KYUNGPOOK Math. J. 52(2012), 209-222
http://dx.doi.org/10.5666/KMJ.2012.52.2.209

Sandwich Results for Certain Subclasses of Multivalent Analytic Functions Defined by Srivastava–Attiya Operator

M. K. Aouf, A. Shamandy, A. O. Mostafa and Eman A. Adwan*
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
e-mail : mkaouf127@yahoo.com, shamandy16@hotmail.com, adelaeg254@yahoo.com *and* eman.a2009@yahoo.com

Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results involving the operator $J_{s,b}^{\lambda;p}$ for certain normalized p-valent analytic functions associated with that operator.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z : z \in \mathbb{C},\ |z| < 1\}$ and let $H[a,p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C}; p \in \mathbb{N} = \{1, 2, ...\}). \tag{1.1}$$

Also, let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N}), \tag{1.2}$$

and let $A_1 = A(1)$.

If f , $g \in A(p)$, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [5], [9] and [10]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

* Corresponding Author.
Received December 23, 2010; accepted September 23, 2011.
2010 Mathematics Subject Classification: 30C45.
Key words and phrases: Multivalent functions, differential subordination, superordination, sandwich theorems, Srivastava-Attiya operator.

Let $p, h \in H(U)$ and let $\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is a solution of the differential superordination (1.3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant of (1.3), if $q(z) \prec p(z)$ for all functions p satisfying (1.3). An univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all subordinants of (1.3) is called the best subordinant. Recently, Miller and Mocanu [11] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi\left(p(z), zp'(z), z^2p''(z); z\right) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [11], Bulboaca [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [1], have used the results of Bulboaca [4] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [20] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [17] obtained sufficient conditions for the normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They [17] also obtained results for functions defined by using Carlson-Shaffer operator.

For functions f given by (1.1) and $g \in A(p)$ given by $g(z) = z^p + \sum\limits_{k=1}^{\infty} b_{k+p} z^{k+p}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

We begin our investigation by recalling that a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (see [19])

$$\Phi(z,s,a)=\sum_{k=0}^{\infty}\frac{z^k}{(k+a)^s}\ , \tag{1.5}$$

$$a\in\mathbb{C}\backslash\mathbb{Z}_0^-=\{0,-1,-2,...\};\mathbb{Z}_0^-=\mathbb{Z}\backslash\mathbb{N},\mathbb{Z}=\left\{0,\pm 1,\pm 2,...\right\};s\in\mathbb{C}$$
$$when\ \ |z|<1;R\{s\}>1when\ \ |z|=1.$$

Recently, Srivastava and Attiya [18] (see also [8], [13] and [14]) introduced and investigated the linear operator $J_{s,b}(f):A_1\to A_1$, defined in terms of the Hadamard product by

$$J_{s,b}f(z)=G_{s,b}(z)*f(z)\ (z\in U;b\in\mathbb{C}\backslash\mathbb{Z}_0^-;s\in\mathbb{C}),$$

where for convenience,

$$G_{s,b}=(1+b)^s[\Phi(z,s,b)-b^{-s}]\ (z\in U).$$

In [21], Wang et al. defined the operator $J_{s,b}^{\lambda,p}:A\left(p\right)\to A\left(p\right)$ by

$$J_{s,b}^{\lambda,p}f(z)=f_{s,b}^{\lambda,p}(z)*f(z) \tag{1.6}$$

$$(z\in U;b\in\mathbb{C}\backslash\mathbb{Z}_0^-;s\in\mathbb{C};\lambda>-p;p\in\mathbb{N};f\in A\left(p\right)),$$

where

$$f_{s,b}^{p}(z)*f_{s,b}^{\lambda,p}(z)=\frac{z^p}{(1-z)^{\lambda+p}} \tag{1.7}$$

and

$$f_{s,b}^{p}(z)=z^p+\sum_{k=1}^{\infty}\left(\frac{p+k+b}{p+b}\right)z^{k+p}\qquad\left(z\in U;p\in\mathbb{N}\right). \tag{1.8}$$

