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INFINITELY MANY REGULAR SUBNORMAL BINARY HERMITIAN LATTICES OVER IMAGINARY QUADRATIC FIELDS

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Abstract. Finiteness of regular normal binary Hermitian lattices are known in several articles. In this article, we point out that there are infinitely many imaginary quadratic fields that admit a regular subnormal binary Hermitian lattice.

1. Introduction

Let $\mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic field with $\mathbb{Q}$-involution and $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{-m})$ with a positive square-free integer m. A Hermitian space V is a vector space over $\mathbb{Q}(\sqrt{-m})$ with a Hermitian map $H : V \times V \to \mathbb{Q}(\sqrt{-m})$. A Hermitian lattice L is defined as a finitely generated $\mathcal{O}$-module in the Hermitian space (V, H) over $\mathbb{Q}(\sqrt{-m})$. If $a = H(v, v)$ for some $v \in L$, we say that a is represented by L. For brevity, we write $H(v) = H(v, v)$ for $v \in V$.

The lattice can be written as

$$L = \mathcal{A}_1 v_1 + \mathcal{A}_2 v_2 + \cdots + \mathcal{A}_n v_n$$

with ideals $\mathcal{A}_i \subset \mathcal{O}$ and vectors $v_i \in V$. The *norm ideal* $\mathfrak{n}L$ of L is an $\mathcal{O}$-ideal generated by the set $\{H(v) \mid v \in L\}$. The *scale ideal* $\mathfrak{s}L$ of L is an $\mathcal{O}$-ideal generated by the set $\{H(v, w) \mid v, w \in L\}$. It is clear that $\mathfrak{n}L \subseteq \mathfrak{s}L$. If $\mathfrak{n}L = \mathfrak{s}L$, then we call L *normal*. Otherwise, we call L *subnormal*. If $\mathfrak{s}L \subseteq \mathcal{O}$, then we call L *integral*. We call a Hermitian lattice L *primitive* if the scaled lattice $L^{\frac{1}{a}}$ is not integral for any non-unit $a \in H(L)$. Through this article, we always assume that (Hermitian) lattices are positive definite and primitive. See O'Meara's book [13] for unexplained terms and notations.

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An integral quadratic form f is called regular if f represents all integers that are represented by the genus of f. A systematic study of regular forms was initiated by Dickson [5]. In the 1950's G. L. Watson, in his Ph.D. dissertation [14] and then in [15], proved that there exist only finitely many inequivalent primitive integral regular ternary quadratic forms over $\mathbb{Z}$ by exhibiting an upper bound for discriminants of those quadratic forms. This approach has been adopted to obtain finiteness results for quadratic forms satisfying the various regularity properties; see [3] for recent results related to these.

It is a natural attempt to extend these to positive definite lattices over number fields. If it is a real number field, the known result is that there are finitely many classes of regular ternary lattices over $\mathbb{Q}(\sqrt{5})$ (see [1]). On the contrary, a little more results are known for imaginary quadratic fields. Earnest and Khosravani [7] showed that there are only finitely many classes of regular *normal* binary Hermitian lattices over a fixed imaginary quadratic field $\mathbb{Q}(\sqrt{-m})$. More precisely, they showed that for a binary Hermitian lattice L, the cardinality of $E(L)$ tends to infinity as the volume of L tends to infinity, where $E(L)$ is the set of integers represented by genus of L but not represented by L itself. Chan and Rokicki [2] showed that for a fixed totally real field F of odd degree over $\mathbb{Q}$, there are only finitely many CM extensions E/F for which there exists a positive definite regular normal Hermitian lattice over the ring of integers of E. In particular, a binary *normal* regular Hermitian lattice exists over the field $\mathbb{Q}(\sqrt{-m})$ if and only if m is

$$1, 2, 3, 5, 6, 7, 10, 11, 15, 19, 23 \text{ or } 31$$

from works on universal binary Hermitian lattices of Earnest and Khosravani [6], Iwabuchi [8], Kim and Park [12]. The authors [9] succeeded in finding such all 68 regular Hermitian lattices as follows, where $\omega_m = \sqrt{-m}$ if $m \equiv 1, 2 \pmod 4$ and $\omega_m = \frac{1+\sqrt{-m}}{2}$ if $m \equiv 3 \pmod 4$. Besides † denotes the universal lattice.

