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A HYBRID PROJECTION METHOD FOR RELAXED COCOERCIVE MAPPINGS AND STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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Abstract. The purpose of this paper is to introduce a hybrid projection method for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of a variational inclusion problem and the set of common fixed points of a finite family of strict pseudo-contractions in Hilbert spaces.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $B : H \to H$ be a single-valued mapping and $M : H \to 2^H$ be a multivalued mapping. Then, we consider the following variational inclusion problem which is to find $u \in H$ such that

$$0 \in B(u) + M(u). \tag{1.1}$$

The set of solutions of the variational inclusion(1.1) is denoted by $VI(H, B, M)$. Special Cases.

(1) When M is a maximal monotone mapping and B is a strongly monotone and Lipschitz continuous mapping, problem (1.1) has been studied by Huang [14].

(2) If $M = \partial\phi$, where $\partial\phi$ denotes the subdifferential of a proper, convex and lower semi-continuous function $\phi : H \to \mathbb{R}\bigcup\{+\infty\}$, then problem (1.1) reduces to the following problem: find $u \in H$ such that

$$\langle B(u), v - u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{1.2}$$

which is called a nonlinear variational inequality and has been studied by many authors; see, for example [4-5].

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(3) If $M = \partial\delta_C$, where δ_C is the indicator function of C, then problem (1.1) reduces to the following problem: find $u \in C$ such that

$$\langle B(u), v - u\rangle \geq 0, \quad \forall v \in C, \tag{1.3}$$

which is the classical variational inequality; see, e.g., [13,15].

Recall the following definitions:

(1) A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \forall u, v \in C.$$

(2) A is called $\nu-$strong monotone, if for each $x, y \in C$, we have

$$\langle Ax - Ay, x - y\rangle \geq \nu\|x - y\|^2 \ \text{ for a constant } \ \nu > 0.$$

(3) A is said to be $\mu-$cocoercive, if for each $x, y \in C$, we have

$$\langle Ax - Ay, x - y\rangle \geq \mu\|Ax - Ay\|^2 \quad \text{for a constant} \quad \mu > 0.$$

Clearly, every $\mu-$cocoercive mapping is $\frac{1}{\mu}-$Lipschitz continuous.

(4) A is said to be relaxed $\mu-$cocoercive, if there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\mu)\|Ax - Ay\|^2 \ \ \forall x, y \in C.$$

(5) A is said to be relaxed $(\mu, \nu)-$cocoercive, if there exist two constants $\mu, \nu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\mu)\|Ax - Ay\|^2 + \nu\|x - y\|^2 \quad \forall x, y \in C.$$

(6) A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

(7) A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H, u \in Mx, v \in My$ imply $\langle x - y, u - v\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if the graph $G(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in G(M)$ implies $u \in Mx$.

(8) A mapping $S : C \to C$ is said to be $k-$strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

Note that the class of $k-$strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, S is nonexpansive if and only if S is 0-strictly pseudo-contractive.

Recently, many authors considered the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality (1.3) for an α-cocoercive mapping. They obtained some weak and strong convergence theorems(see, for example [7, 8, 15, 19, 27, 30]).

Let f be a bifunction of $C \times C \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for the bifunction f is to find $z \in C$ such that

$$f(z, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(f)$. Numerous problem in physics, optimization and economics reduce to find a solution of (1.4). Some methods have been proposed to solve the equilibrium(1.4)(see, for instance [9, 10, 12, 21, 28, 29]).

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, we may assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$, for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t \to 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semi-continuous.

In 2007, S. Takashi and W. Takahashi [26] introduced a viscosity approximation method for finding a common element of the set of solutions to the equilibrium problem (1.4) and the set of fixed points of a nonexpansive mapping in Hilbert space.

On the other hand, Y. Liu [16] and L.-C. Ceng, et al. [6] introduced different iterative methods for finding a common element of the set of solutions to the equilibrium problem (1.4) and the set of fixed points of a $k-$strictly pseudo-contractive mapping in Hilbert space.