It is easy to obtain from (1.6), (1.7) and (1.8) that

$$J_{s,b}^{\lambda,p}f(z)=z^p+\sum_{k=1}^{\infty}\frac{(\lambda+p)_k}{k!}\left(\frac{p+b}{k+p+b}\right)^s a_{k+p}z^{k+p}, \tag{1.9}$$

where $(\gamma)_k$, is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$(\gamma)_k=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & (k=0)\\ \gamma(\gamma+1)....(\gamma+k-1) & (k\in\mathbb{N}).\end{cases}$$

We note that

$$J_{0,b}^{1-p,p}f(z)=f(z)\ \ (f\in A(p))\,.$$

Using (1.9), it is easy to verify that (see [21])

$$z\left(J_{s+1,b}^{\lambda,p}f\right)'(z)=(p+b)J_{s,b}^{\lambda,p}(f)(z)-bJ_{s+1,b}^{\lambda,p}(f)(z) \tag{1.10}$$

and

$$z\left(J_{s,b}^{\lambda,p}f\right)'(z)=(p+\lambda)J_{s,b}^{\lambda+1,p}(f)(z)-\lambda J_{s,b}^{\lambda,p}(f)(z). \tag{1.11}$$

It should be remarked that the linear operator $J_{s,b}^{\lambda,p}f(z)$ is generalization of many other linear operators considered earlier. We have:
(1) $J_{0,b}^{\lambda,p}f(z)=D^{\lambda+p-1}f(z)$ $(\lambda>-p,p\in\mathbb{N})$, where $D^{\lambda+p-1}$ is the $(\lambda+p-1)$-th order Ruscheweyh derivative of a function $f(z)\in A(p)$ (see [7]);
(2) $J_{1,v}^{1-p,p}f(z)=J_{v,p}f(z)$ $(v>-p)$, where the generalized Bernardi-Libera-Livingston operator $J_{v,p}$ was studied by Choi et al. [6];
(3) $J_{m,0}^{1-p,p}f(z)=I_p^m f(z)=z^p+\sum_{k=1}^{\infty}\left(\frac{p}{k+p}\right)^m a_{k+p}z^{k+p}$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$, where for $p=1$ the integral operator $I_1^m=I^m$ was introduced and studied by Salagean [15];
(4) $J_{\sigma,1}^{1-p,p}f(z)=I_p^{\sigma}f(z)$ $(\sigma>0)$, where the integral operator I_p^{σ} was studied by Shams et al. [16] and Aouf et al. [2];
(5) $J_{\gamma,\tau}^{0,1}f(z)=P_\tau^\gamma f(z)$ $(\gamma\geq 0,\tau>1)$, where the integral operator P_τ^γ was introduced and studied by Patel and Sahoo [12].

In this paper, we obtain sufficient conditions for the normalized analytic function f defined by using the operator $J_{s,b}^{\lambda,p}$ to satisfy:

$$q_1(z)\prec\left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{\mu}\prec q_2(z)$$

and q_1 and q_2 are given univalent functions in U.

2. Definitions and preliminaries

In order to prove our results, we shall need the following definition and lemmas.

Definition 1([11]). Let Q be the set of all functions f that are analytic and injective on $\overline{U}\setminus E(f)$, where

$$E(f)=\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\},$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$.

Lemma 1([9]). *Let q be univalent in the unit disc U, and let θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w)\neq 0$ when $w\in q(U)$. Set*

$$Q(z)=zq'(z)\varphi(q(z)) \text{ and } h(z)=\theta(q(z))+Q(z) \tag{2.1}$$

suppose that
(i) *Q is a starlike function in U,*
(ii) $Re\left\{\dfrac{zh'(z)}{Q(z)}\right\}>0,\ z\in U.$
If p is analytic in U with $p(0)=q(0)$, $p(U)\subseteq D$ and

$$\theta(p(z))+zp'(z)\varphi(p(z))\prec\theta(q(z))+zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z)\prec q(z)$, and q is the best dominant of (2.2).