$\mathbb{Q}(\sqrt{-1})$: $\langle 1,1\rangle^{\dagger}$, $\langle 1,2\rangle^{\dagger}$, $\langle 1,3\rangle^{\dagger}$, $\langle 1,4\rangle$, $\langle 1,8\rangle$, $\langle 1,16\rangle$, $\begin{pmatrix} 2 & -1+\omega_1 \\ -1+\overline{\omega}_1 & 3 \end{pmatrix}$, $\begin{pmatrix} 3 & -1+\omega_1 \\ -1+\overline{\omega}_1 & 6 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$\mathbb{Q}(\sqrt{-2})$: $\langle 1,1\rangle^{\dagger}$, $\langle 1,2\rangle^{\dagger}$, $\langle 1,3\rangle^{\dagger}$, $\langle 1,4\rangle^{\dagger}$, $\langle 1,5\rangle^{\dagger}$, $\langle 1,8\rangle$, $\langle 1,16\rangle$, $\langle 1,32\rangle$, $\begin{pmatrix} 2 & \omega_2 \\ \overline{\omega}_2 & 5 \end{pmatrix}$

$\mathbb{Q}(\sqrt{-3})$: $\langle 1,1\rangle^{\dagger}$, $\langle 1,2\rangle^{\dagger}$, $\langle 1,3\rangle$, $\langle 1,4\rangle$, $\langle 1,6\rangle$, $\langle 1,9\rangle$, $\langle 1,12\rangle$, $\langle 1,36\rangle$, $\langle 2,3\rangle$, $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$, $\begin{pmatrix} 3 & 1+\omega_3 \\ 1+\overline{\omega}_3 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$

$\mathbb{Q}(\sqrt{-5})$: $\langle 1,2\rangle^{\dagger}$, $\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix}^{\dagger}$, $\langle 1,8\rangle$, $\langle 1,10\rangle$, $\langle 1,40\rangle$, $\langle 1\rangle \perp 5\begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix} \perp \langle 4\rangle$, $\begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix} \perp \langle 5\rangle$, $\begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix} \perp \langle 20\rangle$

$\mathbb{Q}(\sqrt{-6})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_6 \\ \overline{\omega}_6 & 3 \end{pmatrix}^{\dagger}$, $\langle 1,3\rangle$, $\begin{pmatrix} 2 & \omega_6 \\ \overline{\omega}_6 & 3 \end{pmatrix} \perp 3\begin{pmatrix} 2 & \omega_6 \\ \overline{\omega}_6 & 3 \end{pmatrix}$

$\mathbb{Q}(\sqrt{-7})$: $\langle 1,1\rangle^\dagger$, $\langle 1,2\rangle^\dagger$, $\langle 1,3\rangle^\dagger$, $\langle 1,7\rangle$, $\langle 1,14\rangle$, $\begin{pmatrix} 3 & \omega_7 \\ \overline{\omega}_7 & 3 \end{pmatrix}$

$\mathbb{Q}(\sqrt{-10})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{10} \\ \overline{\omega}_{10} & 5 \end{pmatrix}^\dagger$, $\langle 1,5\rangle$

$\mathbb{Q}(\sqrt{-11})$: $\langle 1,1\rangle^\dagger$, $\langle 1,2\rangle^\dagger$, $\langle 1,4\rangle$, $\langle 1,11\rangle$, $\langle 1,44\rangle$

$\mathbb{Q}(\sqrt{-15})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix}^\dagger$, $\langle 1,3\rangle$, $\langle 1,5\rangle$, $\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix} \perp \langle 5\rangle$,
$\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix} \perp 3\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix} \perp \langle 9\rangle$, $\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix} \perp \langle 15\rangle$

$\mathbb{Q}(\sqrt{-19})$: $\langle 1,2\rangle^\dagger$

$\mathbb{Q}(\sqrt{-23})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{23} \\ \overline{\omega}_{23} & 3 \end{pmatrix}^\dagger$, $\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{23} \\ -1+\overline{\omega}_{23} & 3 \end{pmatrix}^\dagger$