Recently, Takahashi and Takahashi [25] considered the following generalized equilibrium problem:

$$\text{Find } z \in C, \text{ such that } f(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by EP. More precise, they proved the following result:

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-cocoercive mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \bigcap EP \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} f(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S[\alpha_n u + (1 - \alpha_n)z_n], \quad \forall n \in N, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy

$$0 < c \leq \beta_n \leq d < 1, \quad 0 < a \leq \lambda_n \leq b < 2\alpha,$$

$$\lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ *converges strongly to* $z = P_{F(S)\bigcap EP}u$*, where* $P_{F(S)\bigcap EP}$ *is the metric projection from* C *onto* $F(S)\bigcap EP$*.*

It's easy to see that in the case of $A \equiv 0$, this problem (1.5) reduces to the equilibrium problem (1.4).

In this paper, we consider the following generalized equilibrium problem:

$$\text{Find } z \in H, \text{ such that } f(z,y) + \langle Az, y-z\rangle \geq 0, \quad \forall y \in H. \tag{1.6}$$

Denote the set of solutions of (1.6) by Ω.

On the other hand, G. L. Acedo and H. K. Xu [1] introduced iterative methods for finding a common fixed point of a finite family of $k-$strict pseudo-contractions in Hilbert spaces. More precisely, they proposed the following iterative algorithm:

$$\begin{cases} x_1 = \alpha_0 x_0 + (1-\alpha_0)T_0 x_0, \\ x_2 = \alpha_1 x_1 + (1-\alpha_1)T_1 x_1, \\ \quad \cdots \\ x_N = \alpha_{N-1}x_{N-1} + (1-\alpha_{N-1})T_{N-1}x_{N-1}, \\ x_{N+1} = \alpha_N x_N + (1-\alpha_N)T_0 x_N, \\ \quad \cdots . \end{cases}$$

In a more compact form, x_{n+1} can be written as $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}x_n$, where $T_{[n]} = T_i$, with $i = n(modN)$, $0 \leq i \leq N-1$, $N \geq 1$ is a positive integer and $\{T_i\}_{i=0}^{N-1}$ are N strict pseudo-contractions defined on C. Then, they showed that the above algorithm is weakly convergent if the sequence $\{\alpha_n\}$ is appropriately chosen.

In this paper, we introduce a hybrid projection method for the variational inclusion problem (1.1), the generalized equilibrium problem (1.6) and a fixed point problem for a finite family of $k-$strict pseudo-contractions. Then, we obtain a strong convergence theorem. In addition, about the hybrid projection method, we can also see [22, 23].

2. Preliminaries

Throughout this paper, we always let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. We denote by Z^+ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C x \Leftrightarrow \langle x - u, u - y\rangle \geq 0, \quad \forall y \in C.$$

It is also known that $\|x-y\|^2 \geq \|x-P_Cx\|^2 + \|y-P_Cx\|^2$, for all $x \in H$ and $y \in C$.

The following definitions and lemmas are useful for our paper.

Definition 1. ([2, 20]) If M is a maximal monotone mapping on H, then the resolvent operator associated with M is defined by

$$J_{M,\lambda}(u) = (I+\lambda M)^{-1}u, \quad \forall u \in H,$$

where $\lambda > 0$ is a constant and I is the identity operator. Furthermore, $J_{M,\lambda}$ is single-valued.

Definition 2. ([11]) A single-valued operator $A : H \to H$ is said to be hemi-continuous if for any fixed $x, y, z \in H$, the function $t \to \langle A(x+ty), z\rangle$ is continuous at 0^+. It is well known that a continuous mapping must be hemi-continuous.

Definition 3. ([11]) A set-valued mapping $A : X \to 2^{X^*}$ is said to be bounded if $A(B)$ is bounded for every bounded subset B of X.

