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Modification of Unit-Segmenting Schemes for Division Problems Involving Fractional Quantities

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In the field of arithmetic in mathematics education, there has been lack of fine-grained investigations addressing the relationship between students' construction of division knowledge with fractional quantities and their whole number division knowledge. This study, through the analysis of part of collected data from a year-long teaching experiment, presents a possible constructive itinerary as to how a student could modify her unit-segmenting scheme to deal with various fraction measurement division situations: 1) unit-segmenting scheme with a remainder, 2) fractional unit-segmenting scheme. Thus, this study provides a clue for curing a fragmentary approach to teaching whole number division and fraction division and preventing students' fragmentary understanding of the same arithmetical operation in different number systems.

I. Background

Fractions have been considered to be the most intricate numbers to deal with in arithmetic and there have been tremendous efforts in mathematics education research for investigating children's learning of fractions. For instance, Steffe and Olive (1990) conducted *The Fraction Project* to investigate children's construction of operations that generate their fraction schemes. The consecutive studies following *The Fraction Project* by their colleagues as well as Steffe's & Olive's study were based on the *Reorganization Hypothesis*, the view that 'children's fraction schemes are generated through modifications of their abstract whole number sequences' (Biddlecomb, 1994; Olive, 1999; Steffe, 2002; Steffe & Olive 2010; Steffe & Tzur, 1994).

Children's mature understanding of fractions should be viewed as a synthesis of their understanding of multiplication, division, and ratio via measurement (Thompson & Saldanha, 2003). Mack (2000, 2001) conducted a two-year study with children from fifth to sixth grade the purpose of which was to examine the long-term effect of learning with understanding in multiplication of fractions. Especially, students' ability to reconceptualize and partition different types of units was the focus of the study because it was believed to be essential for students to determine the appropriate unit to be partitioned in a problem situation as well as the unit upon which the results of partitionings are based. The findings proposed distinctive mental processes related to viewing the unit to be partitioned and the results of their partitionings as fractional amounts of a referent

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whole as well as related to different types of problems involving multiplication of fractions. However, her analysis was framed on the basis of different types of problem situations that her students encountered where the perceived relationship between the denominator of the multiplier and the numerator of the multiplicand is varied, rather than the suggested levels of mental processes.

Lamon (1999) introduced two types of whole number division: partitive division and quotitive (measurement) division. The former involves partitioning or determining equal parts or shares whereas in the latter the question is how much of a quantity can be measured out of the other quantity. The difficulties that children might encounter in the latter would be to identify the divisor as a new unit of measure. Bulgar (2003) conducted a year-long teaching experiment with fourth-grade students for understanding children's solving fraction division problems prior to introduction of algorithmic instruction. The task was to determine how many bows of each size (mainly unit fractional quantities) could be made from each (whole number) length of ribbon. As a result, Bulgar reported that three distinct solution methods emerged: (1) justification involving natural number, (2) involving measurement, and (3) involving fractions. She also documented that all methods were related to children's counting and they had difficulty with division involving a non-unit fraction divisor. As to partitive division, Empson, Junk, Dominguez & Turner (2005) analyzed children's coordination of two quantities (number of people sharing and number of things being shared) in their solutions to equal sharing problems to see what extent their coordination was multiplicative. They gave an important implication for research in children's construction of fraction schemes by suggesting that with various number combinations, children's mathematical activities in equal sharing problems involved whole number knowledge constructs such as multiplicative reasoning.

Nevertheless, fewer studies in research on division have been attempted to address the relationship between students' construction of division knowledge with fractional quantities and their whole number division knowledge. Such lack of connections in research might affect school mathematics curriculum that division is merely treated as an operation one performs on whole numbers, and fractions are taught almost exclusively as part-whole concepts in school mathematics (Toluk & Middleton, 2004).

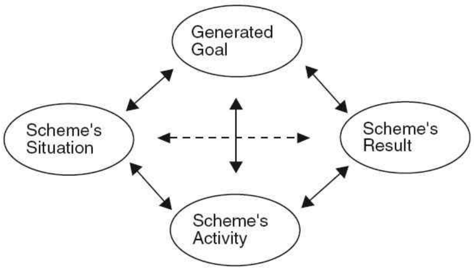
II. Theoretical Constructs

The main concern of this study was to investigate how the participating students modified their whole number knowledge to solve measurement division problems involving fractional quantities. In this section, we explicate necessary theoretical constructs on learning mathematics, which were employed to interpret the students' actions and operations in their problem solving processes.

1. Model of Learning: Scheme Theory

Piaget considered a scheme as a behavior structure within the organism such that the organism can transfer or generalize its action. Schemes that Piaget referred to were more or less action schemes based on repeatable actions patterns, and further generalized through application to new objects

(Steffe, Cobb, & Glasersfeld, 1988). According to von Glasersfeld (1995), a scheme consists of three parts: an 'experiential situation' which is activated or recognized by the child, the specific 'activity' associated with the conceived situation, and a certain 'result' of the activity engendered by the child's prediction. The first part of a scheme consists of a 'recognition template', which contains records of operations used in past experience. Then the activity of a cognitive scheme may consist of an implementation of the assimilation operations, and the result of the scheme may contain an experienced situation (Olive & Steffe, 2002a). In addition to the three components of a scheme, to take into account the goal of an activity is inevitable because, in constructivism, all cognitive activity takes place within the experiential world of a goal directed consciousness. The generated goal has to be associated with the situation of the scheme, the scheme's activity is directed toward that goal, and the results of the scheme are compared to the goal (see Figure II-1). If the newly formed result satisfies the goal, then the scheme is closed. It should be noted that the double arrows linking the three base components imply dynamic nature of a scheme meaning that it is possible for any one of them to be in some way compared or related to either of the two others. Conclusively, the action of a scheme is not sensory motor action, but interiorized action by reflective abstraction with the most minimal sensory motor indication (Olive & Steffe, 2002a).



<Figure II-1> A diagram for the structure of a scheme (Steffe, 2010c, p. 23)

2. Schemes of Whole Number Knowledge

Steffe et al. (1988) conducted longitudinal teaching experiments[1] with young children for their development of whole number knowledge and identified five distinct types of counting schemes: two pre-numerical counting schemes and three successive number sequences - *initial number sequence* (INS), *tacitly nested number sequence* (TNS), and *explicitly nested number sequence* (ENS).[2]

> A 'number sequence' is the recognition template of a numerical counting scheme, that is, its assimilating structure. A number sequence is a discrete numerical structure; it is a sequence of arithmetical unit items that contain records of counting acts (Steffe & Olive, 2010, p. 27).

