

## On Rings Containing a Non-essential $nil$ -Injective Maximal Left Ideal

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ABSTRACT. We investigate in this paper rings containing a non-essential  $nil$ -injective maximal left ideal. We show that if  $R$  is a left  $MC2$  ring containing a non-essential  $nil$ -injective maximal left ideal, then  $R$  is a left  $nil$ -injective ring. Using this result, some known results are extended.

### 1. Introduction

Throughout this paper,  $R$  denotes an associative ring with identity and all modules are unitary. For any nonempty subset  $X$  of a ring  $R$ ,  $r(X) = r_R(X)$  and  $l(X) = l_R(X)$  denote the right annihilators of  $X$  and the left annihilators of  $X$ , respectively. We write  $J(R)$ ,  $P(R)$ ,  $Z_l(R)$  ( $Z_r(R)$ ),  $N(R)$ ,  $U(R)$ ,  $E(R)$ ,  $Soc({}_R R)$  and  $Soc(R_R)$  for the Jacobson radical, the prime radical, the left (right) singular ideal, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements, the left socle and the right socle of  $R$ , respectively. It is well-known that every maximal left ideal of a ring  $R$  is injective if and only if  $R$  is semisimple Artinian. Osofsky [8] showed that if  $R$  is a left self-injective left hereditary ring, then  $R$  is semisimple Artinian. Based on these results, Yue chi ming [19] proposed the following question: If  $R$  is a left hereditary ring containing an injective maximal left ideal, is  $R$  semisimple Artinian? However, Zhang and Du [20] constructed a counterexample to settle in the negative, and then they proved that a ring  $R$  is semiprime left hereditary containing an injective maximal left ideal if and only if  $R$  is semisimple Artinian. As the same direction to Zhang and Du, Kim [4] showed that if  $R$  is a semiprime ring containing a finitely generated  $p$ -injective maximal left ideal, then  $R$  is a left  $p$ -injective ring. And Kim and Baik [3] showed that If  $R$  is a left idempotent reflexive ring containing an injective maximal left

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ideal, then  $R$  is a left self-injective.

We investigate in this paper rings containing a non-essential  $nil$ -injective maximal left ideal. Using left  $MC2$  rings, we show that if  $R$  is a left  $MC2$  ring containing a non-essential  $nil$ -injective maximal left ideal, then  $R$  is a left  $nil$ -injective ring. As a byproduct of this result, we obtain a new characterization of regular left self-injective rings with nonzero socle. This characterization is then used to prove that left  $MC2$  left  $HI$ -rings are semisimple Artinian. Consequently we extend nontrivially some results appeared in [3], [4] and [20].

An element  $k$  of  $R$  is called left minimal if  $Rk$  is a minimal left ideal of  $R$ . An element  $e$  of  $R$  is called left minimal idempotent if  $e^2 = e$  is left minimal. Similarly, the notion of right minimal (idempotent) element can be defined. We denote  $M_l(R)$ ,  $ME_l(R)$ ,  $M_r(R)$  and  $ME_r(R)$  for the set of left minimal elements, the set of left minimal idempotent elements, the set of right minimal elements and the set of right minimal idempotent elements of  $R$ , respectively. A ring  $R$  is called left  $MC2$  if every minimal left ideal which is isomorphic to a summand of  ${}_R R$  is a summand. Left  $MC2$  rings were initiated by Nicholson and Yousif in [6], related to the left mininjective rings. Where a ring  $R$  is called left mininjective if  $rl(k) = kR$  for every  $k \in M_l(R)$ . A ring  $R$  is said to be left minsymmetric [6] if  $M_l(R) \subseteq M_r(R)$ , and  $R$  is left universally mininjective [6] if for any  $k \in M_l(R)$ ,  $Rk$  is a summand of  ${}_R R$ . According to [6], left universally mininjective  $\implies$  left mininjective  $\implies$  left minsymmetric  $\implies$  left  $MC2$  and the converse are not true.