Lemma 2.1. ([2]) *The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive, that is,*

$$\|J_{M,\lambda}(u) - J_{M,\lambda}(v)\| \leq \|u-v\|, \quad \forall u, v \in H.$$

Lemma 2.2. ([17]) *The resolvent operator $J_{M,\lambda}$ is firmly nonexpansive, that is*

$$\langle J_{M,\lambda}u - J_{M,\lambda}v, u-v\rangle \geq \|J_{M,\lambda}u - J_{M,\lambda}v\|^2, \quad \forall u, v \in H.$$

Lemma 2.3. ([20]) *If $T : X \to 2^{X^*}$ is a maximal monotone mapping and $P : X \to X^*$ is a hemi-continuous bounded monotone operator with $D(P) = X$, then the sum $S = T + P$ is a maximal monotone mapping.*

Lemma 2.4. ([18]) *If $S : C \to C$ is a $k-$strict pseudo-contraction, then the mapping $I - S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-S)x_n \to 0$, then $(I-S)\tilde{x} = 0$.*

Lemma 2.5. ([18]) *If $S : C \to C$ is a $k-$strict pseudo-contraction, then the fixed point set $F(S)$ of S is closed and convex.*

Lemma 2.6. ([24]) *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.7. *The function $u \in H$ is a solution of variational inclusion (1.1) if and only if $u \in H$ satisfies the relation*

$$u = J_{M,\lambda}[u - \lambda Bu],$$

where $\lambda > 0$ is a constant, M is a maximal monotone mapping and $J_{M,\lambda} = (I + \lambda M)^{-1}$ is the resolvent operator.

Proof. Using Definition 2.1, we can obtain the desired result. $\square$

Lemma 2.8. ([10, 3]) *Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if $T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$, then the following hold:

(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e.,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H;$$

(3) $F(T_r) = EP(f)$;
(4) $EP(f)$ is closed and convex.

Lemma 2.9. *Let $A : H \to H$ be a relaxed (μ_1, ν_1)-cocoercive and s_1-Lipschitz continuous mapping, $r > 0$ be a positive number satisfying $0 \leq r \leq \frac{2(\nu_1 - \mu_1 s_1^2)}{s_1^2}$, then $I - rA$ is nonexpansive.*

Proof. For all $x, y \in H$, we have

$$\begin{aligned}
\|(I - rA)x - (I - rA)y\|^2 &= \|(x - y) - r(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2r\langle x - y, Ax - Ay\rangle + r^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2r\big((-\mu_1)\|Ax - Ay\|^2 + \nu_1\|x - y\|^2\big) \\
&\quad + r^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + 2r\mu_1 s_1^2\|x - y\|^2 - 2r\nu_1\|x - y\|^2 \\
&\quad + s_1^2 r^2\|x - y\|^2 \\
&= (1 + 2r\mu_1 s_1^2 - 2r\nu_1 + s_1^2 r^2)\|x - y\|^2 \\
&\leq \|x - y\|^2,
\end{aligned}$$

which implies the mapping $I - rA$ is nonexpansive. $\square$

3. Main results

For any $x_0 \in H$, we define the iteration process $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in H \quad \text{chosen arbitrarily,} \\ y_n = J_{M,\lambda_n}(x_n - \lambda_n B x_n), \\ f(u_n, y) + \langle A y_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in H, \\ z_n = \gamma_n u_n + (1 - \gamma_n) T_{[n]} u_n, \\ v_n = \alpha_n x_n + (1 - \alpha_n) z_n, \\ C_0 = H, \\ C_{n+1} = \{z \in C_n : \|v_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases} \tag{$*$}$$

In this section, we will employ the above iterative process $\{x_n\}$ for finding a common element of the set of solutions of the generalized equilibrium problem (1.6), the set of solutions of the variational inclusion problem (1.1) and the set of common fixed points of a finite family of strict pseudo-contractions in Hilbert spaces. By the construction of C_n and $\{x_n\}$, we can prove $x_n \to w \in H$. Using the nonexpansivity of $I - r_n A$ and $I - \lambda_n B$, the firmly nonexpansivity of J_{M,λ_n} and T_{r_n}, we can obtain $\lim\limits_{n\to\infty} \|x_n - y_n\| = \lim\limits_{n\to\infty} \|y_n - u_n\| = \lim\limits_{n\to\infty} \|z_n - u_n\| = 0$. It follows from lemma 2.4, conditions (A1)-(A4), lemma 2.3 and properties of a maximal monotone mapping that $w \in F$, where $F := \bigcap\limits_{i=0}^{N-1} F(T_i) \bigcap \Omega \bigcap VI(H, B, M) \neq \emptyset$. Finally, by the continuity of $\|\cdot\|$ and the uniqueness of $P_F x_0$, we can obtain $w = P_F x_0$.

