_{j=1}^{l} E\|X_{ij}\|^p. \tag{3.7}$$

It follows from (3.5), (3.6) and (3.7) that

$$P\left(\max_{\substack{m\le p\le k\\ n\le q\le l}} \|S_{pq} - S_{mn}\| > \varepsilon\right) \le \frac{C}{\varepsilon^p}\sum_{i=1}^{k}\sum_{j=n}^{l} E\|X_{ij}\|^p + \frac{C}{\varepsilon^p}\sum_{i=m}^{k}\sum_{j=1}^{l} E\|X_{ij}\|^p.$$

This implies, by letting $k \wedge l \to \infty$, that

$$P\left(\sup_{\substack{m\le p\\ n\le q}} \|S_{pq} - S_{mn}\| > \varepsilon\right) \le \frac{C}{\varepsilon^p}\sum_{i=1}^{\infty}\sum_{j=n}^{\infty} E\|X_{ij}\|^p + \frac{C}{\varepsilon^p}\sum_{i=m}^{\infty}\sum_{j=1}^{\infty} E\|X_{ij}\|^p.$$

We have by (3.1) that

$$\sum_{i=1}^{\infty}\sum_{j=n}^{\infty} E\|X_{ij}\|^p \to 0 \text{ as } n \to \infty$$

and

$$\sum_{i=m}^{\infty}\sum_{j=1}^{\infty} E\|X_{ij}\|^p \to 0 \text{ as } m \to \infty,$$

hence,

$$P\left(\sup_{\substack{m\le p\\ n\le q}} \|S_{pq}-S_{mn}\|>\varepsilon\right)\to 0 \text{ as } m\wedge n\to\infty,$$

which implies S_{mn} converges a.s. as $m\wedge n\to\infty$ (by Lemma 2.1).

We now prove (3.3). For each $m\ge 1$, set $\mathcal{H}_{m,1}=\{\Omega;\emptyset\}$ and $\mathcal{H}_{mn}$ is the σ-algebra generated by the family of random elements $\{X_{mj}; 1\le j<n\}$ for $n\ge 1$, we have that $\{S_n^m=\sum_{j=1}^n X_{mj}, \mathcal{H}_{mn}; n\ge 1\}$ is a martingale satisfying $\sum_{n=1}^\infty E\|S_{n+1}^m-S_n^m\|^p<\infty$ (by (3.1)). Applying Theorem 2.2 of Woyczyński [14] we obtain the conclusion (3.3).

For proof of (3.4) is similar to that of (3.3). The proof is completed. □

Remark 3.2. Noting that (3.2), (3.3) and (3.4) imply $X_{mn}\to 0$ a.s. as $m\vee n\to\infty$. Hence, under the condition (3.1) we obtain $\lim_{m\vee n\to\infty}\|X_{mn}\|=0$ a.s. This remark will be used in Theorem 3.4 and Theorem 3.6.

Theorem 3.1 can be applied to obtain a version of the three-series theorem for double random series.

Theorem 3.3. *Let $\mathcal{X}$ be a p-uniformly smooth Banach space for some $1\le p\le 2$ and c be a positive constant. Set $Y_{mn}=X_{mn}I(\|X_{mn}\|>c)$. Suppose that $E(Y_{ij}|\mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $i\le m$ or $j\le n$. If*

(i) $\sum_{m=1}^\infty\sum_{n=1}^\infty P(\|X_{mn}\|>c)<\infty$,

(ii) $\sum_{m=1}^\infty\sum_{n=1}^\infty E(Y_{mn}|\mathcal{F}_{mn})$ *converges a.s., and*

(iii) $\sum_{m=1}^\infty\sum_{n=1}^\infty E\|(Y_{mn}-E(Y_{mn}|\mathcal{F}_{mn}))\|^p<\infty$,

then $\sum_{m=1}^\infty\sum_{n=1}^\infty X_{mn}$ converges a.s.