Lemma 2([4]). *Let q be a convex univalent function in U and θ and φ be analytic in a domain D containing $q(U)$. Suppose that*
(i) $Re\left\{\dfrac{\theta'(q(z))}{\varphi(q(z))}\right\}>0$ *for* $z\in U,$
(ii) $Q(z)=zq'(z)\varphi(q(z))$ *is starlike univalent in* U.
If $p\in H[q(0),1]\cap Q$, with $p(\mathrm{U})\subseteq D$, $\theta(p(z))+zp'(z)\varphi(p(z))$ is univalent in U, *and*

$$\theta(q(z))+zq'(z)\varphi(q(z))\prec\theta(p(z))+zp'(z)\varphi(p(z)), \tag{2.3}$$

then $q(z)\prec p(z)$, and q is the best subordinant of (2.3).

3. Applications to the operator $J_{s,b}^{\lambda,p}$ and sandwich theorems

Unless otherwise mentioned, we shall assume in the reminder of this paper that $b\in\mathbb{C}\backslash\mathbb{Z}_0^-, s\in\mathbb{C}, p\in\mathbb{N}, \lambda>-p,\ \gamma,\tau,\zeta\in\mathbb{C},\Omega,\mu\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}, z\in U$ and the powers are understood as principle values.

Theorem 1. *Let $q(z)$ be analytic and univalent in U with $q(z)\neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let*

$$Re\{1+\frac{\gamma}{\Omega}q(z)+\frac{2\zeta}{\Omega}\left(q(z)\right)^2-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\}>0, \tag{3.1}$$

and

$$\begin{aligned}\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)=&\tau+\gamma\left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{\mu}+\zeta\left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{2\mu}\\&+\Omega\mu\left(p-\frac{z\left(J_{s,b}^{\lambda,p}f(z)\right)'}{J_{s,b}^{\lambda,p}f(z)}\right).\end{aligned} \tag{3.2}$$

If q satisfies the following subordination:

$$\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)\prec\tau+\gamma q(z)+\zeta\left(q(z)\right)^2+\Omega\frac{zq'(z)}{q(z)}. \tag{3.3}$$

Then

$$\left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{\mu} \prec q(z) \tag{3.4}$$

and q is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z) = \left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{\mu} \quad (z \in U). \tag{3.5}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.10) in the resulting equation, we have

$$\tau + \gamma\left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{\mu} + \zeta\left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{2\mu} + \Omega\mu\left(p - \frac{z\left(J_{s,b}^{\lambda,p}f(z)\right)'}{J_{s,b}^{\lambda,p}f(z)}\right)$$

$$= \tau + \gamma p(z) + \zeta\left(p(z)\right)^2 + \Omega\tfrac{zp'(z)}{p(z)}. \tag{3.6}$$

Using (3.3) and (3.6), we have

$$\tau + \gamma p(z) + \zeta\left(p(z)\right)^2 + \Omega\tfrac{zp'(z)}{p(z)} \prec \tau + \gamma q(z) + \zeta\left(q(z)\right)^2 + \Omega\frac{zq'(z)}{q(z)}. \tag{3.7}$$

Setting

$$\theta(w) = \tau + \gamma w + \zeta w^2 \text{ and } \varphi(w) = \frac{\Omega}{w} \tag{3.8}$$

it can be easily observed that θ is analytic in $\mathbb{C}$, φ is analytic in $\mathbb{C}^*$ and $\varphi(w) \neq 0$ $(w \in \mathbb{C}^*)$. Hence, the result now follows by using Lemma 1. □

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 1, the condition (3.1) reduces to

$$Re\{1 + \frac{\gamma}{\Omega}\left(\frac{1+Az}{1+Bz}\right) + \frac{2\zeta}{\Omega}\left(\frac{1+Az}{1+Bz}\right)^2 - \frac{(A-B)z}{(1+Az)(1+Bz)} - \frac{2Bz}{1+Bz}\} > 0. \tag{3.9}$$

hence, we obtain the following corollary.