Theorem 3.1. *Let H be a real Hilbert space. Let $A : H \to H$ be relaxed (μ_1, ν_1)-cocoercive and s_1-Lipschitz continuous. Let $B : H \to H$ be relaxed (μ_2, ν_2)-cocoercive and s_2-Lipschitz continuous. Let $f : H \times H \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $M : H \to 2^H$ be a maximal monotone mapping. Let $N \geq 1$ be an integer. Let, for each $0 \leq i \leq N-1$, $T_i : H \to H$ be a k_i-strict pseudo-contraction for some $0 \leq k_i < 1$. Let $k = \max\{k_i : 0 \leq i \leq N-1\}$. Assume that $F := \bigcap\limits_{i=0}^{N-1} F(T_i) \bigcap \Omega \bigcap VI(H, B, M) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $(*)$ and $\{\alpha_n\}$, $\{r_n\}$, $\{\lambda_n\}$, $\{\gamma_n\}$ satisfy the following conditions:*

(B1) $0 \leq \alpha_n \leq a < 1$,
(B2) $0 \leq b \leq r_n \leq c \leq \frac{2(\nu_1 - \mu_1 s_1^2)}{s_1^2}$,
(B3) $0 \leq d \leq \lambda_n \leq e \leq \frac{2(\nu_2 - \mu_2 s_2^2)}{s_2^2}$,
(B4) $0 \leq k \leq \gamma_n \leq \gamma < 1$, *for some* $a, b, c, d, e, \gamma \in \mathbb{R}$.

Then, the sequence $\{x_n\}$ converges strongly to $P_F x_0$, where P_F is the metric projection of H onto F.

Proof. Firstly, we show that F is closed, convex. By lemma 2.5, we have $\bigcap_{i=0}^{N-1} F(T_i)$ is closed and convex. It follows from Lemma 2.7 that $VI(H,B,M) = F(J_{M,\lambda}(I-\lambda B))$ (the set of fixed points of $J_{M,\lambda}(I-\lambda B)$), where $0 \le \lambda \le \frac{2(\nu_2-\mu_2 s_2^2)}{s_2^2}$. By Lemma 2.1 and Lemma 2.9, we have $J_{M,\lambda}(I-\lambda B)$ is a nonexpansive mapping of H into itself. Thus, $VI(H,B,M)$ is closed and convex. It follows from Lemma 2.8 that $\Omega = F(T_r(I-rA))$ (the set of fixed points of $T_r(I-rA)$), where $0 \le r \le \frac{2(\nu_1-\mu_1 s_1^2)}{s_1^2}$. By Lemmas 2.8 and 2.9, we have $T_r(I-rA)$ is a nonexpansive mapping of H into itself. Thus, Ω is closed and convex. Therefore, F is closed, convex.

The rest of the proof will be split into six steps.

Step 1. Show that $P_{C_{n+1}}x_0$ is well defined.

Now, we show that C_n is closed and convex for all $n \ge 0$. It's obvious that $C_0 = H$ is closed and convex. Suppose that C_k is closed and convex for some $k \in Z^+$. For $z \in C_k$, we obtain that $\|v_k - z\| \le \|x_k - z\|$ is equivalent to $2\langle z, x_k - v_k\rangle \le \|x_k\|^2 - \|v_k\|^2$. It's easy to see that C_{k+1} is closed and convex. Then, for all $n \ge 0$, C_n is closed and convex. This shows that $P_{C_{n+1}}x_0$ is well defined.

Step 2. Show that $F \subset C_n$ and $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{z_n\}$, $\{v_n\}$ are bounded.