Each new number sequence is the result of a reinteriorization of the previous number sequence and generates more abstract units with which the child can operate (Olive, 1999). That is, a gradual decrease in children's dependence on their immediate experimental world can characterize the learning stages of number

1) Teaching experiment is more than Piaget's clinical interview in that clinical interview aims at establishing 'where the child is' but the experiment aims at ways and means of getting children on (Glasersfeld, 1983: Steffe & Thompson, 2000)

2) Detailed explanations for each type of counting scheme are beyond the range of this paper.

sequences and it is the operations that children can perform using their number sequences that distinguish among distinct stages of the number sequences. Later, the notion of the *generalized number sequence* (GNS) ensued while seeing how children who had constructed the ENS might use that number sequence to construct schemes to solve situations that can be regarded as multiplying and dividing situations (Steffe & Olive, 2010).

A crucial step for the construction of an ENS is the establishment of an abstract unit item 'one' as an iterable unit (Olive, 1999). The iterable one can be produced through repeatedly applying the 'one more item' operation when double counting. After the construction of an iterable unit item, a child can engage in part whole reasoning. When the unit of one is iterable, a number word refers to a composite unit containing a unit, which can be iterated the number of times indicated by the number word. This iterability of one "opens the possibility for a child to 'collapse' a composite unit into a unit structure containing a singleton unit, which can be iterated so many times." (Steffe & Olive, 2010, p. 42) This characteristic of the ENS enables children to establish multiplicative schemes that involve two levels of units. The ENS provides children with the necessary operations to engage in multiplicative reasoning. Further, they can generate a numerical composite of composite unit items as a result of those operations, but they have yet to interiorize or symbolize them so that the numerical composite of composite unit items can be used as given input for further operations (Olive, 1999).

The reinteriorization of the ENS results in iterable composite units. When children have constructed composite units as iterable, they can be regarded as at least in the process of reorganizing their ENS into a GNS (Steffe, 1992). In other words, the GNS is a generalization of the operations on units of the ENS to composite units. "Speaking metaphorically, children are in a 'composite units' world rather than a 'units of one' world." (Steffe & Olive, 2010, p. 43) In a GNS, a composite unit is iterable, that is, any composite unit can be taken as the basic unit of the sequence. For a composite unit to be judged as iterable, a child should be able to represent and combine iterations of the composite unit prior to activity.

3. Connected Number Sequence & Fractional Connected Number Sequence

When the situations of the counting scheme involves a connected but segmented quantity through an awareness of figurative length and figurative density, a unification of discrete and continuous quantity begins. Further, constructing connected numerical composites opens the way for the construction of a connected number sequence, which is "a number sequence whose countable items are the elements of a connected but segmented continuous unit" (Steffe & Olive, 2010, p. 56). For the construction of a *connected number sequence* (CNS), a child should build awareness of indefinite length as well as of indefinite numerosity as quantitative properties of a connected number, which means the incorporation of a notion of unit length into the abstract unit items of their ENS (Olive & Steffe, 2002b). Thus, children's construction of a CNS plays a crucial role in making sense of fractions. It enables children to "use their discrete adding, subtracting, and multiplying schemes to find unknown lengths using known lengths, and thus establish part whole relations in the context of continuous quantities" (Olive & Lobato, 2008, p. 9).

A *fractional connected number sequence* (FCNS) is a connected number sequence, in which unit fractions are the units of the connected numbers (Steffe, 2002). The construction of such fractional numbers is made possible when their fractional meaning would no longer be directly dependent on its relation to the whole of which it is part (Steffe & Olive, 2010). Analogously, the iterability of unit fractions with a FCNS is on a par with that of a unit, one with an ENS. That is, children can use, say, one eleventh as they use the unit of one and it can be operated with in a way that is analogous to how the child operates with the ENS involving the unit of one (Steffe, 2002).

III. Method of Inquiry & Research Question

A teaching experiment was conducted with a pair of eighth-grade students at a rural middle school in north Georgia. The teaching experiment is a methodology for conducting scientific research on mathematics learning. A primary purpose for using a teaching experiment methodology is for researchers to experience students' mathematical learning and reasoning. The teaching experiment methodology is deeply rooted in radical constructivism in the sense that researchers in teaching experiments attribute mathematical realities to students that are independent of their own mathematical realities and, therefore, a primary goal of the teacher in a teaching experiment is to establish living models of students' mathematics (Steffe & Thompson, 2000). Learning involved in a teaching experiment is to be regarded as accommodation in the context of scheme theory. That is, what students learn is defined in terms of the modifications of their current schemes using available operations in a new way rather than in terms of the mathematical knowledge of the researchers. Therefore, the attention would be focused on understanding the students' assimilating schemes and how these schemes might change as a result of their mathematical activity.

A teaching experiment consists of a sequence of teaching episodes. A teaching episode includes a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what happens during each episode (Steffe & Thompson, 2000). The important duty of the teacher-researcher in the teaching experiment is to attempt to put aside his or her own concepts and operations and not to insist that the students learn what he or she knows. The research hypotheses one formulates prior to a teaching experiment usually guide the initial selection of the students and the researchers' overall general intentions, but also new hypotheses are to be generated and tested during the teaching episodes. Through generating and testing hypotheses, boundaries of the students' ways and means of operating can be formulated (Steffe & Thompson, 2000).

The overarching goal of our teaching experiment was to establish models for the participating students' construction of Rational Numbers of Arithmetic (RNA)[3] because 'RNA involves a multiplicative

3) RNA is a final form of construction of students' fraction knowledge in *The Fraction Project* (Steffe & Olive,

transformation[4] between two fractional quantities, which is expected to provide a constructive basis for students' construction of their proportional reasoning[5]' (Shin & Lee, 2010). Two eighth-grade participants for the present study were chosen based on their ability to use composite units as iterable units as an indication of the GNS. Carol joined our teaching experiment from her seventh grade and thus the teaching experiment with Rosa was the second year of her experience with our research team. During the first year we posed various problems in several mathematical topics such as basic combinatorial problems, calendar problems related to modular arithmetic, and cooking recipe problems for proportional reasoning.

During the teaching experiment, we met once or twice a week in about 40-minute teaching episodes in which the first author participated mostly as a teacher-researcher. All teaching episodes were videotaped with two cameras for on-going and retrospective analysis. The ongoing analysis, the first type of analysis, occurred by watching videos of the teaching episodes and debating and planning future episodes. For the most part, the resources from two cameras were mixed for a single, digitalized video file on the day of each teaching episode. In this way, we created a restored view (Olive & Vomvoridi, 2006) of our teaching experiment. Then a sequence of summaries for the teaching episodes were created in consecutive time, each of which provided not only a written description of students' mathematical activities and interactions with the teacher, but also emerging key points in students' thinking and learning that were taken into account for the next teaching episode. The second type of analysis, which has to be conducted later, is retrospective analysis. The purpose of the retrospective analysis of the sequence of teaching episodes is to make models of students' ways of operating mathematically through conceptual analysis of students' mathematical activities.