Corollary 1. *Let $f(z) \in A(p)$, assume that (3.9) holds true, $-1 \leq B < A \leq 1$ and*

$$\chi(f, s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma\left(\frac{1+Az}{1+Bz}\right) + \zeta\left(\frac{1+Az}{1+Bz}\right)^2 + \Omega\frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{3.10}$$

where $\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)$ *is given by (3.2), then*

$$\left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of (3.10).*

Taking $q(z)=\left(\frac{1+z}{1-z}\right)^{v}$ $(0<v\leq 1)$ in Theorem 1, the condition (3.1) reduces to

$$Re\{1+\frac{\gamma}{\Omega}\left(\frac{1+z}{1-z}\right)^{v}+\frac{2\zeta}{\Omega}\left(\frac{1+z}{1-z}\right)^{2v}-\frac{2z^2}{1-z^2}\}>0, \tag{3.11}$$

hence, we obtain the following corollary.

Corollary 2. *Let* $f(z)\in A(p)$*, assume that (3.11) holds true,* $0<v\leq 1$ *and*

$$\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)\prec \tau+\gamma\left(\frac{1+z}{1-z}\right)^{v}+\zeta\left(\frac{1+z}{1-z}\right)^{2v}+\Omega\frac{2vz}{(1-z)^2}, \tag{3.12}$$

where $\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)$ *is given by (3.2), then*

$$\left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^{\mu} \prec \left(\frac{1+z}{1-z}\right)^{v},$$

and $\left(\frac{1+z}{1-z}\right)^{v}$ *is the best dominant of (3.12).*

Putting $s=0$ *and* $\lambda=1-p\,(p\in\mathbb{N})$ in Theorem 1, we obtain the following corollary.

Corollary 3. *Let* $q(z)$ *be analytic and univalent in* U *with* $q(z)\neq 0$*. Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U*. If* $f(z)\in A(p)$*, assume that (3.1) holds true and*

$$G(f,p,\gamma,\tau,\zeta,\Omega,\mu)=\tau+\gamma\left(\frac{z^p}{f(z)}\right)^{\mu}+\zeta\left(\frac{z^p}{f(z)}\right)^{2\mu}+\Omega\mu\left(p-\frac{zf'(z)}{f(z)}\right). \tag{3.13}$$

If q *satisfies the following subordination:*

$$G(f,p,\gamma,\tau,\zeta,\Omega,\mu)\prec \tau+\gamma q(z)+\zeta\left(q(z)\right)^2+\Omega\frac{zq'(z)}{q(z)}. \tag{3.14}$$

Then

$$\left(\frac{z^p}{f(z)}\right)^{\mu}\prec q(z)$$

and q is the best dominant of (3.14).

Putting $p = 1$ in Corollary 3, we obtain the following corollary.

Corollary 4. *Let $q(z)$ be analytic and univalent in U with $q(z) \neq 0$. Suppose that $\frac{zq^{'}(z)}{q(z)}$ is starlike univalent in U. If $f(z) \in A$, assume that (3.1) holds true and*

$$K(f, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left(\frac{z}{f(z)} \right)^{\mu} + \zeta \left(\frac{z}{f(z)} \right)^{2\mu} + \Omega\mu \left(1 - \frac{zf^{'}(z)}{f(z)} \right). \tag{3.15}$$

If q satisfies the following subordination:

$$K(f, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta \left(q(z)\right)^2 + \Omega \frac{zq^{'}(z)}{q(z)}. \tag{3.16}$$

Then

$$\left(\frac{z}{f(z)} \right)^{\mu} \prec q(z)$$

and q is the best dominant of (3.16).

Putting $s = 0$ in Theorem 1, we obtain the following corollary.

Corollary 5. *Let $q(z)$ be analytic and univalent in U with $q(z) \neq 0$. Suppose that $\frac{zq^{'}(z)}{q(z)}$ is starlike univalent in U. If $f(z) \in A(p)$, assume that (3.1) holds true and*

$$\begin{aligned} &D(f, p, \lambda, \gamma, \tau, \zeta, \Omega, \mu) \\ &= \tau + \gamma \left(\tfrac{z^p}{D^{\lambda+p-1}f(z)} \right)^{\mu} + \zeta \left(\tfrac{z^p}{D^{\lambda+p-1}f(z)} \right)^{2\mu} + \Omega\mu \left(p - \tfrac{z\left(D^{\lambda+p-1}f(z)\right)^{'}}{D^{\lambda+p-1}f(z)} \right). \end{aligned} \tag{3.17}$$

If q satisfies the following subordination:

$$D(f, p, \lambda, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta \left(q(z)\right)^2 + \Omega \frac{zq^{'}(z)}{q(z)}, \tag{3.18}$$

then

$$\left(\frac{z^p}{D^{\lambda+p-1}f(z)} \right)^{\mu} \prec q(z)$$

and q is the best dominant of (3.18).