Note that u_n can be rewritten as $u_n = T_{r_n}(y_n - r_n A y_n)$ for each $n \ge 0$. By Lemma 2.9 and conditions (B2) and (B3), we know that $I - r_n A$ and $I - \lambda_n B$ are nonexpansive. So, for any $p \in F$, we have

$$\begin{aligned} \|y_n - p\| &= \|J_{M,\lambda_n}(x_n - \lambda_n B x_n) - J_{M,\lambda_n}(p - \lambda_n B p)\| \\ &\le \|(x_n - \lambda_n B x_n) - (p - \lambda_n B p)\| \le \|x_n - p\|, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|u_n - p\| &= \|T_{r_n}(y_n - r_n A y_n) - T_{r_n}(p - r_n A p)\| \\ &\le \|(y_n - r_n A y_n) - (p - r_n A p)\| \le \|y_n - p\|. \end{aligned} \tag{3.2}$$

On the other hand, from Lemma 2.6 and (B4), we have

$$\begin{aligned} \|z_n - p\|^2 &= \gamma_n\|u_n - p\|^2 + (1-\gamma_n)\|T_{[n]}u_n - p\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\|u_n - T_{[n]}u_n\|^2 \\ &\le \gamma_n\|u_n - p\|^2 + (1-\gamma_n)(\|u_n - p\|^2 + k\|u_n - T_{[n]}u_n\|^2) \\ &\quad - \gamma_n(1-\gamma_n)\|u_n - T_{[n]}u_n\|^2 \\ &= \|u_n - p\|^2 - (1-\gamma_n)(\gamma_n - k)\|u_n - T_{[n]}u_n\|^2 \\ &\le \|u_n - p\|^2. \end{aligned} \tag{3.3}$$

Next, we show that $F \subset C_n$ for each $n \geq 0$. $F \subset C_0 = H$ is obvious. Suppose $F \subset C_m$ for some $m \geq 1$. Then, for any $p \in F \subset C_m$, from (3.1)-(3.3), we have

$$\begin{aligned}\|v_m - p\| &\leq \alpha_m \|x_m - p\| + (1-\alpha_m)\|z_m - p\| \\ &\leq \alpha_m \|x_m - p\| + (1-\alpha_m)\|u_m - p\| \\ &\leq \alpha_m \|x_m - p\| + (1-\alpha_m)\|x_m - p\| \\ &= \|x_m - p\|.\end{aligned}$$

It follows that $p \in C_{m+1}$. This implies $F \subset C_n$ for each $n \geq 0$. Since $x_n = P_{C_n} x_0$ and $F \subset C_n$, we have $\|x_n - x_0\| \leq \|p - x_0\|$ for all $p \in F$. In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \tag{3.4}$$

where $q = P_F x_0$. Hence, $\{y_n\}$, $\{u_n\}$, $\{z_n\}$, $\{v_n\}$ are also bounded.

Step 3. Show that $x_n \to w \in H$ as $n \to \infty$.

Noticing that $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \text{for all } \ n \geq 0.$$

Therefore, $\{\|x_n - x_0\|\}$ is nondecreasing. It follows that the limit of $\{\|x_n - x_0\|\}$ exists. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = P_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\|x_m - x_n\|^2 = \|x_m - P_{C_n} x_0\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.5}$$

Letting $m, n \to \infty$ in (3.5), we have $x_m - x_n \to 0$, as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. we can assume that $x_n \to w$, as $n \to \infty$.

