

## THE DIMENSION REDUCTION ALGORITHM FOR THE POSITIVE REALIZATION OF DISCRETE PHASE-TYPE DISTRIBUTIONS

KYUNGSUP KIM<sup>1</sup>

<sup>1</sup>DEPARTMENT OF COMPUTER ENGINEERING, CHUNGNAM NATIONAL UNIVERSITY, YUSEONG-GU, DAEJEON, KOREA

*E-mail address:* sclkim@cnu.ac.kr

**ABSTRACT.** This paper provides an efficient dimension reduction algorithm of the positive realization of discrete phase type(DPH) distributions. The relationship between the representation of DPH distributions and the positive realization of the positive system is explained. The dimension of the positive realization of a discrete phase-type realization may be larger than its McMillan degree of probability generating functions. The positive realization with sufficient large dimension bound can be obtained easily but generally, the minimal positive realization problem is not solved yet. We propose an efficient dimension reduction algorithm to make the positive realization with tighter upper bound from a given probability generating functions in terms of convex cone problem and linear programming.

### 1. INTRODUCTION

This paper handles the dimension reduction algorithm of the positive realization with discrete phase-type distributions. Continuous and Discrete phase type(DPH) distributions have been introduced in [1]. Many research activities and applications have been devoted to the field of continuous PH distributions. Recent, a new attention has been devoted to discrete model since they are more closely related to physical observations, the numerical solution of non-Markovian processes, stochastic modeling and applications [2]. The representation of DPH distributions is closely related to the positive realization of the positive systems, which have been researched wide in control and system theory communities. The representation problem finding a Markov chain associated with phase-type distribution has a lot of links with positive realization problem in the control theory [3]. The relationship between the probability generating functions of the probability mass function of a DPH-distribution and a corresponding representation is very similar to the relationship between a transfer function and a corresponding state space realization. The DPH representation are a special form of state space representations with particular constraints on the Markov generators.

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The parameter estimation algorithm for a DPH distribution has been introduced and formalized in [3]. It is well known that general PH distributions are considerably over-parameterized and have high order representation in fitting or approximation problems[4][5]. The problem of over-parameterizations and high order are due to the constraints of the PH-distribution. The Matrix Exponential (ME) approach can solve the problems and the fitting methods with ME get better results[4][5]. The realization of ME distribution can be converted into the PH realization with higher order bound( so called, positive realization) [6]. From this motivation, we will handle the positive realization of DPH distribution from discrete ME distribution under the assumption that the discrete ME distribution can be obtained.

When a non-negativity constraint is imposed on the realizations, the existence and minimality problems becomes more involved. Problems associated with existence and minimality of positive realization have been widely studied in the last decades and a complete answer is not available. The positive constraint may enforce realizations of larger dimension than McMillan degree[7][8]. Therefore, the minimality problem has been dealt with in a number of papers[9][10]. The minimality problem is solved only for particular classes of transfer functions, and a general solution seems to be out of reach in the current state of art [11]. Therefore, in order to induce a minimal positive realization, we need an efficient dimension algorithm to make a positive realization with tighter dimension bound.

We attempt to propose an efficient general algorithm to reduce the dimension of the positive realization of probability generating functions. We note that a polyhedral cone existence gives a necessary and sufficient condition of the positive realization. Using the dual cone, we formulate a proper polyhedral cone. Furthermore, the novel dimension reduction algorithm of the positive realization will be proposed by using linear programming and convex cone. A new numerical example is given to verify that the proposed algorithm works well.

An outline of the paper is as follows: In Section 2, we introduce the relationship between the DPH distribution and the positive realization of positive system. In section 3, some definitions and preliminary results from convex analysis and linear programming are introduced. Section 4 contains the main results such as the positive realization with sufficiently large upper-bound and an dimension reduction algorithm of the positive realization. Finally, the conclusion section follows.

## 2. PROBLEMS IN DISCRETE PHASE DISTRIBUTION AND POSITIVE REALIZATION

**2.1. Discrete Phase distribution.** A discrete phase distribution is the distribution of the time until in a discrete-state discrete-time Markov chain(DTMC) with  $n$  transient states and one absorbing state [1][2]. If the transient states are numbered  $1, 2, \dots, n$  and the absorbing state is numbered  $(n + 1)$ , the one-step transition probability matrix of the corresponding DTMC can be partitioned as

$$\hat{B} = \begin{bmatrix} B & b \\ 0 & 1 \end{bmatrix}$$

where  $B = [b_{ij}]$  is the  $n \times n$  matrix grouping the transition probabilities among the transient states,  $b = [b_i]$  is the  $n$ -dimensional column vector grouping the probabilities from any state

to the absorbing one. Since  $\hat{B}$  is the transition probability matrix of a DTMC, the following relation holds:  $\sum_{j=1}^n b_{ij} = 1 - b_i$ .

The initial probability vector of the DTMC is an  $n + 1$ -dimensional vector  $\hat{\alpha} = [\alpha \ \alpha_{n+1}]$  with  $\sum_1^n \alpha = 1 - \alpha_{n+1}$ . We only consider the class of DPH distributions for which  $\alpha_{n+1} = 0$ .