Putting $s = 1, b = v\,(v > -p)$ *and* $\lambda = 1 - p\,(p \in \mathbb{N})$ in Theorem 1, we obtain the following corollary.

Corollary 6. *Let $q(z)$ be analytic and univalent in U with $q(z) \neq 0$. Suppose that $\frac{zq^{'}(z)}{q(z)}$ is starlike univalent in U. If $f(z) \in A(p)$, assume that (3.1) holds true and*

$$(f, v, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left(\tfrac{z^p}{J_{v,p}f(z)} \right)^{\mu} + \zeta \left(\tfrac{z^p}{J_{v,p}f(z)} \right)^{2\mu} + \Omega\mu \left(p - \tfrac{z(J_{v,p}f(z))^{'}}{J_{v,p}f(z)} \right). \tag{3.19}$$

If q satisfies the following subordination:

$$(f, v, p, \beta, \delta, \alpha, \eta, \mu) \prec \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq^{'}(z)}{q(z)}, \tag{3.20}$$

then

$$\left(\frac{z^p}{J_{v,p} f(z)} \right)^{\mu} \prec q(z)$$

and q is the best dominant of (3.20).

Putting $s = m\,(m \in \mathbb{N}_0)\,, b = 0$ *and* $\lambda = 1 - p\,(p \in \mathbb{N})$ in Theorem 1, we obtain the following corollary.

Corollary 7. *Let $q(z)$ be analytic and univalent in U with $q(z) \neq 0$. Suppose that $\frac{zq^{'}(z)}{q(z)}$ is starlike univalent in U. If $f(z) \in A(p)$, assume that (3.1) holds true and*

$$S(f, m, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left(\frac{z^p}{I_p^m f(z)} \right)^{\mu} + \zeta \left(\frac{z^p}{I_p^m f(z)} \right)^{2\mu} + \Omega\mu \left(p - \frac{z\left(I_p^m f(z)\right)^{'}}{I_p^m f(z)} \right). \tag{3.21}$$

If q satisfies the following subordination:

$$S(f, m, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq^{'}(z)}{q(z)}, \tag{3.22}$$

then

$$\left(\frac{z^p}{I_p^m f(z)} \right)^{\mu} \prec q(z)$$

and q is the best dominant of (3.22).

Putting $s = \sigma\,(\sigma > 0)\,, b = 1$ *and* $\lambda = 1 - p\,(p \in \mathbb{N})$ in Theorem 1, we obtain the following corollary.

Corollary 8. *Let $q(z)$ be analytic and univalent in U with $q(z) \neq 0$. Suppose that $\frac{zq^{'}(z)}{q(z)}$ is starlike univalent in U. If $f(z) \in A(p)$, assume that (3.1) holds true and*

$$\varphi(f, \sigma, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left(\frac{z^p}{I_p^\sigma f(z)} \right)^{\mu} + \zeta \left(\frac{z^p}{I_p^\sigma f(z)} \right)^{2\mu} + \Omega\mu \left(p - \frac{z\left(I_p^\sigma f(z)\right)^{'}}{I_p^\sigma f(z)} \right). \tag{3.23}$$

If q satisfies the following subordination:

$$\varphi(f, \sigma, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq^{'}(z)}{q(z)}, \tag{3.24}$$

then

$$\left(\frac{z^p}{I_p^\sigma f(z)} \right)^{\mu} \prec q(z)$$

and q is the best dominant of (3.24).