Proof. We have by (i) that

$$\sum_{m=1}^\infty\sum_{n=1}^\infty P(X_{mn}\ne Y_{mn})\le\sum_{m=1}^\infty\sum_{n=1}^\infty P(\|X_{mn}\|>c)<\infty.$$

By virtue of Borel-Cantelli lemma, we have

$$P(X_{mn}\ne Y_{mn}\text{i.o.})=0.$$

So, to prove theorem, it suffices to show

$$\sum_{m=1}^\infty\sum_{n=1}^\infty Y_{mn} \text{ converges a.s.} \tag{3.8}$$

In view of Theorem 3.1, we have by (iii) that

$$\sum_{m=1}^\infty\sum_{n=1}^\infty (Y_{mn}-E(Y_{mn}|\mathcal{F}_{mn})) \text{ converges a.s.} \tag{3.9}$$

Combining (ii) and (3.9) yields (3.8) holds.

The proof is completed. □

The following theorem is a version of Theorem 4.2 of Su and Tong [12] for double arrays of random elements in p-uniformly smooth Banach spaces.

Theorem 3.4. *Let $\mathcal{X}$ be a p-uniformly smooth Banach space for some $1 \leq p \leq 2$ and let $\{b_{mn}; m \geq 1, n \geq 1\}$ be an array of positive numbers satisfying for each $m \geq 1$ and $n \geq 1$, $b_{ij} \leq b_{mn}$ for all $(i,j) \prec (m,n)$ and $b_{mn} \to \infty$ as $m \wedge n \to \infty$. Suppose that Suppose that $E(Y_{ij}|\mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $i \leq m$ or $j \leq n$. Set*

$$N(x) = card\{(m,n) : b_{mn} \leq x\} \ \forall x > 0.$$

If $\{X_{mn}; m \geq 1, n \geq 1\}$ is weakly mean dominated by random element X such that

$$E(\|X\|^p R_p(\|X\|)) < \infty \tag{3.10}$$

and

$$E(\|X\| L(\|X\|)) < \infty, \tag{3.11}$$

then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{X_{mn}}{b_{mn}} \text{ converges a.s.} \tag{3.12}$$

And if $\{b_{mn}; m \geq 1, n \geq 1\}$ is an array of positive numbers satisfying for each $m \geq 1$ and $n \geq 1$, $b_{ij} < b_{mn}$ for all $(i,j) \prec (m,n)$ and $(i,j) \neq (m,n)$, $b_{mn} \to \infty$ as $m \wedge n \to \infty$, then

$$\lim_{m \wedge n \to \infty} b_{mn}^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} = 0 \quad a.s. \tag{3.13}$$

Proof. For each m, n, set $Y_{mn} = X_{mn} I(\|X_{mn}\| \leq b_{mn}), Z_{mn} = X_{mn} I(\|X_{mn}\| > b_{mn}), U_{mn} = Y_{mn} - E(Y_{mn}|\mathcal{F}_{mn}), V_{mn} = Z_{mn} - E(Z_{mn}|\mathcal{F}_{mn})$. It is clear that $X_{mn} = U_{mn} + V_{mn}$. Moreover, $E(U_{mn}|\mathcal{F}_{mn}) = E(V_{mn}|\mathcal{F}_{mn}) = 0$ for $m \geq 1$, $n \geq 1$. If $\mathcal{G}'_{kl}$ and $\mathcal{G}''_{kl}$ are the σ-algebras generated by the family of random elements $\{U_{ij} : i < k \text{ or } j < l\}$ and $\{V_l : i < k \text{ or } j < l\}$, respectively, then $\mathcal{G}'_{kl} \subset \mathcal{F}_{kl}$ and $\mathcal{G}''_{kl} \subset \mathcal{F}_{kl}$ for all $(k,l) \prec (m,n)$, which imply that $E(U_{kl}|\mathcal{G}'_{kl}) = E(V_{kl}|\mathcal{G}''_{kl}) = 0$ for all $(k,l) \prec (m,n)$. Hence, in order to prove (3.12) we prove