Let  $\tau$  be the time till absorption into state  $n + 1$  in the DTMC. We say that  $\tau$  is a random variable of order  $n$  and representation  $(\alpha, B, b)$  [1]. Let  $f(k)$ ,  $F(k)$  and  $\mathfrak{F}(s)$  be the probability mass, cumulative probability and probability generating function of  $\tau$ , respectively, as follows:

$$f(k) = Pr\{\tau = k\} = \alpha B^{k-1} b \geq 0, \quad \text{for } k > 0 \quad (2.1)$$

$$F(k) = Pr\{\tau \leq k\} = \alpha \sum_{i=0}^{k-1} B^i b = \alpha (I - B^k)(I - B)^{-1} b \leq 1 \quad (2.2)$$

$$\mathfrak{F}(z) = E(z^\tau) = \alpha (zI - B)^{-1} b = \frac{U(z)}{V(z)} \quad (2.3)$$

where  $\mathbf{e}$  is an  $n$ -dimensional column vector with all the entries equal to 1,  $I$  is the  $(n \times n)$  identity matrix and the realization  $(\alpha, B, b)$  has constraints such that  $\alpha \geq 0$ ,  $0 \leq b_{ij} \leq 1$  and  $b_i \geq 0$  for all  $i, j$  (i.e.,  $(\alpha, B, b)$  is denoted by a positive realization). In DPH realization, an additional condition  $(I - B)\mathbf{e} = b$  should be needed where  $\mathbf{e}$  is an  $n$ -dimensional matrix column vector with all the entries equal to 1.  $U(z)$  and  $V(z)$  are polynomials with respect to  $s$ . When DPHs are used to approximate general distributions to transform a nonMarkovian process into a DTMC, a very crucial feature is to keep the order of the positive DPH realization as low as possible [2].

**Theorem 2.1.** *Assume that the realization  $(\alpha, B, b)$  of a probability generating function is a positive realization satisfying Eq. (2.1)(2.2) and (2.3). Then there is a nonsingular matrix  $M$  such that another positive realization  $(\hat{\alpha}, \hat{B}, \hat{b})$  transformed by  $\hat{\alpha} = \alpha M$ ,  $\hat{B} = M^{-1} B M$  and  $\hat{b} = M^{-1} b$  satisfies that  $\hat{b} = (I - \hat{B})\mathbf{e}$ .*

*Proof.* Since  $A$  is positive valued matrix and absolute values of its eigenvalues are less than 1, the all entries  $(I - B)^{-1} = \sum_{k=0}^{\infty} B^k$  are positive-valued. The entries of  $x = (I - B)^{-1} b$  are positive. A similarity transform matrix  $M$  is defined by  $M = \text{diag}(x)$ . We can verify easily  $(I - \hat{B})\mathbf{e} = \hat{b}$ .  $\square$

From Theorem 2.1, if a positive realization of a probability generating function is constructed, we can induce a DPH realization defined in [2]. We introduce the discrete Matrix Exponential distribution(DME) as a general version of DPH distributions.

**Definition 2.1** (Discrete Matrix Exponential Distribution). *The representation  $(\alpha, B, b)$  is called as discrete Matrix Exponential Distribution (DME) if it satisfies the following conditions*

- (1) *The induced probability mass function are nonnegative (i.e.,  $f(k) = \alpha B^k b \geq 0$  for all  $k > 0$ )*
- (2) *The induced cumulative probability distribution is less than or equal to 1 for  $k < \infty$  (i.e.,  $F(k) = Pr\{\tau \leq k\} \leq 1$ ).*

(3) The total sum of  $\{f(k)\}$  is 1 (i.e.,  $\alpha(I - B)^{-1}b = 1$ ).

The triple  $(\alpha, B, b)$  is called a representation of the DME distribution.

In the definition of DME, there is no nonnegative constraint in the generator triple  $(\alpha, B, b)$ . Then we can obtain a realization with minimal dimension. Given a probability generating function corresponding to DME distribution, we can easily find a canonical DME realization  $(\alpha, B, b)$  such as

$$\alpha = [a_1 \ a_2 \ \cdots \ a_n] \quad (2.4)$$

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ -b_1 & -b_2 & -b_3 & \cdots & -b_n \end{bmatrix} \quad (2.5)$$

$$b = [0 \ \cdots \ 0 \ 1]^T \quad (2.6)$$

where  $U(z) = a_n z^{n-1} + a_{n-1} z^{p-2} + \cdots + a_1$ ,  $V(z) = z^p + b_n z^{n-1} + \cdots + b_1$  and  $b^T$  is defined by the transpose of  $b$ . The nonnegative conditions of the DME representation  $(\alpha, B, b)$  are not necessary. Therefore, we can expect that the fitting or identification problems with DME distributions are more efficient and easier than those of DPH distribution. However, we assume that a general DME distribution can be obtained. The main problem in this paper is to find as minimal positive realization of DPH distribution as possible.