A ring  $R$  is called left min-abel [12] if for each  $e \in ME_l(R)$ ,  $e$  is left semicentral in  $R$ , and  $R$  is said to be strongly left min-abel [12] if every element of  $ME_l(R)$  is contained in the central of  $R$ . [12, Theorem 1.8] showed that  $R$  is strongly left min-abel if and only if  $R$  is left  $MC2$  left min-abel.

A ring  $R$  is called reflexive [5] if  $aRb = 0$  implies  $bRa = 0$  for all  $a, b \in R$ , and  $R$  is said to be left idempotent reflexive [3] if  $aRe = 0$  implies  $eRa = 0$  for all  $a \in R$  and  $e \in E(R)$ . Clearly, semiprime  $\implies$  reflexive  $\implies$  left idempotent reflexive  $\implies$  left  $MC2$ . But the converse are not true.

$R$  is called reduced if it contains no nonzero nilpotent element. Clearly, reduced  $\implies$  semiprime  $\implies$  left universally mininjective.

Recall that a ring  $R$  is left  $NPP$  [14] if for any  $a \in N(R)$ ,  ${}_R Ra$  is projective, and  $R$  is said to be  $n$ -regular if  $a \in aRa$  for all  $a \in N(R)$ . A left  $R$ -module  $M$  is called left  $nil$ -injective [14] if for any  $a \in N(R)$  and every left  $R$ -homomorphism from  $Ra$  to  $M$  extends to one from  ${}_R R$  to  $M$ . If  ${}_R R$  is  $nil$ -injective, then  $R$  is called left  $nil$ -injective ring. According to [14]  $R$  is a  $n$ -regular ring if and only if every cyclic left  $R$ -module is  $nil$ -injective if and only if  $R$  is a left  $nil$ -injective left  $NPP$  ring.

## 2. Main results

We start with the following lemma.

**Lemma 2.1.** (1) *The following conditions are equivalent for a left  $R$ -module  $M$ :*

- (a)  $M$  is *nil*-injective;
  - (b) For any  $a \in N(R)$ ,  $r_M l_R(a) = aM$ ;
  - (c) For any  $a \in N(R)$  and  $m \in M$ ,  $l_R(a)m = 0$  implies  $m \in aM$ ;
  - (d) For any  $a \in N(R)$  and  $m \in M$ , if  $m \notin aM$ , then there exists  $c \in R$  such that  $ca = 0$  and  $cm \neq 0$ .
- (2) Let  $M_1$  and  $M_2$  be left  $R$ -modules. Then  $M_1$  and  $M_2$  are *nil*-injective if and only if  $M_1 \oplus M_2$  is *nil*-injective.

*Proof.* It is routine. □

**Lemma 2.2.** *Let  $R$  be a ring and  $e \in ME_l(R)$ . If  $e$  is central in  $R$ , then  ${}_R Re$  is injective.*

*Proof.* Let  $e \in ME_l(R)$  be contained in the central of  $R$  and  $I$  a left ideal of  $R$  with a nonzero  $R$ -homomorphism  $f : I \rightarrow Re$ . Since  ${}_R Re$  is projective,  $I = \ker f \oplus V$ , where  $V \cong Re$ . Let  $\sigma$  be the isomorphism and let  $0 \neq v \in V$  such that  $\sigma(v) = e$ , then  $V = Rv$  and  $v = ev$ . Since  $e$  is central in  $R$ ,  $v = ev = ve$ , which implies  $Re = Rv = V$ . Therefore  $I = \ker f \oplus Re$ . Now for any  $x \in I$ , let  $x = y + z$ , where  $y \in \ker f$  and  $z = ze \in Re$ . Since  $yf(e) = f(ye) = f(ey) = ef(y) = 0$ ,  $f(x) = f(y) + f(z) = f(z) = f(ze) = zf(e) = yf(e) + zf(e) = (y + z)f(e) = xf(e)$ . Hence  ${}_R Re$  is injective. □

[3, Proposition 5] showed that if  $R$  is a left idempotent reflexive ring containing an injective maximal left ideal, then  $R$  is a left self-injective ring. [4, Theorem 2] showed that if  $R$  is a semiprime ring containing a finitely generated  $p$ -injective maximal left ideal, then  $R$  is a left  $p$ -injective ring. We have the following theorem.