Theorem 2. *Let q be a convex univalent function in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$\text{(3.25)} \qquad Re\left\{\frac{2\zeta}{\Omega}(q(z))^2 + \frac{\gamma}{\Omega}q(z)\right\} > 0.$$

If $f \in A(p), 0 \neq \left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^\mu \in H[q(0),1] \cap Q$, $\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)$ univalent in U, and

$$\text{(3.26)} \qquad \tau + \gamma q(z) + \zeta(q(z))^2 + \Omega\frac{zq'(z)}{q(z)} \prec \chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu),$$

where $\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)$ is given by (3.2), then

$$q(z) \prec \left(\frac{z^p}{J_{s,b}^{\lambda,p}f(z)}\right)^\mu,$$

and q is the best subordinant of (3.26).

Proof. Taking

$$\theta(w) = \tau + \gamma w + \zeta w^2 \text{ and } \varphi(w) = \frac{\Omega}{w},$$

it is easily observed that θ is analytic in $\mathbb{C}$, φ is analytic in $\mathbb{C}^*$ and $\varphi(w) \neq 0 \ (w \in \mathbb{C}^*)$. Since q is a convex (univalent) function it follows that

$$Re\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} = Re\left\{\frac{2\zeta}{\Omega}(q(z))^2 + \frac{\gamma}{\Omega}q(z)\right\}q'(z) > 0.$$

Thus the assertion (3.26) of Theorem 2 follows by an application of Lemma 2. □

Putting $s = 0$ *and* $\lambda = 1 - p\,(p \in N)$ in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

Corollary 9. *Let q be a convex univalent function in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. If $f \in A(p), 0 \neq \left(\frac{z^p}{f(z)}\right)^\mu \in H[q(0),1] \cap Q$ and $G(f,p,\gamma,\tau,\zeta,\Omega,\mu)$univalent in U, where $G(f,p,\gamma,\tau,\zeta,\Omega,\mu)$ is given by (3.13), then*

$$\text{(3.27)} \qquad \tau + \gamma q(z) + \zeta(q(z))^2 + \Omega\frac{zq'(z)}{q(z)} \prec G(f,p,\gamma,\tau,\zeta,\Omega,\mu),$$

implies

$$q(z) \prec \left(\frac{z^p}{f(z)}\right)^\mu$$

and qis the best dominant of (3.27).

Putting $s = 0$ in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

Corollary 10. *Let q be a convex univalent function in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. If $f \in A(p), 0 \neq \left(\frac{z^p}{D^{\lambda+p-1}f(z)}\right)^\mu \in H[q(0),1] \cap Q$ and $D(f,p,\lambda,\gamma,\tau,\zeta,\Omega,\mu)$ univalent in U, where $D(f,p,\lambda,\gamma,\tau,\zeta,\Omega,\mu)$ is given by (3.17), then*

$$\tau + \gamma q(z) + \zeta\,(q(z))^2 + \Omega\frac{zq'(z)}{q(z)} \prec D(f,p,\lambda,\gamma,\tau,\zeta,\Omega,\mu), \tag{3.28}$$

implies

$$q(z) \prec \left(\frac{z^p}{D^{\lambda+p-1}f(z)}\right)^\mu$$

and q is the best dominant of (3.28).

Putting $s = 1, b = v\,(v > -p)$ *and* $\lambda = 1 - p\,(p \in \mathbb{N})$ in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

Corollary 11. *Let q be a convex univalent function in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. If $f \in A(p), 0 \neq \left(\frac{z^p}{J_{v,p}f(z)}\right)^\mu \in H[q(0),1] \cap Q$ and $(f,v,p,\gamma,\tau,\zeta,\Omega,\mu)$ univalent in U, where $(f,v,p,\gamma,\tau,\zeta,\Omega,\mu)$ is given by (3.19), then*

$$\tau + \gamma q(z) + \zeta\,(q(z))^2 + \Omega\frac{zq'(z)}{q(z)} \prec (f,v,p,\gamma,\tau,\zeta,\Omega,\mu), \tag{3.29}$$

implies

$$q(z) \prec \left(\frac{z^p}{J_{v,p}f(z)}\right)^\mu$$

and q is the best dominant of (3.29).