As a preliminary step of our teaching experiment, we needed to investigate the current fraction multiplication and division knowledge of our participating students because it was closely related to their construction of RNA. We expected that the students were well-versed in fraction multiplication and division because such topics were already taught at their elementary schools and their academic records were over an average level. However, when the two students were asked division problems generating fractional quantities, unexpected difficulties (from the researchers' point of view) were indicated while their dealing with fractional quotients. Therefore, the retrieved data analysis for this paper is derived from our one-year teaching experiment with the two 8th grade students, with which we intend to explore the following question: <u>How did the students modify their whole number division knowledge when solving division problems involving fractional quantities?</u>

1990). A student can be judged to have constructed the RNA when "the child is aware of the operations needed not only to reconstruct the unit whole from any one of its parts but also to produce any fraction of the unit whole from any other fraction" (Olive, 1999, p. 281).

4) A student can be attributed to the construction of a scheme for multiplicative transformation if the student is able to transform a fractional quantity into any other fractional quantity, and if the student is explicitly aware of the involved fractional operator.

5) A proportional reasoning can be understood as a form of a mathematical reasoning that involves a sense of multiplicative co-variation of two different quantities under the same multiplicative transformation.

IV. Data Analysis

1. Measuring-out a Whole Number Quantity with a Remainder

For a situation to be established as divisional, it is always necessary to establish at least two composite units, one composite unit to be segmented and the other composite unit to be used in segmenting. The goal is to find how many times one can use the measuring unit [the unit to be used in segmenting] with a given unit to be segmented. However, when the composite unit to be segmented is not completely measured out by the unit used in segmenting [when producing a remainder], the divisional situation might be assimilated as novel and lead to perturbation in a student's use of her unit-segmenting scheme because the result of a unit-segmenting scheme produces a fractional quantity in terms of the segmenting unit. Since Carol and Rosa were eighth-grade students, we already knew that they had a certain amount of fractional knowledge. Through a year of teaching experiment (with Carol) and pre-interview (with Rosa), we attributed to the students construction of (at least) partitive fraction schemes.[6] Therefore, we decided to begin our teaching experiment with a whole number division problem that produces a remainder. In all of the following protocols, R stands for Rosa, C for Carol, T for the teacher-researcher (the first author), and W for a witness-researcher.

Protocol IV-1: Finding how many times 3 meters is contained in 5 meters.

T: How many times is three meters contained in five meters?

(Both students write down the problem on their own paper. Rosa divides five by three using a division algorithm and gets 1.6 as an answer.)

C: Hmm… this is hard.

R: You want this in fraction form?

T: Yeah, I prefer to fraction form.

R: Okay.

T: You don't necessarily calculate in decimal form.

R: Yeah. I don't, how would, hmm…in decimal, okay. (Rosa draws a 5-part bar, shades three parts out of the five parts and writes down '3/5' under the circled three parts and '1 time' over the shaded three parts. Then, she writes 2/5 under the left two parts. See Figure IV-1a). (To Carol) how did you work it out? I don't know.

C: I divided it.

R: I divided it too. But you have to show in fraction form.

C: I got point six.

R: You got point six? I got one point six because three…

C: Because five goes into thirty six times and then I plotted a decimal point.

R: Yeah, but three can go into five one time.

C: But your… but it's not three going into five. It's three meters in five meters.

R: So you do five divided by three.

C: Oh, wait. I did it backwards. You're right. Don't listen to me.

R: Okay. (Rosa resumes her work. She divides each part of the whole 5-part bar into two and writes down '6/10' right to the next of '3/5' and '4/10' to the next of '2/5')

T: Carol, you don't necessarily calculate in decimal form. Just think about in fraction form.

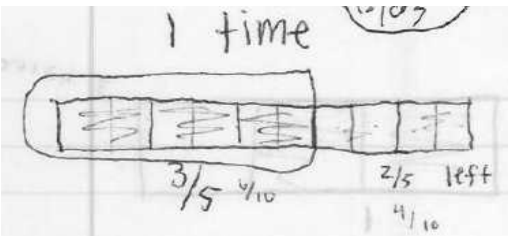
C: Okay.

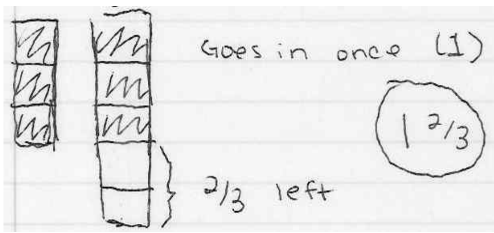
6) A partitive fraction scheme is the first scheme to be a genuine fractional scheme (Steffe, 2002). It enables a student to establish a substantial but limited understanding of fractions as parts of a specific partitioned whole (Tzur, 1999)

R: In fraction form.
T: Yeah.

Carol and Rosa assimilated the problem as a divisional situation. Such assimilation of the problem situation activated their division algorithm although Carol conducted division calculation in a reverse way, that is, three divided by five rather than five divided by three. However, since they already knew that the teacher-researcher always prefers a fraction form to a decimal form as an answer, Rosa's attempt to find her fractional answer prompted her to think about the part-whole relationship between five meters and three meters and she drew a 5-part bar with three parts shaded. Rosa seemed to try to find her decimal answer,1.6, in her drawing by dividing each of the five parts into two (cf. Figure IV-1a), which gave her two fractions with a denominator, '10' [6/10 for 3/5 and 4/10 for 2/5] for each part. However, she could not connect her decimal answer 1.6 to her drawing because she took the five parts as a referent whole rather than the shaded three parts. Considering their multiplicative reasoning with whole numbers, they must have been able to find the answer very easily for a simple whole number division problem, say, to find how many 3 meters are contained in 6 meters. Therefore, the struggle that the students demonstrated above indicates that to find a fractional answer for a whole number division problem involving a remainder was a novel situation for them, which led them to experience perturbation in using their unit-segmenting scheme.

Protocol IV-1: (Cont.)
R: It's gonna be one and then something fraction.
C: This is hard.
T: It's not an easy problem.
R: All right. Let me try something.
W: (Witness-researcher intervenes.) Can you draw five meters, Carol?
C: I am.
W: Okay. (Carol draws two different-sized bars and divides the smaller bar into three parts and the larger bar into five parts.)
R: Is that it? One and six-tenths?
T: One and six-tenths?
R: Is that what you're looking for?
W: (Witness-researcher points out '2/5' on Rosa's paper.) You have two-fifths here.
R: See. What I did just um…I doubled it. So I did six meters in ten meters because ten can go into a hundred and then percents can go into decimals or fractions or….
W: I want you, (Witness-researcher points out Rosa's 5-part bar) I want you to use this up here, not (inaudible) decimal.
R: Well if you…
T: Use these numbers in a similar way.
C: Would it be one and two-thirds?
T: One and two-thirds?
C: Yeah.