**Theorem 2.3.** *Let  $R$  be a left MC2 ring. If  $R$  contains a non-essential *nil*-injective maximal left ideal  $M$ , then  $R$  is a left *nil*-injective ring.*

*Proof.* Let  $M$  be a non-essential *nil*-injective maximal left ideal of  $R$ . Then  $R = M \oplus L$ , where  $L = Re$  is a minimal left ideal of  $R$  and  $M = R(1 - e)$ . If  $ML = 0$ , then  $M = l(L)$  since  $M$  is maximal. So  $M$  is an ideal of  $R$ . Since  $R$  is left MC2 and  $(1 - e)Re = ML = 0$ , by [13, Theorem 1.3],  $eR(1 - e) = 0$ . Hence  $e$  is a central idempotent of  $R$ . By Lemma 2.2,  ${}_R Re$  is injective. Now suppose that  $ML \neq 0$ . Then there exists  $u \in L$  such that  $Mu \neq 0$ , whence  $L = Mu$ . Let  $f : M \rightarrow L$  be the map defined by  $f(x) = xu$  for each  $x \in M$ . Since  $f$  is an epimorphism and  ${}_R L$  is projective,  $M \cong \ker f \oplus L$ . Hence, by Lemma 2.1(2),  ${}_R L$  is *nil*-injective. In any case,  ${}_R L$  is *nil*-injective. By Lemma 2.1(2),  $R$  is left *nil*-injective. □

According to [15], a ring  $R$  is called  $n$ -regular if for any  $a \in N(R)$ ,  $a = aba$  for some  $b \in R$  and a right  $R$ -module  $M$  is said to be Nflat if for any  $a \in N(R)$ , the map  $1_M \otimes i : M \otimes_R Ra \rightarrow M \otimes_R R$  is monic, where  $i : Ra \hookrightarrow R$  is the inclusion map. [15, Theorem 4.7] showed that  $R$  is  $n$ -regular if and only if every cyclic right  $R$ -module is Nflat.  $R$  is said to be *abelian* if every idempotent element of  $R$  is central, and  $R$  is said to be semiabelian [21] if every idempotent element of  $R$  is

either left semicentral or right semicentral. A ring  $R$  is called quasi-normal [16] if for any  $e \in E(R)$ ,  $eR(1 - e)Re = 0$ . A ring  $R$  is called left min-abel if every element of  $ME_l(R)$  is left semicentral. [16, Theorem 2.5] showed that quasi-normal rings are left min-abel. [16, Theorem 2.6] showed that  $R$  is *abelian* if and only if  $R$  is quasi-normal left idempotent reflexive. Clearly, *abelian*  $\implies$  *semiabelian*  $\implies$  quasi-normal.

**Theorem 2.4.** (1) *A ring  $R$  is  $n$ -regular if and only if every cyclic singular right  $R$ -module is  $Nflat$ .*

(2) *The following conditions are equivalent for a ring  $R$ .*

(a)  *$R$  is reduced.*

(b)  *$R$  is  $n$ -regular and quasi-normal.*

(c)  *$R$  is left  $NPP$  and for any  $a \in R$ ,  $al(a) = 0$ .*

(3) *Let  $M$  be a maximal left ideal of  $R$  which is also an ideal. Then  $R/M$  is  $Nflat$  as right  $R$ -module if and only if  $R/M$  is *nil*-injective as left  $R$ -module.*

*Proof.* (1) The necessity is evident by [15, Theorem 4.7].

The sufficiency part: Let  $a \in N(R)$  and  $T$  be a complement right ideal such that  $aR \oplus T$  is essential in  $R$ . Then  $R/(aR \oplus T)$  is  $Nflat$  by hypothesis. Since  $a \in N(R)$  and  $a \in aR \oplus T$ ,  $a = (ac + t)a$  for some  $c \in R$  and  $t \in T$  by [15, Theorem 4.6]. Since  $ta = a - aca \in T \cap aR = 0$ ,  $a = aca$ . Hence  $R$  is  $n$ -regular.

(2) (a)  $\implies$  (b) and (a)  $\implies$  (c) are trivial.