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{U_{mn}}{b_{mn}} \text{ and } \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{V_{mn}}{b_{mn}} \text{ converge a.s.}$$

Applying the strangle inequality and inequality (1.6) of Lemma 1.2 [3] we have

$$\begin{aligned}
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E\|V_{mn}\|}{b_{mn}} &\leq 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E\|Z_{mn}\|}{b_{mn}} \\
&\leq 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{b_{mn}} \int_{b_{mn}}^{\infty} P(\|X_{mn}\| > s) ds
\end{aligned}$$

$$
\begin{aligned}
&+ 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(\|X_{mn}\| > b_{mn}) \\
&\leq C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{b_{mn}} \int_{b_{mn}}^{\infty} P(\|X\| > s) ds \\
&+ C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(\|X\| > b_{mn}) \\
&< \infty \text{ (by Lemma 2.4)}
\end{aligned}
$$

which implies by Theorem 3.1 that

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{V_{mn}}{b_{mn}} \text{ converges a.s.} \tag{3.14}
$$

Again applying the strangle inequality and equality (1.5) of Lemma 1.2 [3] we have

$$
\begin{aligned}
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E\|U_{mn}\|^p}{b_{mn}^p} &\leq C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E\|Y_{mn}\|^p}{b_{mn}^p} \\
&= C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{b_{mn}^p} \int_{b_{mn}}^{\infty} s^{p-1} P(\|X_{mn}\| > s) ds \\
&\quad - C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(\|X_{mn}\| > b_{mn}) \\
&\leq C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{b_{mn}^p} \int_{b_{mn}}^{\infty} s^{p-1} P(\|X\| > s) ds \\
&\quad - C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(\|X\| > b_{mn}) \\
&< \infty \text{ (by Lemma 2.4)}
\end{aligned}
$$

which implies by Theorem 3.1 that

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{U_{mn}}{b_{mn}} \text{ converges a.s.} \tag{3.15}
$$

Now we prove (3.13). Since (3.14) and (3.15) we have by Theorem 3.1 that $b_{mn}^{-1} V_{mn} \to 0$ a.s. and $b_{mn}^{-1} U_{mn} \to 0$ a.s. as $m \vee n \to \infty$. Hence,

$$
\lim_{m \vee n \to \infty} b_{mn}^{-1} \|X_{mn}\| = 0 \text{ a.s.}
$$

Applying Lemma 2.2 with $a_{mnij} = \frac{b_{ij}}{b_{mn}}$ we have

$$
\lim_{m \wedge n \to \infty} b_{mn}^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} \|X_{ij}\| \to 0 \text{ a.s.,}
$$

and using the strangle inequality

$$\|b_{mn}^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}\| \le b_{mn}^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\|X_{ij}\|$$

we obtain (3.13). □

Corollary 3.5. *Let $\mathcal{X}$ be a p-uniformly smooth Banach space for some $1 \le p \le 2$. Let $\{a_{mn}; m \ge 1, n \ge 1\}$ be an array of real numbers such that $a_{mn} \ne 0$, let $\{b_{mn}; m \ge 1, n \ge 1\}$ be an array of positive numbers satisfying for each $m \ge 1$ and $n \ge 1$, $b_{ij} < b_{mn}$ and $b_{ij}/|a_{ij}| < b_{mn}/|a_{mn}|$ for all $(i,j) \prec (m,n)$ and $(i,j) \ne (m,n)$, $b_{mn}/|a_{mn}| \to \infty$ as $m \wedge n \to \infty$. Suppose that $E(X_{ij}I(\|X_{ij}\| \le b_{ij})|\mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $i \le m$ or $j \le n$. Set*

$$N(x) = card\{(m,n) : \frac{b_{mn}}{|a_{mn}|} \le x\}\ \forall x > 0.$$

If $\{X_{mn}; m \ge 1, n \ge 1\}$ is weakly mean dominated by random element X such that (3.10) *and* (3.11) *hold, then*

$$\lim_{m\wedge n\to\infty} b_{mn}^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}X_{ij} = 0 \quad a.s.$$

Finally, we extend Theorem 2.1 of Hong and Tsay [4] to double array of random elements. It is the same Theorem 3.4, we establish convergence of double random series before obtaining strong laws of large numbers.