<Figure IV-1a & 1b> Rosa's (left) & Carol's (right) drawing to find her fraction answer

T: How did you figure out it?

C: Um, because it goes in once and there is two-thirds of that leftover (see Figure IV-1b). Is that right?

R: I don't know how you...

T: Yeah, can you explain it to Rosa?

C: It's like what you are doing except that I compared the two, like if you have three and you have five (pointing out the 3-part bar) it goes in once and since it was three pieces there is two of three pieces left.

R: There is two-thirds, okay I see now. All right. Okay.

W: What is that two-thirds out there?

R: Two-thirds of the total three-thirds that are (inaudible). Okay. I got it now. I see. I got it.

Carol seemed to be stuck with the problem indicated by the repeated comments, "This is hard" However, Rosa' comment, "It's gonna be one and then something fraction." seemed to activate Carol's unit-segmenting operation as an assimilating operation for this problem and lead her to draw a 3-part bar and a 5-part bar to compare two quantities (cf. Figure IV-1b). We make this conjecture based on the fact that Carol was already drawing her two bars on her paper when the witness-researcher attempted to help her by asking "Can you draw five meters?" Once Carol set the goal of measuring the 5-meter bar with the 3-meter bar, she realized the answer, one and two-thirds and her explanation indicated that she constructed a three-levels-of-units structure as a result of her unit-segmenting operation. In other words, the 5-meter bar was not only five units of 1-meter, but also one unit of a 3-meter and two-thirds of the 3-meter for Carol. It can be viewed as similar to construction of a multiplicatively nested three-levels-of-units structure, say, six as two units of three, but it should be more than that because five was not a multiple of three, which means it necessarily produced a fractional remainder as a part of the measuring unit, three. We conjecture that her partitive fraction scheme contributed to her construction of such a three-levels-of-units structure. When she got 2 meters as a result of her unit-segmenting operation by 3 meters from 5 meters, the result of the unit-segmenting operation turned into a situation for her partitive fraction scheme, which led her to view 2 meters as two-thirds of 3 meters. We hypothesize that the goal of measuring a quantity that produces a remainder, can be fulfilled by a generalizing assimilation of students' unit-segmenting scheme. When assimilating operations of a unit-segmenting scheme are modified and contain a partitive fraction scheme as a sub-scheme, the unit-segmenting scheme is generalized to include the records of the operation of constructing a part-whole fractional relationship with any two quantities. In other words, a student becomes aware of the need to measure the remainder of the division problems by establishing a part-whole relationship between the leftover and the measuring unit [the unit used in segmenting].

On the other hand, Rosa, although she initiated Carol's unit-segmenting scheme, did not seem to associate the result of her unit-segmenting scheme and her fraction scheme. Since Rosa joined our teaching experiment as a new partner of Carol, we did not have enough information about her construction of fraction schemes at this point in the teaching episode, although we hypothesized that she had constructed a splitting operation based on her pre-interview. What can be conjectured from this protocol is that her attempt to find a fraction answer enticed her to take the biggest quantity (five meters) as a referent whole. Since she already knew that 1.6 was an answer, Rosa kept trying to convert her drawing into a decimal form regardless of the teacher-researcher's insistence of a fraction form. Until Carol provided her explanation, Rosa did not realize that the 3 meters should be a referent whole nor associate her unit-segmenting scheme and her fraction scheme. Even though Rosa immediately assimilated Carol's explanation later, whether the relation that Rosa established between her unit-segmenting scheme and her partitive fraction scheme was an embedding of her partitive fraction scheme into her unit-segmenting scheme or just a sequential chain of associations was still to be investigated. She may have constructed an associative chain of schemes, where any scheme in the chain was triggered by the results of the scheme immediately preceding, but she was unable to independently use her scheme.

When Carol was asked a similar question, "How many times is five meters contained in seven meters?" right after the protocol IV-1, she demonstrated that her partitive fraction scheme was embedded in the assimilating part of her unit-segmenting scheme. She said, "it's gonna be the same" and immediately drew a 5-part bar and a 7-part bar to get one and two-fifths. It also indicated that Carol anticipated that the remainder could be dealt with by using her partitive fraction scheme prior to conducting an actual unit-segmenting operation. In contrast to Carol, Rosa used the division algorithm for her answer again and got 1.4. Although she easily converted 1.4 to one and four-tenths and then to one and two-fifths, whether her division algorithm was used as a part of assimilating operations for her unit-segmenting scheme was uncertain at that time.

2. Measuring a Whole Number Quantity with a Fractional Quantity, which does not Evenly Divide the Whole Number Quantity

The teacher-researcher decided to ask a division problem involving a fractional divisor. They might be able to solve it if the whole number quantity to be segmented was evenly divisible by the fractional divisor used in segmenting, but what we were interested in was the problem situation where the fractional divisor does not evenly divide the whole number quantity and produces a remainder.

Protocol IV-2: Measuring 4 gallons of water with 3/4 gallon of a container.

T: I have only four gallons of water, how many times do I take out?

R: How many three-fourths?

T: Yeah, with three-fourths.

(Rosa starts writing numerical expressions of a conventional division algorithm, but seems to hesitate to go on with her calculation process. On the other hand, Carol draws a 4-part bar for 4 gallons on her paper and put a line for every one and a half gallons and shades two of one and a half gallons on her 4-part bar. See Figures IV-2a & IV-2b).

R: Okay, going to have to be...

C: Is it four and two-thirds?

R: See, that's what I'm about to get. Cause I know four scoops is three hundred which is like...

T: That's very close.

C: (Disappointedly) Oh... (Carol and Rosa resume their solving activity.) It's three and two-thirds? (Teacher answers in the negative.) Oh...

T: (To Carol) What are you trying to do?

R: Is it five and one-third?

T: Five and one-third.

R: No, it's not five. It's no way.

C: It could be five.

T: I didn't say no.

C: It could be five. Because one, two, three, four (Carol seems to count the number of three-fourths in her 4-part bar for 4 gallons of water.)

T: Five and one-third is right.

R: It is? Oh, do you want me to show you how I did it? I have to do everything like in a problem situation like I can't just use pictures. (To Carol) do you want me to show how I did it?.

C: (Twenty seconds elapse) yeah, that's what I got. Five and one-third. I'll tell you what I did after you're done.