(b)  $\implies$  (a) Let  $a \in R$  such that  $a^2 = 0$ . Then  $a = aba$  for some  $b \in R$ . Let  $e = ba$ , then  $e \in E(R)$ ,  $a = ae$ ,  $ea = 0$  and  $Ra = Re$ . Therefore  $eRe = eRa = eR(1 - e)ae \subseteq eR(1 - e)Re$ . Since  $R$  is quasi-normal,  $eR(1 - e)Re = 0$ , which implies  $eRe = 0$ . Hence  $e = 0$  and so  $a = 0$ .

(c)  $\implies$  (a) Let  $a \in R$  with  $a^2 = 0$ . Since  $R$  is left  $NPP$ ,  $l(a) = Re$  for some  $e \in E(R)$ . Hence  $a = ae \in aRe$  because  $a \in l(a)$ . By hypothesis,  $aRe = al(a) = 0$ , which implies  $a = 0$ .

(3) Assume that  $R/M$  is  $Nflat$  right  $R$ -module and  $a \in N(R)$  with a nonzero  $R$ -homomorphism  $f : Ra \rightarrow R/M$ . If  $a \in M$ , then by [15, Theorem 4.6],  $a = ma$  for some  $m \in M$ . Hence  $f(a) = f(ma) = mf(a) = 0$  because  $M$  is an ideal of  $R$ . This is impossible because  $f \neq 0$ . So  $a \notin M$ , which implies  $Ra + M = R$ . Since  $R/M$  is division ring,  $aR + M = R$ . Let  $ab + m = 1$  for some  $b \in R$  and  $m \in M$ . Hence  $abf(a) = (1 - m)f(a) = f(a) - mf(a) = f(a)$  because  $mf(a) = 0$ . This shows that  $R/M$  is *nil*-injective left  $R$ -module.

Conversely, assume that  $R/M$  is *nil*-injective left  $R$ -module and  $a \in N(R)$ . Then  $Ma \subseteq M \cap Ra$ . Now let  $x = ca \in M \cap Ra$  where  $c \in R$ . If  $l(a) \not\subseteq M$ , then  $l(a) + M = R$ . Let  $1 = y + m$ , where  $y \in l(a)$  and  $m \in M$ . So  $a = ma$  and  $x = ca = cma \in Ma$ . If  $l(a) \subseteq M$ , then the left  $R$ -homomorphism  $f : Ra \rightarrow R/M$  defined by  $f(ra) = r + M, r \in R$  can be extended to  $R \rightarrow R/M$ , which implies there exists  $b \in R$  such that  $1 - ab \in M$ . Since  $cab = xb \in M$ ,  $c = c - cab + cab = c(1 - ab) + cab \in M$ . Therefore  $x = ca \in Ma$ . In any case, we obtain  $M \cap Ra = Ma$ . By [15, Theorem 4.6],  $R/M$  is  $Nflat$  right  $R$ -module.  $\square$

As an application of Theorem 2.3, we have the following result.

**Theorem 2.5.** *The following conditions are equivalent for a ring  $R$ :*

- (1)  $R$  is a  $n$ -regular ring with nonzero left socle.
- (2)  $R$  is a left MC2 left NPP ring containing a non-essential  $nil$ -injective maximal left ideal.
- (3)  $R$  is a left idempotent reflexive left NPP ring containing a non-essential  $nil$ -injective maximal left ideal.
- (4)  $R$  is a reflexive left NPP ring containing a non-essential  $nil$ -injective maximal left ideal.
- (5)  $R$  is a semiprime left NPP ring containing a non-essential  $nil$ -injective maximal left ideal.
- (6)  $R$  is a left universally mininjective left NPP ring containing a non-essential  $nil$ -injective maximal left ideal.
- (7)  $R$  is a left mininjective left NPP ring containing a non-essential  $nil$ -injective maximal left ideal.
- (8)  $R$  is a left minsymmetric left NPP ring containing a non-essential  $nil$ -injective maximal left ideal.
- (9)  $Soc({}_R R) \subseteq Soc(R_R)$  and  $R$  is a left NPP ring containing a non-essential  $nil$ -injective maximal left ideal.
- (10)  $R$  is a left NPP ring containing a non-essential  $nil$ -injective maximal left ideal which is  $n$ -regular.