Theorem 3.6. *Let $\mathcal{X}$ be a p-uniformly smooth Banach space for some $1 \le p \le 2$ and let $\{b_{mn}; m \ge 1, n \ge 1\}$ be an array of positive numbers. Suppose that $E(Y_{ij}|\mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $i \le m$ or $j \le n$. Let $\{\Phi_{mn}; m \ge 1, n \ge 1\}$ be an array of positive Borel functions and let $C_{mn} \ge 1$, $D_{mn} \ge 1$, $b_{mn} \ge 1$, $0 < \beta_{mn} \le p$ be constants satisfying for $u \ge v > 0$,*

$$C_{mn}\frac{u^{b_{mn}}}{v^{b_{mn}}} \le \frac{\Phi_{mn}(u)}{\Phi_{mn}(v)} \le D_{mn}\frac{u^{\beta_{mn}}}{v^{\beta_{mn}}}.$$

If

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\frac{E\Phi_{mn}(\|X_{mn}\|)}{\Phi_{mn}(b_{mn})} < \infty,$$

where $A_{mn} = \max\{\frac{1}{C_{mn}}, D_{mn}\}$, then (3.12) *holds. And if $\{b_{mn}; m \ge 1, n \ge 1\}$ is an array of positive numbers satisfying for each $m \ge 1$ and $n \ge 1$, $b_{ij} \le b_{mn}$ for all $(i,j) \prec (m,n)$ and $b_{mn} \to \infty$ as $m \wedge n \to \infty$, then* (3.13) *holds.*

Proof. Set the same Y_{mn}, Z_{mn}, U_{mn} and V_{mn} as in the proof of Theorem 3.4. It is similar to the proof of Theorem 3.4, we show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|V_{mn}\|}{b_{mn}} < \infty \tag{3.16}$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\frac{\|U_{mn}\|^p}{b_{mn}^p} < \infty. \tag{3.17}$$

First we prove (3.16).

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\|V_{mn}\|}{b_{mn}} &\le 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\frac{\|Z_{mn}\|}{b_{mn}} \\
&\le 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\left(\frac{\|Z_{mn}\|}{b_{mn}}\right)^{\alpha_{mn}} \\
&\le 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{C_{mn}} E\frac{\Phi_{mn}(\|Z_{mn}\|)}{\Phi_{mn}(b_{mn})} \\
&\le 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} E\frac{\Phi_{mn}(\|Z_{mn}\|)}{\Phi_{mn}(b_{mn})} \\
&\le 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} E\frac{\Phi_{mn}(\|X_{mn}\|)}{\Phi_{mn}(b_{mn})} < \infty.
\end{aligned}$$

Finally we prove (3.17).

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\frac{\|U_{mn}\|^p}{b_{mn}^p} &\le 2^p\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\frac{\|Y_{mn}\|^p}{b_{mn}^p} \\
&\le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\left(\frac{\|Y_{mn}\|}{b_{mn}}\right)^{\beta_{mn}} \\
&\le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn} E\frac{\Phi_{mn}(\|Y_{mn}\|)}{\Phi_{mn}(b_{mn})} \\
&\le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} E\frac{\Phi_{mn}(\|Y_{mn}\|)}{\Phi_{mn}(b_{mn})} \\
&\le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} E\frac{\Phi_{mn}(\|X_{mn}\|)}{\Phi_{mn}(b_{mn})} < \infty.
\end{aligned}$$

The proof is completed. □

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