**2.2. Positive systems: Definitions and problems.** The representation of DPH distributions is closely related to the positive realization of the positive systems. We set  $z = s^{-1}$  and  $H(z) = \mathfrak{F}(s)$ . Assume that there exists the rational transfer function of a given positive system with McMillon degree  $n$  such as

$$H(z) = \frac{a_{n-1}z^{n-1} + \cdots + a_0}{z^n + b_{n-1}z^{n-1} + \cdots + b_0} = \sum_{k>0} h_k z^{-k}. \quad (2.7)$$

Then The impulse response sequence  $h_k$  of  $H(z)$  are nonnegative (i.e.,  $h_k \geq 0$ ). A positive linear system is denoted by a linear dynamical system in which the input  $u$ , state  $x$  and output  $y$  are nonnegative real-valued. We deal with time-invariant finite-dimensional positive in discrete time. The linear system will be represented by

$$\begin{aligned} x(t+1) &= Ax(t) + bu(t) \\ y(t) &= cx(t) \end{aligned} \quad (2.8)$$

where  $A \in R^{N \times N}$ ,  $b \in R^N$ , and  $c^T \in R^N$ . The rational transfer function has a state space realization  $H(z) = c^T(zI - A)^{-1}b$ . The state space realization  $(A, b, c)$  is denoted by a positive realization if there exists a triple  $A \in R_+^{N \times N}$ ,  $b \in R_+^N$ , and  $c^T \in R_+^N$  where  $R_+$  denote nonnegative real number. It is known that the linear system with positive realization  $(A, b, c)$  is a positive system. The positive realization problem is to find a triple  $(A, b, c)$  with nonnegative

entries for a given positive system. The positive realization problem has some important issues. Necessary and sufficient conditions for the existence of a positive linear system are completely presented in [12][13]. It is known that the constraint of positivity may force the dimension  $N$  to be strictly larger than McMillon degree  $n$  [7][10]. The positive minimality problem is to find the lowest possible value of  $N$ . It seems to be very difficult to give tight lower bounds and upper bounds of the dimension of positive realization [10]. The minimality problems were handled in [9][10][11]. We attempt to propose an efficient general algorithm to find the minimal positive realization for a special class of transfer functions by using the convex polyhedral cone approach and linear programming.

### 3. POLYHEDRAL CONE APPROACH FOR DPH DISTRIBUTION

**3.1. Convex Polyhedral cone.** The concepts and results related to convex polyhedral cones are reviewed in this section (refer to references [14] and [15]). A set  $\mathcal{K}$  is a cone, if  $\alpha\mathcal{K} \subset \mathcal{K}$  for all  $\alpha > 0$ . If  $\mathcal{K}$  contains an open ball of  $R^n$  ( $\mathcal{K} \subset R^n$ ), then  $\mathcal{K}$  is solid. A cone that is closed, convex, and solid and pointed is denoted by a proper cone. If  $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$ , then  $\mathcal{K}$  is a pointed cone.  $\text{cone}(v_1, \dots, v_n)$  denotes the polyhedral closed convex cone consisting of all nonnegative linear combinations of vectors  $v_1, \dots, v_n$ . A set  $\mathcal{K} \subset R_+^k$  is denoted by a polyhedral cone if an  $n \in Z_+$  and a  $V = [v_1 \ \dots \ v_n] \in R_+^{n \times k}$  exist in order that

$$\mathcal{K} = \{Vx | x \in R_+^n\} = \text{cone}(V) \quad (3.1)$$

The description in equation (3.1) is a vertex description [14]. A polyhedral cone can also be described by using the intersection of a finite number of half-spaces and hyperplanes such as  $\mathcal{K} = \{y \in R^n | Ay \succeq 0, Cy = 0\}$ , where  $A$  and  $C$  denote matrices with finite dimensions. This is denoted by the half-space description.

Given a closed convex cone  $\mathcal{K}$  in a subspace of  $R^n$ ,  $V$  is called a generator of  $\mathcal{K}$ . For any set  $\mathcal{K}$ , the dual cone is defined by

$$\mathcal{K}^* = \{y \in R^n | x^T y \geq 0, \text{ for all } x \in \mathcal{K}\}. \quad (3.2)$$

A finite set of vectors,  $\{v_1, \dots, v_n\} \subset R^k$ , with a columnar matrix,  $V = [v_1 \ \dots \ v_n] \in R_+^{n \times k}$ , is said to be positively dependent if there exists a  $v_i$  such that it can be written as a nonnegative linear combination of  $\{v_j, j \neq i\}$ ; otherwise, it is positively independent. Consider the extreme direction for a convex cone set (i.e. refer to reference [14]). For a given convex cone,  $\mathcal{K}$ , a nonzero direction  $v \in \mathcal{K}$  is called an extreme point if  $v$  is positively independent of the elements of  $\mathcal{K} - \{cv | c \geq 0\}$ . Recursively, a set of extreme points can be found from  $\{v_1, \dots, v_n\} \subset R^k$ , which is called a frame set. So called, a frame set is an extreme point set spanning the convex cone set  $\mathcal{K}$ . The frame columnar matrix of induced from  $V$  is denoted by  $V_1 = [v_{i_1} \ v_{i_2} \ \dots \ v_{i_{n_1}}]$  with  $n_1 \leq n$ . There are a permutation matrix  $P$  and a nonnegative matrix  $Q$  such that the columnar matrix  $V$  is divided by

$$PV = [V_1 \ V_2], \quad V_2 = V_1Q. \quad (3.3)$$

Using linear programming, An algorithm of a conic independency in Algorithm 1 is derived, which removes all positively dependent points from the given generator. A generator set for a

convex cone is arranged in a matrix columnar,  $V = [v_1 \ v_2 \ \cdots \ v_n] \in R^{m \times n}$ . Select an extreme point set  $V_1$  from  $V$ .