*Proof.* (5)  $\implies$  (4)  $\implies$  (3)  $\implies$  (2); (5)  $\implies$  (6)  $\implies$  (7)  $\implies$  (8)  $\implies$  (9) and (1)  $\implies$  (10) are trivial.

(9)  $\implies$  (2) Let  $k \in PM_l(R)$ . Then  $l(k) = l(e)$  for some  $e \in ME_l(R)$ . If  $(Rk)^2 = 0$ , then  $RkR \subseteq l(k)$ . Let  $T$  be a right ideal of  $R$  such that  $RkR \cap T = 0$  and  $RkR \oplus T$  is an essential right ideal of  $R$ . Hence  $e \in Soc({}_R R) \subseteq Soc(R_R) \subseteq RkR \oplus T \subseteq l(k) = l(e)$ , which is a contradiction. Hence  $(Rk)^2 \neq 0$ , which implies  $R$  is left MC2.

(2)  $\implies$  (1) By (2) and Theorem 2.3,  $R$  is a left  $nil$ -injective ring with  $Soc({}_R R) \neq 0$ . Since  $R$  is left NPP,  $R$  is  $n$ -regular by [14, Theorem 2.18].

(1)  $\implies$  (5) Suppose that  $R$  is a  $n$ -regular ring with nonzero left socle. By [14, Theorem 2.18],  $R$  is a semiprime left NPP ring. Since  $Soc({}_R R) \neq 0$ , there exists  $k \in M_l(R)$ , If  $k^2 \neq 0$ , then  $(Rk)^2 \neq 0$ , hence  $Rk = Re, e \in ME_l(R)$ . If  $k^2 = 0$ ,  $k = kbk$  for some  $b \in R$  because  $R$  is a  $n$ -regular ring. Hence  $Rk = Re$ , where  $e = bk \in ME_l(R)$ . In any case, we have  $R(1 - e)$  is a non-essential  $nil$ -injective maximal left ideal of  $R$  by [14, Theorem 2.18].

(10)  $\implies$  (1) Let  $M$  be a non-essential  $nil$ -injective maximal left ideal which is  $n$ -regular. Then  $R = M \oplus R(1 - e)$  for some  $e \in ME_l(R)$ . So  $M = Re$  and  $R$  has a nonzero socle because  $R(1 - e)$  is minimal left ideal of  $R$ .

If  $eR(1 - e) = 0$ , then  $M = l(R(1 - e))$  is an ideal of  $R$ . Now we show that  $(1 - e)Re = 0$ . Otherwise there exists  $a \in R$  such that  $h = ae - eae \neq 0$ . Clearly,  $he = h, eh = 0, h^2 = 0$  and  $h \in M$ . Since  $M$  is  $n$ -regular, there exists  $c \in R$  such that  $h = hch$ . Since  $eR(1 - e) = 0, hch = hec(1 - e)h = 0$ , so  $h = 0$  which

is a contradiction. Hence  $(1 - e)Re = 0$ , which implies  $e$  is central. Now for any  $u \in N(R)$ , since  $M$  is an ideal of  $R$ ,  $Mu \subseteq M \cap Ru$ . Let  $x \in M \cap Ru$ , then  $x = du$  for some  $d \in R$ . Since  $x = xe$ ,  $x = xe = due = deu \in Mu$ , which implies that  $M \cap Ru \subseteq Mu$ . Hence  $M \cap Ru = Mu$ , by [15, Theorem 4.6],  $R/M$  is *Nflat* right  $R$ -module. By Theorem 2.4(3),  $R/M$  is left *nil*-injective left  $R$ -module, which implies  $R(1 - e)$  is *nil*-injective left  $R$ -module. If  $eR(1 - e) \neq 0$ , then by the same method in the proof of Theorem 2.3,  ${}_R R(1 - e)$  is also *nil*-injective. Therefore  $R$  is left *nil*-injective and hence  $R$  is  $n$ -regular.  $\square$

The following example shows that the condition: “ $R$ ” is left *MC2* is not superfluous in Theorem 2.5.