R: Okay, what I did is I knew that four hundreds was a whole so I just did four over one and multiplied it times four over three, which is like the... which is basically four over one divided by three-fourths. I did four hundreds divided by point seventy five. And then I got sixteen over three (see Figure IV-2a) and I just made it...

T: Why did you multiply four-thirds?

R: It's what I was intending to do was four over one divided by three-fourths but you have to change it to four over one and make it it's like multiplied by its reciprocal.

C: What I did is, I have four gallons and then I did like, I couldn't [do what I] did with the last one, and divided it into one and a half on each but then I just started off doing I found two whole ones and times two and I got four and two-thirds [of one and a half.] Then I was thinking what you were saying like how you get. And then I realize you can have an extra one because you have an extra. Then I got six, then there is the two extra leftover one from the cup. So I minused two-thirds from the six from the leftovers, and I got five and one-third.

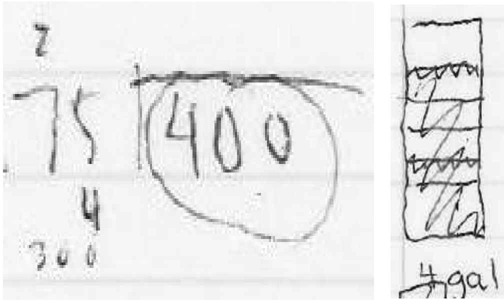
T: Can you say that again? How did you figure out the last extra one?

C: The left extra one? Because you only had so much left of the top of the gallon you could scoop out, then you have left over two-thirds.

T: Two-thirds? Two-thirds of what?

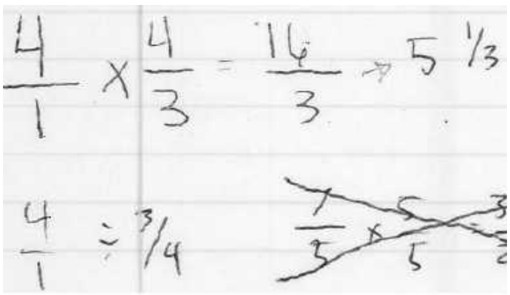
C: Two-thirds of a little cup.

R: I see what you're saying, but I'm just don't understand your drawing, but I know what you're saying.



<Figure IV-2a & IV-2b> Rosa's division algorithm (left) & Carol's drawing for measuring out 4 gallons with 3/4 of a gallon (right)

Rosa could associate a measuring-out situation with a division problem situation. However, she seemed stuck when she realized that four cannot be evenly measured out by three-fourths. That is, she seemed to feel perturbation associating the result of her numerical division calculation with that of her unit-segmenting scheme when the result produced a remainder. It corroborated that the relationship between her unit-segmenting scheme and her partitive fraction scheme was just an association, not that her partitive fraction scheme was embedded into the first part of her unit-segmenting scheme. The fact that Rosa independently could not deal with the remainder of her numerical division calculation result indicated that her partitive fraction scheme (if she had constructed one) was not embedded as an assimilating operation of her unit-segmenting scheme. Rather, the result of her unit-segmenting scheme was possibly associated to the situation of her partitive fraction scheme by the additional cue from an external source like Carol's mathematical behavior. Her comment, "That's what I'm about to get. Cause I know four scoops is three hundred" also corroborated that her use of a division algorithm still symbolized her unit-segmenting operations with a fractional quantity. However, the realization that her unit-segmenting scheme did not work for this problem induced Rosa to try another division algorithm, the invert-and-multiply algorithm for fraction division problems (cf. Figure IV-3). Even though Rosa found the answer, five and one-third, her fraction division algorithm seemed disconnected to her unit-segmenting scheme. The result of her numeric calculation no longer seemed to symbolize her unit-segmenting operations. Her insecurity about the answer right after she provided it, "No, it's not five. It's no way." and her justification about the answer depending only on the invert-and-multiply algorithm process for fraction division corroborated that her unit-segmenting scheme was yet to undergo an accommodation whereby her partitive fraction scheme was embedded in it as a sub-scheme. Further, Rosa's comment at the end of the protocol after Carol explained the answer using her drawing, indicated that such accommodation was not a simple process for her.



<Figure IV-3> Rosa's invert-and-multiply algorithm

Interestingly, Rosa's answer of division calculation using an invert- and-multiply method helped Carol reflect her unit-segmenting operation and find the right answer. Carol's drawing definitely indicated that she conducted her unit-segmenting operations. Her first answer was four and two-thirds rather than five and one-third. Her mistake appears to have come from the conflation of units used in segmenting. As in Figure IV-2b, her unit to be used in measuring out 4 gallons of water was 1 and 1/2 gallons of water, rather than 3/4 of a gallon given in the problem. The reason she chose 1 and 1/2 gallons as a segmenting unit seemed due to the convenience of actual drawing on the paper, that is, putting a line for a half was much easier than finding a mark for 3/4 of a gallon. She measured 3 gallons as twice of 1 and 1/2 gallons and doubled it because her unit was originally 3/4 of a gallon. However, the change of her segmenting unit caused a conflation of units in segmenting when dealing with the remainder as a result of her unit-segmenting operations. She accidently measured the remaining 1 gallon with 1 and 1/2 gallons, rather than measuring it with 3/4 of a gallon, which led her to get four and two-thirds for the final answer. Even with the inaccurate use of her unit-segmenting operations, we do not believe that the mistake alleviated our conjecture that Carol's partitive fraction scheme was already embedded in her unit-segmenting scheme and she had constructed a newly modified unit-segmenting scheme at a higher level because she immediately self-corrected her answer from Rosa's new answer. In spite of Rosa's uncertainty about the answer, Carol independently assimilated Rosa's answer as a new possibility. Rosa's answer seemed to help Carol re-conceive her segmenting unit and re-initiate her unit-segmenting operations. Then she realized that three of 1 and 1/2 gallons of water, which also were six of 3/4 gallon of water, covered more extras in addition to 4 gallons of water. This time, she explicitly measured the surplus with 3/4 of a gallon and finally got five and one-third by subtracting two-thirds from six.

3. Rosa's Construction of a Unit-Segmenting Scheme with a Remainder through Retrospective Accommodation

Protocol IV-3: Finding how many times 3/5-meter is contained in 1 meter.

T: How many times is three-fifths meter contained in one meter?

R: Three-fifths of a meter?

T: Um-hm.

(Both students draw a 5-part bar and shade three parts of the 5-part bar on their own paper. However, Rosa starts writing numerals for calculation of fraction multiplication as '5/5x5/3=25/15')

C: One and two-thirds?

T: (Teacher nods his head) one and two-thirds.

R: Hold on.

T: (To Rosa) that's fine. Take your time.

R: Did you get five and three-thirds?

C: No.

T: Five and three-thirds?