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**Algorithm 1** Algorithm of conic independency (CI)

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**Require:** A generator set for a convex cone is arranged in a matrix columnar,  $V = [v_1 \ v_2 \ \cdots \ v_n] \in R^{m \times n}$ .

**Ensure:** Select an extreme point set  $V_1$  from  $V$ .

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1: procedure CI( $V, V_1, P, Q$ )                                     ▷ Input:  $V$ , Output:  $V_1, P, Q$ 
2:   Set  $V_1 = [ \ ]$  and  $\hat{V} = V$ 
3:   for  $1 \leq k \leq n$  do
4:     • The positive independency test may be expressed as a set of the linear feasibility
       problem:
                               find     $\alpha \in R^n$ 
                               subject to  $V\alpha = v_k$ 
                                        $\alpha \succeq 0$ 
                                        $\alpha_k = 0$ .
5:     if a feasible solution  $\alpha$  exists then
6:       • a direction  $v_k$  must be removed from the generator set  $\hat{V}$  since  $v_k$  is positive
         dependent.
7:     else
8:        $V_1 = [V_1 \ v_k]$ 
9:     end if
10:  end for
11:  return  $V_1$ .
12: end procedure

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Through the fundamental theorem of linear programming presented in reference [16], the basic feasible solutions in solving linear programs are established.

**Theorem 3.1** ([16]). *Let  $A$  be an  $m \times n$  matrix of rank  $m$  and let  $b$  be an  $m$ -vector. Let  $\mathcal{K}$  be the convex polyhedral consisting of all  $n$ -vectors  $x$  satisfying*

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \tag{3.4}$$

*A vector  $x$  is an extreme point of  $\mathcal{K}$  if and only if  $x$  is a basic feasible solution for equation (3.4).*

**3.2. Dual cone generator.** It is noted that the concepts of the polyhedral cone and dual polyhedral cone are useful in analyzing the existence conditions of the positive realization. There are two types of descriptions for polyhedral cones: the vertex-description and the half-space-description. It is relatively easy for the half-space-description (i.e., use generator matrix) to be considered in the computation. The half-space-description of the dual cone is transformed from a generator matrix of a given polyhedral cone [14]. Consider a given matrix  $A$  and closed

convex cone  $\mathcal{K}$ . The membership relationship is  $\{y|Ay \in \mathcal{K}^*\} = \{A^T x|x \in \mathcal{K}\}^*$ . When  $\mathcal{K}$  is  $R_+^n$ , then  $\{y|Ay \geq 0\} = \{A^T x|x \geq 0\}^*$  is obtained. A theorem can be arranged for a generator of a dual cone (i.e., to convert the half-space-description) from a given generator matrix of a convex cone. For more detailed proof, refer to reference [14].

**Theorem 3.2** ([14]). *Assume that there exists a half space-description such that  $\mathcal{K} = \{x \in R^n | Xx \geq 0\}$  for a pointed polyhedral cone  $\mathcal{K}$  where its extreme directions  $X_i$  arranged columnar is given by  $X = [x_1 \ x_2 \ \cdots \ x_N] \in R^{n \times N}$  with  $\text{rank}(X) = N$  and  $n \geq N$ . Define a matrix  $X^\perp = \text{basis}(\mathcal{N}(X^T))$ . A columnar basis  $X^\perp$  is the orthogonal complement of a range space  $\mathcal{R}(X)$ . Then the vertex-description generator  $X^*$  of  $\mathcal{K}^*$  (i.e.  $\mathcal{K}^* = \{X^*b|b \geq 0\}$ ) is given by*

$$X^* = [X^{\dagger T} \quad X^\perp \quad -X^\perp]$$

where  $X^\dagger$  is defined by the pseudoinverse of  $X$  with the same dimensions.

A simplicial convex cone with  $\text{rank}(X) = N = n$  is considered [14]. It is known that the dual cone  $\mathcal{K}^*$  is pointed and  $X^\perp$  is null. Thus  $X^*$  is  $X^* = X^{\dagger T}$ . A proper non-simplicity  $\mathcal{K}$  with a generator  $X$  fat full-rank is considered. Assume that there exists a set of  $N$  conically independent generators of an arbitrary proper polyhedral  $\mathcal{K}$  in  $R^n$  with columnar  $X \in R^{n \times N}$  such that  $N > n$  and  $\text{rank}(X) = n$ . A simpler way for locating a vertex-description of the proper dual cone  $\mathcal{K}^*$  is to first decompose  $\mathcal{K}$  into the simplicity cones  $\mathcal{K}_i$  so that  $\mathcal{K} = \bigcup \mathcal{K}_i$ . The generator  $X^*$  of the dual of  $\mathcal{K}$  amounts to each generator of the dual of each simplicial part. Suppose the proper cone  $\mathcal{K} \subset R^n$  equals the union of  $M$  simplicial cones  $\mathcal{K}_i$  whose extreme directions are all those of  $\mathcal{K}$ . Then the proper dual cone  $\mathcal{K}^*$  is the intersection of the  $M$  dual simplicial cones  $\mathcal{K}_i^*$ :

$$\mathcal{K} = \bigcup_{i=1}^M \mathcal{K}_i \quad \Rightarrow \quad \mathcal{K}^* = \bigcap_{i=1}^M \mathcal{K}_i^*$$