Step 4. Show that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|y_n - u_n\| = \lim_{n\to\infty} \|u_n - z_n\| = 0.$$

Noticing that $x_{n+1} \in C_{n+1}$, we obtain $\|v_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. It follows that

$$\|v_n - x_n\| \leq \|v_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad \text{as } \ n \to \infty. \tag{3.6}$$

On the other hand, from $v_n - x_n = (1-\alpha_n)(z_n - x_n)$, we have

$$\|z_n - x_n\| = \frac{1}{1-\alpha_n}\|v_n - x_n\| \leq \frac{1}{1-a}\|v_n - x_n\| \to 0, \quad \text{as } \ n \to \infty. \tag{3.7}$$

For any $p \in F$, from (3.3), we have

$$\begin{aligned}\|v_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(z_n - p)\|^2 \\ &= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - z_n\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2.\end{aligned} \tag{3.8}$$

Observing that

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(y_n - r_nAy_n) - T_{r_n}(p - r_nAp)\|^2 \\
&\le \|(y_n - r_nAy_n) - (p - r_nAp)\|^2 \\
&= \|(y_n - p) - r_n(Ay_n - Ap)\|^2 \\
&= \|y_n - p\|^2 + r_n^2\|Ay_n - Ap\|^2 - 2r_n\langle y_n - p, Ay_n - Ap\rangle \\
&\le \|y_n - p\|^2 + r_n^2\|Ay_n - Ap\|^2 + 2r_n\mu_1\|Ay_n - Ap\|^2 \\
&\quad - 2r_n\nu_1\|y_n - p\|^2 \\
&\le \|y_n - p\|^2 + r_n^2\|Ay_n - Ap\|^2 + 2r_n\mu_1\|Ay_n - Ap\|^2 \\
&\quad + \frac{-2r_n\nu_1}{s_1^2}\|Ay_n - Ap\|^2 \\
&= \|y_n - p\|^2 + (2r_n\mu_1 + r_n^2 - \frac{2r_n\nu_1}{s_1^2})\|Ay_n - Ap\|^2.
\end{aligned} \tag{3.9}$$

Similarly, we have

$$\|y_n - p\|^2 \le \|x_n - p\|^2 + (2\lambda_n\mu_2 + \lambda_n^2 - \frac{2\lambda_n\nu_2}{s_2^2})\|Bx_n - Bp\|^2. \tag{3.10}$$

It follows from (3.8), (3.3), (3.9) and (3.10) that

$$\begin{aligned}
\|v_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\big(\|y_n - p\|^2 \\
&\quad + (2r_n\mu_1 + r_n^2 - \frac{2r_n\nu_1}{s_1^2})\|Ay_n - Ap\|^2\big) \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\big(\|x_n - p\|^2 \\
&\quad + (2\lambda_n\mu_2 + \lambda_n^2 - \frac{2\lambda_n\nu_2}{s_2^2})\|Bx_n - Bp\|^2 \\
&\quad + (2r_n\mu_1 + r_n^2 - \frac{2r_n\nu_1}{s_1^2})\|Ay_n - Ap\|^2\big) \\
&= \|x_n - p\|^2 + (1-\alpha_n)(2\lambda_n\mu_2 + \lambda_n^2 - \frac{2\lambda_n\nu_2}{s_2^2})\|Bx_n - Bp\|^2 \\
&\quad + (1-\alpha_n)(2r_n\mu_1 + r_n^2 - \frac{2r_n\nu_1}{s_1^2})\|Ay_n - Ap\|^2.
\end{aligned} \tag{3.11}$$

So, we have

$$\begin{aligned}
&-(1-a)(2e\mu_2 + e^2 - \frac{2d\nu_2}{s_2^2})\|Bx_n - Bp\|^2 \\
&\le -(1-\alpha_n)(2\lambda_n\mu_2 + \lambda_n^2 - \frac{2\lambda_n\nu_2}{s_2^2})\|Bx_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - \|v_n - p\|^2,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&-(1-a)(2c\mu_1+c^2-\frac{2b\nu_1}{s_1^2})\|Ay_n-Ap\|^2\\
&\le -(1-\alpha_n)(2r_n\mu_1+r_n^2-\frac{2r_n\nu_1}{s_1^2})\|Ay_n-Ap\|^2\\
&\le \|x_n-p\|^2-\|v_n-p\|.
\end{aligned} \tag{3.13}$$