R: (Rosa compares her answers with Carol's) or one and two-thirds. That's the same thing.

T: Yeah, one and two-thirds is right, but um··· Can you explain your way, Carol?

C: Um··· I just started up by drawing a bar, divided it into fifths. And then I have three of the fifths, which is what I was asked for. So I already knew I had one and there are two leftover out of the three pieces so I have two-thirds.

T: So two pieces is··· Can you say that again why two pieces is two-thirds?

C: Two pieces is two-thirds because, if that (three parts)'s one, and it's three-thirds or three over one. Then you only have two. It would be two-thirds.

T: Okay.

R: Okay. I do, I have to do everything like with an equation. So I just did five over five which is the one meter times five over three or just divided by three-fifths and I just reduced it down and got one and two-thirds.

T: Um-hm. Did you see...

R: I know, I know how she did it. Because this (three parts) is one and then there is two here. So there is two out of that three. I see how she did it, but I would like this anyway.

T: So, what do you think, this one (two unshaded parts) is two-thirds of what?

R: Two-thirds of this way here (three shaded parts).

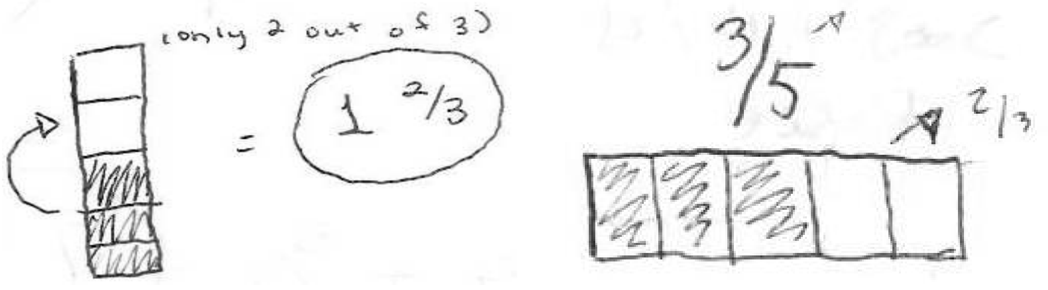
T: Yeah, two-thirds of three-thirds. Right?

R: Yeah.

W: Rosa, can you tell us how you thought about that?

R: First I started drawing it. But then I thought, you know, it's three-fifths into one, five-fifths. So I just thought it is a division problem and um··· I just did five over five divided by three-fifths. You have to multiply and make the reciprocal. Then I just reduced it down and that's how many times three-fifths is in one meter.

In Protocol IV-1, Rosa seemed to construct a unit-segmenting scheme with remainder or at least associate a result of her unit-segmenting scheme with a situation for her partitive fraction scheme. At that time, although the construction was initiated by Carol, Rosa exclaimed that she understood what Carol did. However, this protocol demonstrated that Rosa had yet to construct a unit-segmenting scheme with a remainder as an anticipatory scheme. Even though there was no difference between the two students' drawings of a 5-part bar for one



<Figure IV-4a & IV-4b> Carol's (left) and Rosa's (right) drawings for measuring 1-meter with 3/5-meter

meter and three shaded parts for 3/5-meter (see Figures IV-4a & IV-4b), Rosa failed to conceive the remaining two-fifths of a meter as two-thirds of the segmenting unit, 3/5-meter and went back to relying on an invert-and-multiply algorithm for fraction division calculation in contrast to Carol's immediate realization of the answer from her drawing.

When Rosa got '25/15' and '5/3' in her calculation, she did not know the meaning of the result of her fraction division calculation in a quantitative sense. This was corroborated by her comments, "Did you get five and three-thirds?" when Rosa was asking Carol for the confirmation of her answer. We do not believe Rosa was unable to distinguish five-thirds from five and three-thirds. Rather, even though she got the right answer using an invert-and-multiply algorithm for fraction division, she did not seem to realize that the answer [five-thirds] times the 3/5-meter should be 1-meter quantitatively because the answer was what she got by dividing 1-meter by 3/5-meter. That is, the result of her calculation did not stand in for the multiplicative relationship between 1-meter and 3/5-meter that 1-meter could be constructed by multiplying by five-thirds the 3/5-meter even though she was actually taking perceptual information from her drawing of a 5-part bar with three parts shaded. Therefore, such lack of confidence for her answer gave her a temporary confusion in identification of the answer, and the confusion was not eliminated until Rosa compared her answer with Carol's answer. As indicated in the Protocol IV-1, Rosa quickly assimilated Carol's way of construction and was able to explain with her own words. Since we knew that she had already constructed necessary conceptual elements for a unit-segmenting scheme with remainder, a unit-segmenting scheme and a partitive fractions scheme, somehow we can attribute construction of a unit-segmenting scheme with a remainder to Rosa. However, if the conceptual elements were selected and used by herself only as a result of interactive communication with Carol, we would attribute to her the construction of a unit-segmenting scheme with a remainder through retrospective accommodation.[7]

7) A retrospective accommodation involves selecting and using conceptual elements already constructed. From the student's perspective, a retrospective accommodation is self-initiated in that it is the student who must select and use the concept. From an observer's perspective, the conceptual elements may by selected as result of interactive communication (Steffe & Wiegel, 1994)

4. Construction of a Fractional Unit-Segmenting Scheme

The following protocol was extracted from a teaching episode almost five months after the previous Protocol IV-3. Although the overarching goal of the teaching episode was to investigate the students' mathematical actions and operations emerging in their transformation activities between two (fractional) quantities, at that time we also desired to attempt several fraction measurement division problems before the academic semester was over.

Protocol IV-4: Measuring an 11/19-meter bar with a 4/19-meter bar.

(The problem is "If you measure an 11/19-meter bar with a 4/19-meter bar, how many 4/19-meter bars are contained in the 11/19-meter bar?" Rosa already wrote down her answer for the problem on paper.)

T: Rosa, you said that the answer was...

R: Oh, it's two and three-fourths.

T: Why do you think like that? Can you tell me why...

R: Okay, um... I, when I took its numerators, how many times four go into eleven, and that's two times with three remaining...

T: Um-hm.

R: Actually, it's not three-fourths. It's three-nineteenths.

C: Yeah, because there is three pieces of a... We got the four because yours just...

T: Three-nineteenths?

C&R: Yeah, three-nineteenths.

T: Why did you change?

C: Because nineteenth is the denominator from all of them.

R: Yeah, and that's what you're measuring with, not fourth.

T: So, if you measure four with, measure eleven-nineteenths with four-nineteenths...

R: If you get two and three-nineteenths...

T: Two and three-nineteenths of what?

R: That's how many times four-nineteenths can go into eleven-nineteenths.

T: So, you said that three and, three-nineteenths of...four-nineteenths is contained in eleven-nineteenths?