In order to locate the extreme directions of the dual cone, some facets of each simplicial part  $\mathcal{K}_i$  with the generator  $X_i$  are common to facets of  $\mathcal{K}$ . The generators of the dual cones  $\mathcal{K}_i^*$  are computed using  $X^* = X^{\dagger T}$ . The set of extreme directions  $\{X_i^*\}$  of the proper dual cone  $\mathcal{K}^*$  is constituted by the conically independent generators, from the columns of all the dual simplicial matrices  $\{X_i^*\}$ . The extreme directions of the dual cone are orthogonal to the facets of  $\mathcal{K}$ . A generator matrix  $X^*$  of the dual cone  $\mathcal{K}^*$  can be selected as follows:

$$X^* = CI\{X_i^{\dagger T}(:, j) | X_i^{\dagger T} x_l \geq 0\} \quad (3.5)$$

where  $1 \leq i \leq M$ ,  $1 \leq j \leq n$ , and  $1 \leq l \leq N$ .  $CI$  function is defined in Algorithm 1 as the selection method of the columnar matrix of the only conically independent vectors induced from the given set.

## 4. DIMENSION REDUCTION OF A POSITIVE REALIZATION

In this section, we discuss the dimension reduction of the positive realization of DPH from the induced generating functions or DME distributions. The existence problems of positive realizations of discrete or continuous Phase-type distributions have been extensively researched [6] [3] [2]. One of the most interesting outstanding question is to find the minimal order or minimal parameter number of the phase-type distribution. The problem of finding the minimal order of a representation for a given PH distribution is open. The PH representation problem is strongly connected with the positive realization problem which received a great deal of attention in the last theory in control theory [17]. We propose the dimension reduction of the positive realization of DPH distribution by using the concepts of the positive realization in positive system.

**4.1. Positive realizations with upper bound.** An upper bound on the size of nonnegative realizations for a primitive transfer functions with simple poles was presented in construction method [8]. If  $H(z) = c(zI - A)^{-1}b$ , then we have  $H(\alpha z) = c(\alpha zI - A)^{-1}b = \alpha^{-1}c(zI - A/\alpha)^{-1}b$ . Thus, we can remark that the transfer function  $H(z)$  has a nonnegative realization if and only if  $\alpha H(\alpha z)$  has a nonnegative realization for all  $\alpha > 0$ .

**Lemma 4.1.** [8] *A positive impulse response  $h(k)$  is given by  $h(k) = 1 + c_2\lambda_2^k$  for  $k \geq 0$  where  $|\lambda_2| < 1$ . Then there always exists a positive realization  $(A_2, b_2, c_2)$  of dimensions 2.*

We derive a positive realization with upper bound introduced by Hadjicostis [8].

**Theorem 4.1.** *Let  $H(z) = \sum_{j=1}^{\infty} h_j z^{-j}$  be a rational transfer function such that  $h(k) = 1 + \sum_{i=2}^n c_i \lambda_i^k \geq 0$  for  $k \geq 0$ . Let  $H_m(z)$  denote the transfer function corresponding to the shifted sequence  $\{h_m, h_{m+1}, \dots\}$ , i.e.,  $H_m(z) = \sum_{j=m}^{\infty} h_j z^{-j}$ .  $H_m(z)$  admits a positive realization with dimension  $2(n-1)$ .*

*Proof.*  $h(k)$  is a finite nonnegative impulse response that has an  $N$ -dimensional nonnegative realization  $(A_1, b_1, c_1)$  as follows:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (4.1)$$

$$c_1 = [h(N) \quad h(N-1) \quad \cdots \quad h(2) \quad h(1)] \quad (4.2)$$

and  $(A_2, b_2, c_2)$  is the given positive realization of the truncated transfer matrix  $H_N(z)$ . We construct the following nonnegative realization  $(A, b, c)$  of dimension  $K + s(N-1)$

$$A = \begin{bmatrix} A_1 & 0 \\ [b_2 \quad 0] & A_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}, \quad c = [c_1 \quad c_2] \quad (4.3)$$

□

The constructive positive realization with the upper bound for a transfer function with simple pole can be derived. Furthermore, the construction of the positive realization of the transfer function with multiple pole was handled [18]. The construction of the positive realization of the transfer function with complex pole has been discussed in [19]. Thus we expect that we can construct a positive realization for sufficient large dimension. Therefore, without loss of generality, we can remark that there is a positive realization of the positive system for sufficiently large dimension. From now, we will propose a dimension reduction algorithm of a positive realization. First, we consider an important theorem for the necessary and sufficient conditions of the existence of a positive realization introduced in reference [12] and [13].