Putting $s = m\,(m \in \mathbb{N}_0), b = 0$ *and* $\lambda = 1 - p\,(p \in \mathbb{N})$ in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

Corollary 12. *Let q be a convex univalent function in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. If $f \in A(p), 0 \neq \left(\frac{z^p}{I_p^m f(z)}\right)^\mu \in H[q(0),1] \cap Q$ and $S(f,m,p,\gamma,\tau,\zeta,\Omega,\mu)$ univalent in U, where $S(f,m,p,\gamma,\tau,\zeta,\Omega,\mu)$ is given by (3.21), then*

$$\tau + \gamma q(z) + \zeta\,(q(z))^2 + \Omega\frac{zq'(z)}{q(z)} \prec S(f,m,p,\gamma,\tau,\zeta,\Omega,\mu), \tag{3.30}$$

implies

$$q(z) \prec \left(\frac{z^p}{I_p^m f(z)}\right)^\mu$$

and q is the best dominant of (3.30).

Putting $s = \sigma\,(\sigma > 0)\,, b = 1$ *and* $\lambda = 1 - p\,(p \in \mathbb{N})$ in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

Corollary 13. *Let q be a convex univalent function in U, $q(z) \neq 0$ and $\frac{zq^{'}(z)}{q(z)}$ be starlike univalent in U. If $f \in A(p), 0 \neq \left(\frac{z^p}{I_p^\sigma f(z)}\right)^\mu \in H[q(0),1] \cap Q$ and $\varphi(f,\sigma,p,\gamma,\tau,\zeta,\Omega,\mu)$ univalent in U, where $\varphi(f,\sigma,p,\gamma,\tau,\zeta,\Omega,\mu)$ is given by (3.23), then*

$$\tau + \gamma q(z) + \zeta\,(q(z))^2 + \Omega\frac{zq^{'}(z)}{q(z)} \prec \varphi(f,\sigma,p,\gamma,\tau,\zeta,\Omega,\mu), \tag{3.31}$$

implies

$$q(z) \prec \left(\frac{z^p}{I_p^\sigma f(z)}\right)^\mu$$

and q is the best dominant of (3.31).

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.

Theorem 3. *Let q_1 be convex univalent in U and q_2 be univalent in U,$q_1 \neq 0$ and $q_2 \neq 0$ in U. Suppose that q_1 and q_2 satisfies (3.1) and (3.25), respectively. If $f \in A(p), \left(\frac{z^p}{J_{s,b}^{\lambda,p} f(z)}\right)^\mu \in H[q(0),1] \cap Q$ and $\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)$ is univalent in U, where $\chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu)$ is defined in* (3.2), *then*

$$\tau + \gamma q_1(z) + \zeta\,(q_1(z))^2 + \Omega\frac{zq_1^{'}(z)}{q_1(z)} \prec \chi(f,s,b,\lambda,p,\gamma,\tau,\zeta,\Omega,\mu) \prec \tau + \gamma q_2(z) + \zeta\,(q_2(z))^2 + \Omega\frac{zq_2^{'}(z)}{q_2(z)}, \tag{3.32}$$

implies

$$q_1(z) \prec \left(\frac{z^p}{J_{s,b}^{\lambda,p} f(z)}\right)^\mu \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and dominant of (3.32).

Putting $s = 0$ *and* $\lambda = 1 - p\,(p \in \mathbb{N})$ in Theorem 3, we obtain the following corollary.

Corollary 14. *Let q_1 be convex univalent in U and q_2 univalent in U, $q_1 \neq 0$ and*

$q_2 \neq 0$ *in* U. *Suppose that* q_1 *and* q_2 *satisfies (3.1) and (3.25), respectively. If* $f \in A(p)$, $\left(\frac{z^p}{f(z)}\right)^\mu \in H[q(0),1] \cap Q$ *and* $G(f,p,\gamma,\tau,\zeta,\Omega,\mu)$ *is univalent in* U, *where* $G(f,p,\gamma,\tau,\zeta,\Omega,\mu)$ *is defined in* (3.13), *then*

$$\tau + \gamma q_1(z) + \zeta\left(q_1(z)\right)^2 + \Omega\frac{zq_1'(z)}{q_1(z)} \prec G(f,p,\gamma,\tau,\zeta,\Omega,\mu) \prec \tau + \gamma q_2(z) + \zeta\left(q_2(z)\right)^2 + \Omega\frac{zq_2'(z)}{q_2(z)}, \tag{3.33}$$

implies

$$q_1(z) \prec \left(\frac{z^p}{f(z)}\right)^\mu \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and dominant of (3.33).*