Let  $Z_2$  be the ring of integers modulo 2. We consider the ring  $R = \begin{pmatrix} Z_2 & 0 \\ Z_2 & Z_2 \end{pmatrix}$ . Then  $R$  is a left *NPP* ring but  $R$  is not left *MC2*. Let  $M = \begin{pmatrix} Z_2 & 0 \\ Z_2 & 0 \end{pmatrix}$ . Then  $M = R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  is a non-essential maximal left ideal of  $R$  which is *nil*-injective by [20, Theorem 3]. Since  $R$  is a left quasi-duo ring,  $R$  is left min-abel ring by [12, Theorem 1.2], However  $R$  is not  $n$ -regular. Moreover,  $R$  is not left *nil*-injective. For, if  $R$  is left *nil*-injective, then  $R$  is  $n$ -regular by [14, Theorem 2.18], which is a contradiction.

This example also implies that there exists a left *NPP* ring containing a non-essential *nil*-injective maximal left ideal which is not  $n$ -regular.

Checking carefully the proof of theorem 2.3, we can obtain the following corollaries which generalize [20, Theorem 8] and [4, Theorem 2].

**Corollary 2.6.** *Let  $R$  be a left *MC2* ring. If  $R$  contains an injective maximal left ideal  $M$ , then  $R$  is a left self-injective ring.*

**Corollary 2.7.** *Let  $R$  be a left *MC2* ring. If  $R$  contains a finitely generated  $p$ -injective maximal left ideal  $M$ , then  $R$  is a left  $p$ -injective ring.*

According to [20], a ring  $R$  is called left *HI*-ring if  $R$  is a left hereditary ring containing an injective maximal left ideal. Wei [11, Theorem 5.1] showed that  $R$  is a left *MC2* ring if and only if  $\text{Soc}({}_R R) \cap J(R) = \text{Soc}({}_R R) \cap Z_l(R)$  and Osofsky [8] showed that if  $R$  is a left self-injective left hereditary ring, then  $R$  is semisimple Artinian. Hence by Corollary 2.6, we have the following Theorem.

**Theorem 2.8.** *The following conditions are equivalent for a ring  $R$ :*

- (1)  $R$  is a semisimple Artinian ring.
- (2)  $R$  is a semiprime left *HI*-ring.
- (3)  $R$  is a reflexive left *HI*-ring.
- (4)  $R$  is a left idempotent reflexive left *HI*-ring.
- (5)  $R$  is a left *MC2* left *HI*-ring.
- (6)  $R$  is a left universally mininjective left *HI*-ring.
- (7)  $R$  is a left mininjective left *HI*-ring.

- (8)  $R$  is a left *minsymmetric* left *HI*-ring.
- (9)  $R$  is a left *HI*-ring with  $\text{Soc}({}_R R) \subseteq \text{Soc}(R_R)$ .
- (10)  $R$  is a left *HI*-ring with  $J(R) = Z_l(R)$ .
- (11)  $R$  is a left *HI*-ring with  $J(R) \cap \text{Soc}({}_R R) = Z_l(R) \cap \text{Soc}({}_R R)$ .

$R$  is called a left *C2* ring [7] if any left ideal  $I$  is isomorphic to a summand of  ${}_R R$  then  $I$  is a summand. It is well known that  $R$  is a von Neumann regular ring if and only if  $R$  is a left *pp* left *C2* ring. Since von Neumann regular rings and left  $p$ -injective rings are left *C2* and left *C2* rings are left *MC2*. Hence by Corollary 2.7, we have the following theorem which generalizes [4, Theorem 3].

**Theorem 2.9.** *The following conditions are equivalent for a ring  $R$ :*

- (1)  $R$  is a von Neumann regular ring with nonzero socle.
- (2)  $R$  is a left *MC2* left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal.
- (3)  $R$  is a left idempotent reflexive left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal.
- (4)  $R$  is a reflexive left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal.
- (5)  $R$  is a semiprime left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal.
- (6)  $R$  is a left universally mininjective left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal.
- (7)  $R$  is a left mininjective left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal.
- (8)  $R$  is a left *minsymmetric* left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal.
- (9)  $R$  is a left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal and  $\text{Soc}({}_R R) \subseteq \text{Soc}(R_R)$ .
- (10)  $R$  is a left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal and  $J(R) = Z_l(R)$ .
- (11)  $R$  is a left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal and  $\text{Soc}({}_R R) \cap J(R) = \text{Soc}({}_R R) \cap Z_l(R)$ .