**Theorem 4.2.** *Let  $H(z)$  be a rational transfer function, with minimal realization  $(F, g, h)$ . Then if  $H(z)$  has a positive realization  $(A_+, b_+, c_+)$ , there exists a generator matrix  $K$  (with  $\mathcal{K} = \text{cone}(K)$ ) such that*

- (1)  $FK \subset \mathcal{K}$
- (2)  $\mathcal{R} \subset \mathcal{K}$  where  $\mathcal{R} = \text{span}\{g, Fg, F^2g, \dots\}$ .
- (3)  $h \in \mathcal{K}^*$ .

and there is a positive realization  $\{\tilde{A}_+, \tilde{b}_+, \tilde{c}_+\}$  such that

$$FK = K\tilde{A}_+, \quad g = K\tilde{b}_+, \quad \tilde{c}_+ = hK$$

where  $K$  is such that  $\mathcal{K} = \text{cone}(K)$  and  $\deg(\tilde{A}_+) \leq \deg(A_+)$ . Here,  $\deg(A)$  is defined by the size of matrix  $A$ .

The first step is to demonstrate that a positive realization of  $H(z)$  with sufficient large dimension can be obtained from any minimal realization if a polyhedral proper cone  $\mathcal{K}$  is found. From a given positive realization, a generator  $M$  of a polyhedral cone  $\mathcal{M}$  is constructed creating a minimal positive realization.

**Lemma 4.2.** *Assume that a positive realization of  $H(z)$  is given by  $(A_+, b_+, c_+)$ , there is a nonsingular matrix  $M = [M_1 \ M_2]$  such that*

$$A_+M = [M_1 \ M_2] \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (4.4)$$

$$b_+ = [M_1 \ M_2] \begin{bmatrix} e_1 \\ 0 \end{bmatrix} \quad (4.5)$$

$$c_+[M_1 \ M_2] = [f_1 \ f_2]. \quad (4.6)$$

and a polyhedral cone is defined by  $\mathcal{I} = [\text{Cone}(M_1^T)]^*$ . Then we have

$$A_+M_1 = M_1A_{11}, \quad b_+ = M_1e_1, \quad c_+M_1 = f_1 \quad (4.7)$$

and  $\mathcal{I}$  satisfies the following:

$$\text{cone}(e_1, A_{11}e_1, A_{11}^2e_1, \dots) \subset \mathcal{I} \quad (4.8)$$

$$A_{11}\mathcal{I} \subset \mathcal{I} \quad (4.9)$$

$$f_1^T \in \mathcal{I}^*. \quad (4.10)$$

*Proof.* Equation (4.7) can be derived easily from equations (4.4)-(4.6) using matrix multiplication computations. Using equation (3.5), a generator of the polyhedral cone  $\mathcal{I}$  can be constructed. Set  $X = M_1^T$ . The generator  $X^*$  of  $\mathcal{I}$  is created by equation (3.5); thus,  $\mathcal{I} = \text{cone}(X^*)$ . This ensures that  $\mathcal{I}$  is finitely generated. Since  $M_1 A_{11}^i e_1 = A_+^i b_+ \geq 0$  for all  $i$  from equation (4.7),  $A_{11}^i e_1 \in \mathcal{I} = [\text{cone}(M_1^T)]^*$  is derived for all  $i$ . Then, obtaining Equation (4.8) is straightforward. From  $x \in A_{11}\mathcal{I}$ ,  $x = A_{11}y$  is obtained for  $y \in \mathcal{I}$ . For  $a \geq 0$ ,  $x^T M_1^T a = y^T A_{11}^T M_1^T a = y M_1^T A_+^T a \geq 0$  is derived because  $A_+ M_1 = M_1 A_{11}$ . Equation (4.9) is obtained. Finally, it is verified  $f_1^T = M_1^T c_+^T \in \mathcal{I}^*$  since  $f_1 = c_+ M_1$ .  $\square$

**Lemma 4.3.** *Assume  $\mathcal{I}$  satisfies equations (4.8), (4.9), and (4.10) where  $\mathcal{I} = [\text{cone}(M_1^T)]^* = \text{cone}(X^*)$ ,  $X = M_1^T$  and the dual cone generator  $X^*$  is defined by (3.5). Then there is a positive realization  $(\tilde{A}_+, \tilde{b}_+, \tilde{c}_+)$  such that*

$$A_{11}X^* = X^*\tilde{A}_+, \quad e_1 = X^*\tilde{b}_+, \quad \tilde{c}_+ = f_1X^*. \quad (4.11)$$

*Proof.* From equation (4.8),  $e_1 \in \mathcal{I} = \text{cone}(X^*)$  is derived. So there exists a nonnegative vector  $\tilde{b}_+$  such that  $e_1 = X^*\tilde{b}_+$ . From equation (4.9),  $AX^* \in \mathcal{I} = \text{cone}(X^*)$  is derived. So there exists a nonnegative matrix  $\tilde{A}_+$  such that  $A_{11}X^* = X^*\tilde{A}_+$ . Finally, from equation (4.10),  $f_1^T = M_1^T c_+^T$  is derived and  $M_1X^*$  is nonnegative matrix by the definition of  $X^*$ , i.e.,  $[\text{cone}(M_1^T)]^* = \text{cone}(X^*)$ . Therefore, it can be seen that  $f_1X^* = c_+M_1X^*$  is a nonnegative vector and  $\tilde{c}_+$  is denoted by  $\tilde{c}_+ = c_+M_1X^*$ . Then  $\tilde{c}_+ = f_1X^*$  is derived.  $\square$