R: No, because if you take four-nineteenths times two over one, it's eight-nineteenths and there is three left to get to eleven-nineteenths. So, it's gonna be two and three-nineteenths. That's how many times goes in.

T: So, how many times goes,

R: Two and three-nineteenths.

T: Two and three-nineteenths?

R: Um-hm.

....

T: Okay, let me pose another question and we will back to this problem. So, can you see the numbers here? Seven-seventeenths, yeah, seven over seventeen and sixteen over seventeen. What number do you have to multiply by to seven over seventeen to get sixteen over seventeen?

R: Two... and... two seventeenths?

T: Two and ...

R: Two-seventeenths?

T: Two-seventeenths?

C: Cause seven times two is fourteen and there is two leftover.

T: Two-seventeenths? Can you confirm your answer? Using paper and pencil or whatever, using JavaBars.[8]

C: Can we use the GSP?

T: Uh, I don't think I have. I need a CD. You can, then can you calculate, by calculation can you confirm your answer? What was your answer? Seven and... Two and what?

C: Two and two-seventeenths.

T: Two-seventeenths. So...

(Rosa makes some bars in JavaBars and Carol tries to use calculation for confirmation of her answer.)

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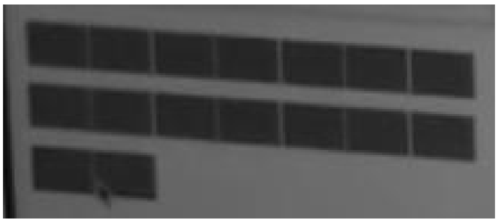
8) JavaBars is a computer program that opens the possibility for students to create an arbitrary rectangle with indefinite area and enact a variety of different operations (i.e., partitioning, disembedding, iterating, etc.)



<Figure IV-5a & IV-5b> Rosa's (left) and Carol's (right) constructions to measure 16/17-meter with 7/17-meter

C: Wait, that will be hard to reduce it.

R: Copy. Pull out. (To Carol) did we say two-seventeenths?

C: Can I do it on JavaBars? This is hard to write it down.

T: Okay.

C: So do we start with seven right?

R: (Rosa has two 7-part bars and one 2-part bar on the screen now. See Figure IV-5a) You know, it might be two-sevenths.

C: (Carol has three 7-part bars on her screen.) Because you do...

R: Cause the seven pieces is what you're starting with.

C: That's what I was thinking eleven-fourths.

R: See, that's what I was thinking four right here. That's what I thought. Two and three-fourths. But, then I was like, but it's four-nineteenths. So I was thinking three-nineteenths. But maybe I was right at the first time.

(Carol now has two 7-part bars and one 2-part bar on the screen.)

R: Because it's two out of the seven pieces. Because you have only seven pieces, not all seventeenths. Right?

....

C: We know it's two and two-sevenths.

T: Two and two-sevenths?

C: Seven-seventeenths to get sixteen-seventeenths.

T: Why do you think seven, two and two-sevenths?

C: Because I was just thinking... (Carol clicks and drags the 7-part bar to measure the 16-part bar by moving the 7-part bar along the 16-part bar from the leftmost part. See Figure IV-5b.) There is one whole, two whole, and then two-sevenths leftover. I'm trying to figure out.

Rosa's first answer was two and three-fourths, which was correct, but she changed her answer into two and three-nineteenths. She seemed to reflect on her unit-segmenting operations while explaining her answer to the teacher-researcher. When she got three pieces leftover as a result of her unit-segmenting operations in reflection, she conflated her unit to be used in segmenting [4/19] with a unit [1] given in the problem to measure the leftover [3/19]. Carol immediately agreed with Rosa's changed answer. It indicated that Carol also conflated units in using her unit-segmenting scheme as Rosa did, which was identified in her comment, "Because nineteenth is the denominator from all of them."

In response to the students' conflation of units in dealing with the leftover on the basis of their unit-segmenting schemes, the teacher-researcher posed another similar question. His intention was to check whether their conflation was lasting because Carol and Rosa had already constructed unit-segmenting schemes with a remainder where the whole number divisor does not evenly divide the whole number dividend. When the teacher-researcher asked them to measure 16/17-meter with 7/17-meter, their answer was two and two-seventeenths, not two and two-sevenths. Until both Carol and Rosa constructed two 7-part bars and one 2-part bar and conducted their unit-segmenting operations with those perceptual materials on JavaBars, they could not realize that the two leftover should be measured in terms of the 7-part bar [7/17-meter], not measured as a length with regard to the given referent whole [1-meter]. Perceptual material [two 7-part bars and one 2-part bar] on JavaBars and implementation of unit-segmenting operations with them obviously helped the students evoke their units-segmenting schemes with a remainder, which worked properly for this problem situation.

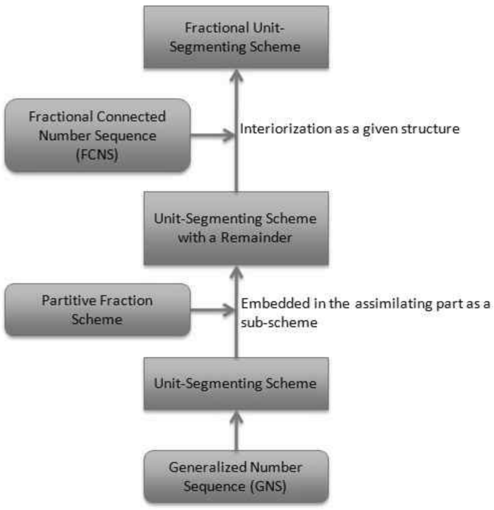
Our conjecture is that Carol's and Rosa's conflation of units might be due to the evocation of their unit-segmenting schemes but without iterability of unit fractions [1/19 and 1/17] in assimilating the fraction measurement division situation. In other words, if they had been able to see 7/17 as seven units of 1/17 each of which can be iterated seven times to make 7/17-meter and also sixteen times to make 16/17-meter prior to activity, they could have assimilated the problem as a situation of their unit-segmenting schemes with remainder as '16÷7'. Actually, they were able to eliminate such confusion through construction of perceptual materials for their unit-segmenting operations using JavaBars. Rosa finally seemed to be explicitly aware of such a relationship between 7/17 and 16/17 with regard to 1/17. She knew that each piece of seven parts was 1/17-meter because "seven pieces is out of the seventeen". Moreover, she also realized that the leftover two parts should be measured in terms of seven-seventeenths because "you only have the seven pieces." Similarly, Carol manifested her unit-segmenting operations with her 7-part bar using JavaBars. She clicked and dragged the 7-part bar to measure the 16-part bar by moving the 7-part bar along the 16-part bar from the leftmost part.