It is well known that von Neumann regular ring satisfying the *ACC* on left annihilators is semisimple Artinian. Hence we have the corollary which generalizes [4, Corollary 4].

**Corollary 2.10.** *The following conditions are equivalent for a ring  $R$ :*

- (1)  $R$  is a semisimple Artinian ring.
- (2)  $R$  is a left *MC2*, left *pp* and left Noetherian ring containing a  $p$ -injective maximal left ideal.
- (3)  $R$  is a left *MC2* left *pp* ring containing a finitely generated  $p$ -injective maximal left ideal and satisfies the *ACC* on left annihilators.

A ring  $R$  is called strongly regular if  $a \in a^2R$  for all  $a \in R$ . It is well known that  $R$  is a strongly regular if and only if  $R$  is an *abelian* von Neumann regular ring. Since von Neumann regular rings are reflexive, by [16, Theorem 2.6], we obtain that  $R$  is strongly regular if and only if  $R$  quasi-normal von Neumann regular. Therefore, by Theorem 2.9, we have the following corollary.

**Corollary 2.11.** *The following conditions are equivalent for a ring  $R$ :*

- (1)  $R$  is a strongly regular ring with nonzero socle.
- (2)  $R$  is a quasi-normal left MC2 left pp ring containing a finitely generated  $p$ -injective maximal left ideal.
- (3)  $R$  is a semiabelian left MC2 left pp ring containing a finitely generated  $p$ -injective maximal left ideal.

$R$  is called left  $V$ -ring [10] if every simple left  $R$ -module is injective, and  $R$  is said to be left  $GV$ -ring if every left  $R$ -module is either projective or injective. Clearly, strongly regular  $\implies$  left  $V$ -rings  $\implies$  left  $GV$ -rings, and by [13, Theorem 1.7], left  $V$ -rings are left MC2.

Let  $M$  be a left  $R$ -module and  $N$  a submodule of  $M$ .  $N$  is called an absolute summand if for any submodule  $T$  of  $M$  such that  $T$  is maximal with respect to  $N \cap T = 0$ , we have  $N \oplus T = M$ .

**Lemma 2.12.** *Let  $e$  be a central idempotent element of  $R$ . Then  $Re$  is an absolute summand of  $R$ .*

*Proof.* Let  $T$  be a left ideal of  $R$  such that  $T$  is maximal with respect to  $Re \cap T = 0$ . Then  $Te = eT \subseteq Re \cap T = 0$  and so  $T = T(1-e) \subseteq R(1-e)$ . Since  $Re \cap R(1-e) = 0$ ,  $R(1-e) = T$ . Therefore  $Re \oplus T = R$ , which shows that  $Re$  is an absolute summand of  $R$ .  $\square$

**Theorem 2.13.**  *$R$  is a left  $V$ -ring if and only if  $R$  is a left MC2 left  $GV$ -ring such that for any  $e \in ME_l(R)$ ,  $Re$  is an absolute summand of  $R$ .*

*Proof.* The necessity follows from [10, Proposition 3.7].

Now let  $R$  be left MC2 left  $GV$ -ring such that for any  $e \in ME_l(R)$ ,  $Re$  is an absolute summand of  $R$ . If  $W$  is a projective simple left  $R$ -module and  $f : I \rightarrow W$  is a nonzero morphism with  $\ker(f) = K$ , then  $K$  is a summand of  $I$ , that is  $I = K \oplus L$  for a left ideal  $L$  of  $R$ . As  $L$  is isomorphic to  $W$ ,  $L = Re$  for some  $e \in ME_l(R)$  because  $R$  is left MC2. By hypothesis,  $L$  is an absolute summand of  $R$ . Then by a same proof of [10, Proposition 3.7], we obtain  $W$  is injective.  $\square$

Since every left minimal idempotent element of left MC2 left min-abel rings is central, by Lemma 2.12 and Theorem 2.13, we have the following corollary.