*Theorem 4.2 Proof.* Next, the unobservable parts induced from a positive realization are eliminated remaining  $A_{11}, e_1, f_1$ . There exists a nonsingular matrix  $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$  such that

$$\begin{aligned} TA_{11} &= \begin{bmatrix} F & 0 \\ F_{21} & F_{22} \end{bmatrix} T \\ Te_1 &= \begin{bmatrix} g \\ g_2 \end{bmatrix} \\ f_1 &= [h \quad 0] T \end{aligned}$$

where  $(F, h)$  is a completely observable pair and  $(F, g)$  is a completely controllable pair. Define  $\mathcal{K} = \text{cone}(T_1\mathcal{I})$ . Then, the following is obtained

$$FK = F\text{cone}(T_1\mathcal{I}) = FT_1\mathcal{I} = T_1A_{11}\mathcal{I} \subset T_1\mathcal{I} = \mathcal{K}$$

Finally, the generator matrix  $K = T_1X^*$  can be constructed. It can be verified that  $K$  satisfies the given conditions:

$$FK = K\tilde{A}_+, \quad g = K\tilde{b}_+, \quad \tilde{c}_+ = hK$$

where  $K$  is such that  $\mathcal{K} = \text{cone}(K)$ .  $\square$

In the next theorem, the dimension reduction problem of the positive realization is discussed from a given positive realization with a sufficiently larger dimension.

**Theorem 4.3.** *Let  $H(z)$  be a strictly proper rational transfer function. It is assumed that there is a positive realization  $(A_+, b_+, c_+)$  of  $H(z)$  divided into*

$$A_+ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad b_+ = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad c_+ = [c_1 \quad c_2]$$

and  $\{F, g, h\}$  is a minimal realization of  $H(z)$  (or, not minimal but  $\deg(F) < \deg(A_+)$ ). Assume that there is a polyhedral proper cone  $\mathcal{K}$  satisfying the condition of Theorem 4.2 where a columnar generator  $K$  satisfies  $\mathcal{K} = \text{cone}(K)$ . Then there is a minimal extreme point set with  $K_1 \in R_+^{n_1 \times n_1}$  such that  $\text{cone}(K_1) = \mathcal{K}$  for  $n_1 \leq n$  and  $K = [K_1 \quad K_2]$  and there is a reduced positive realization  $(\bar{A}, \bar{b}, \bar{c})$  such that

$$FK_1 = K_1\bar{A}, \quad g = K_1\bar{b}, \quad \bar{c} = fK_1 \quad (4.12)$$

where there is a matrix  $Q$  such that  $\bar{A} = A_{11} + QA_{21} \in R_+^{n_1 \times n_1}$ ,  $\bar{b} = b_1 + Qb_2$  and  $\bar{c} = cK_1$ .

*Proof.* Using Algorithm 1, an extreme point set is located from the generator  $K$ . Using permutation  $P$  as in equation (3.3),  $PK = [K_1 \quad K_2]$  can be set where  $K_1$  is an extreme generator of  $\mathcal{K}$ . There is a  $Q \in R_+^{n_1 \times n}$  such that  $K_2 = K_1Q$ . Thus, following is rewritten:

$$A [K_1 \quad K_2] = [K_1 \quad K_2] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Then the positive realization reduced from  $(A_+, b_+, c_+)$  is given by  $(\bar{A}, \bar{b}, \bar{c})$  such that

$$AK_1 = K_1\bar{A}, \quad b = K_1\bar{b}, \quad \bar{c} = cK_1.$$

□

**Remark 1.** *The dimension of the generator matrix  $M_1$  in Lemma 4.2 can be similarly reduced by transposing the matrices in Theorem 4.3 and removing the positive dependent rows of  $M_1$ . The transposed form method is similar to Theorem 4.3.*

The dimension of the positive realization can be reduced depending on the method of choosing the induced positive realization  $(\tilde{A}_+, \tilde{b}_+, \tilde{c}_+)$  in equations (4.11) and (4.12). According to Theorem 3.1, the base feasible solutions are extreme points of feasible solutions. All base feasible solutions can be found from equation (4.11). The uncontrollable section in each induced positive realization can be compared and removed. The reduction algorithm is proposed in order to remove the uncontrollable part in positive realization. All feasible basic solutions are chosen using Theorem 3.1. For a certain  $k$ ,  $(\tilde{A}_+, \tilde{b}_+)$  with zero rows and a polyhedra generator  $X^*$  can be chosen from feasible basic solutions such that

$$\tilde{A}_+ = \begin{bmatrix} A_{11} & a_{1,k} & A_{13} \\ 0 & 0 & 0 \\ A_{31} & a_{2,k} & A_{33} \end{bmatrix}, \quad \tilde{b}_+ = \begin{pmatrix} b_1 \\ 0 \\ b_3 \end{pmatrix}, \quad X^* = [V_1 \quad v_k \quad V_3] \quad (4.13)$$

Then, by removing the  $k$ -th row and column of  $\tilde{A}_+$ ,  $k$ -th row of  $\tilde{b}_+$  and  $k$ -column of  $X^*$  in equation (4.13), a new reduced positive  $(\tilde{A}_+, \tilde{b}_+, \tilde{c}_+)$  and  $X^*$  can be reduced. Recursively, call

it again until the dimension cannot be reduced. Finally, a dimension reduction algorithm of positive realization for a given positive system can be summarized in Algorithm 2.