In sum, comparing with a whole number division problem where the divisor does not evenly divide the dividend, this sort of problem involving two fractions [16/17 and 7/17] seemed to require the students to conduct their unit-segmenting scheme with a remainder based on their use of FCNS as given material. Therefore, when the iterability of a unit fraction is interiorized and embedded in the assimilating part of a unit-segmenting scheme with a remainder, we would call such a modified scheme a *fractional unit-segmenting scheme* in the sense that assimilating situations of the scheme include fraction measurement division situations involving fractional numbers.

V. Conclusion

Students' construction of unit-segmenting schemes has been studied in whole number measurement division situations, where one composite unit to be used in measuring, evenly divided the other composite unit to be measured, and the goal of which was to find how many times the measuring unit was used in segmenting the other unit to be measured (Steffe, 1992). However, relatively little research has been carried out for studying how the unit-segmenting schemes could be modified in measurement divisional situations involving fractional quantities.

Based upon the data analysis in our teaching experiment, the following diagram (Figure V-1) illustrates a possible progression of students' unit-segmenting operations and schemes, on the basis of their generalized number sequences (GNS) and fractional connected number sequences (FCNS).



<Figure V-1> A possible constructive path of modifications of a unit-segmenting scheme for fraction measurement division problems

First, we suggested that by embedding the participating students' fraction schemes into the first part of their unit-segmenting schemes, they were able to deal with a remainder when measuring a whole number quantity with another whole number quantity that did not evenly divide the dividend, producing a unit-segmenting scheme with a remainder (cf. Protocol IV-1). A scheme embedded in another scheme needs to be distinguished from just an association of two schemes. When a scheme embeds another scheme in the first part of the former scheme, it means the latter scheme is ready at hand to be used as the former scheme is activated. On the other hand, when two schemes are associated, those two schemes are executed sequentially, rather than simultaneously. It requires students to re-assimilate the result of the first scheme as a situation of the second scheme. Continuing upward in Figure V-1, as students' fractional connected number sequences become interiorized, that is, available as a given structure prior to actual actions of their unit-segmenting operations, the students' unit-segmenting schemes with a remainder would be modified into fractional unit-segmenting schemes so that they could cope with measurement division problems involving fractional quantities (cf. Protocol IV-4). Further, we conjecture that in order for the fractional unit-segmenting scheme to be generalized to solve a question like 'measuring a 1/3-meter bar with a 1/7-meter bar', it would be necessary for students' common partitioning operations to be associated with the fractional unit-segmenting scheme. That is, the common partitioning operations enable the students to convert the division problem situation between the two unit fractions above [1/3÷1/7] into a situation for their fractional unit-segmenting scheme [7/21 ÷ 3/21].[9)]

The present study confirms the grand assumption of the Fraction Project's Reorganization Hypothesis, which argued that children would construct their fraction schemes through modifications of whole number operations based on their abstract number sequences. Further, this study supports that students' construction of the operations that produced a GNS opened possibilities for their constructive activity that could not be observed in the students to whom the construction of only an ENS was attributed. The two participating students with a GNS in the present

9) We realized that the conjecture regarding common partitioning operations was plausible, but it was not supported by the data in this study.

study demonstrated constructions of more advanced fractional schemes and operations, especially in the context of divisional situations, which were not reported in the previous literature.

In addition, the current research results imply that students' interiorization of an iterability of unit fractions, i.e. to take their FCNS as given prior to activity, needs to be considered as a critical factor in establishing viable second-order models for students' construction of more advanced fractional knowledge. In spite of the great potential for mathematical developments that the participating GNS students indicated in this study, the interiorized use of their FCNS was not evident. Actually, in the previous research literature, to establish a multiplicative relationship between a unit fraction and a referent whole in the process of modification of a partitive fraction scheme into an iterative fraction scheme, has been considered a big leap in the development of students' fractional knowledge. This leap enables the students to expand a fraction concept beyond the whole to include improper fractions (Tzur, 1999; Steffe, 2002; Steffe & Olive, 2010). However, this study additionally implies that 1) students' interiorization of such a multiplicative relationship of unit fractions to a referent whole and further 2) their being able to utilize the iterability of unit fractions as given for other mathematical activities, are not spontaneous transitions from the construction of a FCNS in action. These two developments require another level of vertical learning on the part of the students.

Finally, it is worthy of note that Rosa's outstanding calculation ability for fraction division, which was based on procedural algorithms learned in school, seemed to play as an obstacle for her to develop necessary mathematical schemes and operations for advanced fractional knowledge during the teaching experiment. Rosa's struggles, due to her inclination to rely on her procedural algorithms, implies that the aims and methods for teaching fractions in school mathematics need to be seriously reconsidered. Mathematical competence cannot be reduced to proficiency in calculation. That is, students' mathematical competence is not indicated solely by computational results or performances. Rather, results of students' calculating performances become meaningful only when the results are symbolizing the students' mental mathematical schemes and operations involved in their problem-solving processes. Often, Rosa could not use her results of calculation for fraction division in establishing a quantitative relationship in the context of problem situations. Her experiencing such difficulty casts a question about building a curriculum for school mathematics on the assumption that students' training procedures and skills constitute essential steps in their further mathematical learning. The present study, therefore, implies that we, as mathematics teachers, should be able to provide our students opportunities to construct meaningful mathematical structures, processes and symbols for those processes, based on their own mathematical operations, rather than convey simple operational rules as pre-packaged products for the students.

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단위분할 도식의 재구성을 통한 포함제 분수나눗셈 문제해결에 관한 연구

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학생들의 산술 활동 (수의 사칙연산) 학습에 관한 연구 중 분수량을 포함한 나눗셈 문제의 해결을 위한 자연수 지식의 활용을 상세히 다룬 연구가 매우 부족한 실정이다. 교수실험이 연구방법으로 사용된 본 정성연구에서는, 일년간 행해진 교수실험 중 일부 자료의 분석을 바탕으로 다양한 포함제 분수나눗셈 상황을 해결하기 위해 어떻게 자연수 나눗셈의 기본이 되는 단위분할 도식을 수정, 구성해 나갈 수 있는지에 대한 가능한 발달 경로(나머지가 있는 단위분할 도식, 분수 단위분할 도식)를 제시하고 있다. 따라서 본 연구는 다른 수 체계(자연수, 분수)에서 같은 종류의 연산(나눗셈)에 대한 조작적 연결성을 고찰함으로써 현재 학생들이 가지고 있는 수 연산에 관한 분절적 이해를 올바르게 지도할 수 있는 방안을 제시한다.

*key words : unit-segmenting scheme (단위분할 도식), partitive fraction scheme (부분 분수 도식), fractional connected number sequence (분수 연속 수열)

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