**Corollary 2.14.** *Let  $R$  be a left min-abel ring. Then the following conditions are equivalent:*

- (1)  $R$  is left  $V$ -ring.
- (2)  $R$  is left MC2 left  $GV$ -ring.



As for left min-abel rings and strongly left min-abel rings, we have the following characterization.

**Theorem 2.15.** (1) *The following conditions are equivalent for a ring  $R$ :*

- (a)  *$R$  is left min-abel.*
  - (b) *For any  $e \in ME_l(R)$ ,  $eR(1-e)Re = 0$ .*
  - (c) *For any  $k \in PM_l(R)$ ,  $l(k)Rk = 0$ .*
  - (d) *For any  $a \in N(R)$  and  $e \in ME_l(R)$ ,  $ae = 0$  implies  $eaRe = 0$ .*
  - (e) *For any  $a \in N(R)$  and  $e \in ME_l(R)$ ,  $ea = 0$  implies  $eRae = 0$ .*
- (2) *The following conditions are equivalent for a ring  $R$ :*
- (a)  *$R$  is strongly left min-abel.*
  - (b) *For any  $k \in PM_l(R)$ ,  $kRl(k) = 0$ .*
  - (c)  *$R$  is left min-abel and for any  $e \in ME_l(R)$ ,  ${}_RRe$  is injective.*

*Proof.* (1) (a)  $\implies$  (b) and (a)  $\implies$  (e) are trivial, because every element of  $ME_l(R)$  is left semicentral.

(b)  $\implies$  (c) Let  $k \in PM_l(R)$ . Then  $l(k) = l(e) = R(1-e)$  for some  $e \in ME_l(R)$ , so  $k = ek$ . If  $l(k)Rk \neq 0$ , then  $Rk = l(k)Rk$ , so by (b),  $eRk = el(k)Rk = eR(1-e)Rek = 0$ , which implies  $k = ek = 0$ , a contradiction. Hence  $l(k)Rk = 0$ .

(c)  $\implies$  (d) By (c),  $l(e)Re = 0$  for all  $e \in ME_l(R)$ . Since  $a \in l(e)$ ,  $aRe = 0$  which implies  $eaRe = 0$ .

(d)  $\implies$  (a) and (e)  $\implies$  (a) Let  $e \in ME_l(R)$ . For any  $a \in R$ , write  $h = ae - eae$ . If  $h \neq 0$ , then  $eh = 0, he = h$  and  $h^2 = 0$ , so  $Re = Rh$ . By (e),  $eRhe = 0$ , so  $eRe = eRh = eRhe = 0$ , which is a contradiction. If let  $e = ch, c \in R$  and set  $g = hc$ , then  $g \in ME_l(R)$  and  $h = gh$ . Since  $hg = h^2c = 0$ , by (d),  $ghRg = 0$ . Therefore  $hRg = 0$ . Since  $hR = gR$ ,  $gRg = hRg = 0$ . This is a contradiction. Hence, in any case,  $h = 0$ , which implies  $R$  is left min-abel.

(2) (a)  $\implies$  (b) Let  $k \in PM_l(R)$ . Then  $l(k) = R(1-e)$  for some  $e \in ME_l(R)$ . By (a),  $e$  is central. Hence  $kRl(k) = kR(1-e) = (1-e)kR = 0$ .

(b)  $\implies$  (c) Let  $e \in ME_l(R)$ . Then by (b),  $eRl(e) = 0$ , which implies  $eR(1-e) = 0$ . If  $(1-e)Re \neq 0$ , then  $Re = R(1-e)Re$ . Therefore  $eRe = eR(1-e)Re = 0$ , which is a contradiction. Hence  $(1-e)Re = 0$ , which shows that  $e$  is central. Consequently,  $R$  is left min-abel and  ${}_RRe$  is injective by (2).

(c)  $\implies$  (a) By [13, Theorem 1.7],  $R$  is left *MC2*. By [12, Theorem 1.8],  $R$  is strongly left min-abel.  $\square$

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