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**Algorithm 2** A dimension reduction algorithm of the positive realization

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**Require:** Assume that a positive transfer function  $H(z)$  is given and the positive realization within the upper bound can be induced.

**Ensure:** Reduce the dimension of a positive realization  $(A_+, b_+, c_+)$  as minimal as possible.

- 1: Find a positive realization  $(A_+, b_+, c_+)$  with a sufficiently large upper bound using Lemma 4.1 and Theorem 4.1 for real value  $\lambda_i$  ( $|\lambda_i| < 1$ ).
  - 2: Using the controllability, compute  $(A_{11}, e_1, f_1)$  and  $M_1$  in Lemma 4.2.
  - 3: Remove positive dependent components of  $M_1$  using Remark 1 and Algorithm 1, and recreate  $(A_{11}, e_1, f_1)$  and  $M_1$ .
  - 4: Compute  $X^*$  and  $(\tilde{A}_+, \tilde{b}_+, \tilde{c}_+)$  using Lemma 4.2.
  - 5: Reduce a new positive realization  $(\tilde{A}_+, \tilde{b}_+, \tilde{c}_+)$  and  $X^*$  as (4.13).
  - 6: Compute a generator  $P$ , a minimal realization  $(F, g, h)$  and a positive realization  $(\tilde{A}_+, \tilde{b}_+, \tilde{c}_+)$  by using the observability in Theorem 4.2.
  - 7: Remove positive dependent components of  $P$  using Theorem 4.3 and Algorithm 1, and recreate a reduced realization  $(F, g, h)$ , a positive realization  $(\tilde{A}_+, \tilde{b}_+, \tilde{c}_+)$  and a generator  $P$ .
- 

**Example 4.1.** The probability mass function  $f(k)$  is defined by  $f(k) = \frac{\gamma}{2^k} \{1 + 5(-\frac{1}{8})^k + 3(-\frac{1}{9})^k\}$  for  $k \geq 0$  and a proper  $\gamma = 0.1047$ , so called,  $f(k) \geq 0$  and  $\sum_{k=0}^{\infty} f(k) = 1$ . Using Lemma 4.1 and Theorem 4.1, a positive realization for sufficient large dimension (i.e., the degree is 6) is obtained as follows: for  $\lambda_1 = -1/8$  and  $\lambda_2 = -1/9$

$$A_+ = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1/8 & 0 & 0 \\ 0 & 0 & 1 & 7/8 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 0 & 1 & 8/9 \end{bmatrix}, \quad b_+ = \begin{bmatrix} 2f(1) \\ f(0) \\ 0.5 + 5\lambda_1^2 \\ 0.5 + 5\lambda_1^3 \\ 0.5 + 3\lambda_2^2 \\ 0.5 + 3\lambda_2^3 \end{bmatrix}$$

$$c_+ = \gamma [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$$

Using Algorithm 2, the dimension of a positive realization can be reduced and a new positive realization is obtained

$$\tilde{A}_+ = \begin{bmatrix} 0.1761 & 0.0000 & 0.5294 \\ 0.0163 & 0.0000 & 0.0294 \\ 0.1456 & 0.5000 & 0.2059 \end{bmatrix}, \quad \tilde{b}_+ = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\tilde{c}_+ = [0.0562 \quad 0.9426 \quad 0.0044].$$

By using Theorem 2.1, a positive realization of DPH distribution is obtained as

$$\hat{B} = \begin{bmatrix} 0.1761 & 0 & 0.8239 \\ 0.0075 & 0.0000 & 0.0210 \\ 0.0936 & 0.7005 & 0.2059 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0 \\ 0.9715 \\ 0 \end{bmatrix}$$

$$\hat{\alpha} = [0.0265 \quad 0.9702 \quad 0.0032]$$

where  $\hat{\alpha} = \tilde{c}_+ M$ ,  $\hat{B} = M^{-1} \tilde{A}_+ M$ ,  $\hat{b} = M^{-1} \tilde{b}_+$  and  $M = \text{diag}(x)$  with  $x = (I - \tilde{A}_+)^{-1} \tilde{b}_+$ . It is known that the dimension the minimal realization is larger than or equal to 3 [8]. We can see that the positive realization of the discrete phase type distribution is minimal.

## 5. CONCLUSION

We have investigated the dimension reduction algorithm of the positive realization for a given discrete phase type distribution by using the concepts of the positive system theory. The dimension of a discrete phase-type realization may be larger than its McMillan degree of probability generating functions. The dimension reduction algorithm is to find as minimal dimension of the positive realization as possible. We use the convex cone approach to provide positive realization with a lower-bound for a given positive transfer function.

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