Remark. Combining: (1) Corollary 5 and Corollary 10; (2) Corollary 6 and Corollary 11; (3) Corollary 7 and Corollary 12; (4)Corollary 8 and Corollary 13, we obtain similar sandwich theorems for the corresponding operators.

Acknowledgements The authors would like to thank the referees of the paper for their helpful suggestions.

References

[1] R. M. Ali, V. Ravichandran and K. G. Subramanian, *Differential sandwich theorems for certain analytic functions*, Far East J. Math., Sci., **15**(2004), 87-94.

[2] M. K. Aouf , T. Bulboaca and A. O. Mostafa, *Subordination properties of subclasses of p-valent functions involving certain operators*, Publ. Math. Debrecen, **73**(2008), 401-416.

[3] T. Bulboacă, *A class of superordination-preserving integral operators*, Indag. Math. (N. S.), **13**(2002), 301-311.

[4] T. Bulboacă, *Classes of first order differential superordinations*, Demonstratio Math., **35**(2002), 287-292.

[5] T. Bulboacă, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[6] J. H. Choi, M. Saigo and H. M. Srivastava, *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl., **276**(2002), 432-445.

[7] V. Kumar and S. L. Shakla, *Multivalent functions defined by Ruscheweyh derivatives, I*, Indian J. Pure Appl. Math., **15**(1984), 1216-1227; II, Indian J. Pure Appl. Math., **15**(1984), 1228-1238.

[8] J. L. Liu, *Subordinations for certain multivalent analytic functions associated with the generalized Srivastava-Attiya operator*, Integral Transforms Spec. Funct., **18**(2007), 207-216.

[9] S. S. Miller and P. T. Mocanu, Differential Subordination : Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[10] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), 157-171.

[11] S. S. Miller and P. T. Mocanu, *Subordinates of differential superordinations*, Complex Variables, **48**(2003), 815-826.

[12] J. Patel and P. Sahoo, *Som applications of dierential subordination to certain one-parameter families of integral operators*, Indian J. Pure Appl. Math., **35**(2004), 1167-1177.

[13] J. K. Prajapat and S. P. Goyal, *Applications of Srivastava–Attiya operator to the classes of strongly starlike and strongly convex functions*, J. Math. Inequal., **3**(2009), 129-137.

[14] D. Raducanu and H. M. Srivastava, *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz–Lerch Zeta function*, Integral Transforms Spec. Funct., **18**(2007), 933-943.

[15] G. S. Sălăgean, Subclasses of univalent functions, Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1(Bucharest, 1981), Lecture Notes in Math. (Springer-Verlag), 1013, 362-372.

[16] S. Shams, S. R. Kulkarni and Jay M. Jahangiri, *Subordination properties of p-valent functions defined by integral operators*, Internat. J. Math. Math. Sci., (2006), Art. ID 94572, 1-3.

[17] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, *Differantial sandwich theorems for some subclasses of analytic functions*, J. Austr. Math. Anal. Appl., **3**(2006), Art. 8, 1-11.

[18] H. M. Srivastava and A. A. Attiya, *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transforms Spec. Funct., **18**(2007), 207-216.

[19] H. M. Srivastava and J. Choi, Series associated with the Zeta and related functions, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.

[20] N. Tuneski, *On certain sufficient conditions for starlikeness*, Internat. J. Math. Math. Sci., **23**(2000), 521-527.

[21] Z. G. Wang, Q. G. Li and Y. P. Jiang, *Certain subclasses of multivalent analytic functions involving the generalized Srivastava–Attiya operator*, Integral Transforms Spec. Funct., **21**(2010), 221-234.