$\mathbb{Q}(\sqrt{-31})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{31} \\ \overline{\omega}_{31} & 4 \end{pmatrix}^\dagger$, $\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{31} \\ -1+\overline{\omega}_{31} & 4 \end{pmatrix}^\dagger$

Similarly, authors [10] found that an integral regular *subnormal* binary lattice of $\mathfrak{n}L = 2\mathcal{O}$ exists over the field $\mathbb{Q}(\sqrt{-m})$ if and only if m is

$$1, 2, 5, 6, 10, 13, 14, 17, 21, 22, 29, 34, 37 \text{ or } 38.$$

But it is not yet completed to find such all Hermitian lattices. It is more difficult to find all binary regular subnormal Hermitian lattices L of $\mathfrak{n}L \neq 2\mathcal{O}$.

In the present article, we will show that primitive regular subnormal binary Hermitian lattices of $\mathfrak{n}L = m\mathcal{O}$ appear over infinitely many imaginary quadratic fields $\mathbb{Q}(\sqrt{-m})$.

2. Result

Any square-free integer m of the form $t^2 + k$ with $k = 1, 2, 3$ has asymptotically positive density [11]

$$\prod_{p:\text{prime}} \left(1 - \frac{N_p}{p^2}\right),$$

where $N_p = \#\{0 \le t \le p^2 - 1 \mid t^2 + k \equiv 0 \pmod{p^2}\}$. Thus infinitely many m's are square-free.

Theorem. *Let $m = t^2 + k$ be square-free with $t > 0$ and $k = 1, 2, 3$. Then a binary subnormal Hermitian lattice*

$$L = \begin{pmatrix} m & t\sqrt{-m} \\ -t\sqrt{-m} & m \end{pmatrix}$$

over $\mathbb{Q}(\sqrt{-m})$ is regular. Thus there are infinitely many imaginary quadratic fields $\mathbb{Q}(\sqrt{-m})$ that admit a regular subnormal binary Hermitian lattice.

Proof. It is clear L represents multiples of m. Note that a Hermitian form associated to L is

$$mx\overline{x} + t\sqrt{-m}x\overline{y} - t\sqrt{-m}\overline{x}y + my\overline{y}$$

for $x, y \in \mathcal{O}$. If $x = x_1 + \sqrt{-m}x_2$, $y = y_1 + \sqrt{-m}y_2$, then we know that L represents

$$\begin{aligned} L' &= m(x_1 + x_2\sqrt{-m})(x_1 - x_2\sqrt{-m}) + t\sqrt{-m}(x_1 + x_2\sqrt{-m})(y_1 - y_2\sqrt{-m}) \\ &\quad - t\sqrt{-m}(x_1 - x_2\sqrt{-m})(y_1 + y_2\sqrt{-m}) + m(y_1 + y_2\sqrt{-m})(y_1 - y_2\sqrt{-m}) \\ &= m(x_1^2 + mx_2^2 + 2tx_1y_2 - 2tx_2y_1 + y_1^2 + my_2^2) \\ &= m[\ (x_1 + ty_2)^2 + (tx_2 - y_1)^2 + (m - t^2)x_2^2 + (m - t^2)y_2^2\ ] \\ &= m[\ (x_1 + ty_2)^2 + (tx_2 - y_1)^2 + kx_2^2 + ky_2^2\ ]. \end{aligned}$$

So $\frac{1}{m}L'$ represents the quadratic form

$$X^2 + Y^2 + kZ^2 + kU^2.$$

The quadratic forms $X^2 + Y^2 + kZ^2 + kU^2$ are universal for $k = 1, 2, 3$ [4]. So the Hermitian lattice L is regular.

From the comment at the beginning of this section, we know that there are infinitely many imaginary quadratic fields $\mathbb{Q}(\sqrt{-m})$ that admit a regular subnormal binary Hermitian lattice of $\mathfrak{n}L = m\mathcal{O}$. □

Remark. *In the above theorem, if L is not primitive, we can get a primitive lattice $\widetilde{L} = L^{1/d}$ scaled by $d = (m, t)$.*

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