From (3.6), (3.12) and (3.13), we have

$$\lim_{n\to\infty}\|Bx_n-Bp\|=\lim_{n\to\infty}\|Ay_n-Ap\|=0. \tag{3.14}$$

Using Lemma 2.8, we have

$$\begin{aligned}
\|u_n-p\|^2 &= \|T_{r_n}(y_n-r_nAy_n)-T_{r_n}(p-r_nAp)\|^2\\
&\le \langle (y_n-r_nAy_n)-(p-r_nAp),u_n-p\rangle\\
&= \frac{1}{2}\{\|(I-r_nA)y_n-(I-r_nA)p\|^2+\|u_n-p\|^2\\
&\quad -\|(I-r_nA)y_n-(I-r_nA)p-(u_n-p)\|^2\}\\
&\le \frac{1}{2}\{\|y_n-p\|^2+\|u_n-p\|^2-\|(y_n-u_n)-r_n(Ay_n-Ap)\|^2\}\\
&= \frac{1}{2}\{\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2\\
&\quad +2r_n\langle y_n-u_n,Ay_n-Ap\rangle-r_n^2\|Ay_n-Ap\|^2\},
\end{aligned}$$

which implies

$$\|u_n-p\|^2\le\|y_n-p\|^2-\|y_n-u_n\|^2+2r_n\langle y_n-u_n,Ay_n-Ap\rangle. \tag{3.15}$$

Using Lemma 2.2, we also have

$$\begin{aligned}
\|y_n-p\|^2 &= \|J_{M,\lambda_n}(x_n-\lambda_nBx_n)-J_{M,\lambda_n}(p-\lambda_nBp)\|^2\\
&\le \langle (x_n-\lambda_nBx_n)-(p-\lambda_nBp),y_n-p\rangle\\
&= \frac{1}{2}\{\|(x_n-\lambda_nBx_n)-(p-\lambda_nBp)\|^2+\|y_n-p\|^2\\
&\quad -\|(x_n-\lambda_nBx_n)-(p-\lambda_nBp)-(y_n-p)\|^2\}\\
&\le \frac{1}{2}\{\|x_n-p\|^2+\|y_n-p\|^2-\|(x_n-y_n)-\lambda_n(Bx_n-Bp)\|^2\}\\
&= \frac{1}{2}\{\|x_n-p\|^2+\|y_n-p\|^2-\|x_n-y_n\|^2-\lambda_n^2\|Bx_n-Bp\|^2\\
&\quad +2\lambda_n\langle x_n-y_n,Bx_n-Bp\rangle\},
\end{aligned}$$

which implies

$$\|y_n-p\|^2\le\|x_n-p\|^2-\|x_n-y_n\|^2+2\lambda_n\langle x_n-y_n,Bx_n-Bp\rangle. \tag{3.16}$$

Substitute (3.15) and (3.16) into (3.8) yields that

$$\begin{aligned}
\|v_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\{\|y_n - p\|^2 - \|y_n - u_n\|^2 \\
&\quad + 2r_n \langle y_n - u_n, Ay_n - Ap\rangle\} \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\{\|x_n - p\|^2 - \|x_n - y_n\|^2 \\
&\quad + 2\lambda_n \langle x_n - y_n, Bx_n - Bp\rangle\} - (1-\alpha_n)\|y_n - u_n\|^2 \\
&\quad + 2r_n(1-\alpha_n)\|y_n - u_n\|\|Ay_n - Ap\| \\
&\le \|x_n - p\|^2 - (1-\alpha_n)\|x_n - y_n\|^2 \\
&\quad + 2\lambda_n(1-\alpha_n)\|x_n - y_n\|\|Bx_n - Bp\| - (1-\alpha_n)\|y_n - u_n\|^2 \\
&\quad + 2r_n(1-\alpha_n)\|y_n - u_n\|\|Ay_n - Ap\|.
\end{aligned}$$

This implies

$$\begin{aligned}
(1-a)\|x_n - y_n\|^2 &\le (1-\alpha_n)\|x_n - y_n\|^2 \\
&\le \|x_n - p\|^2 - \|v_n - p\|^2 \\
&\quad + 2\lambda_n(1-\alpha_n)\|x_n - y_n\|\|Bx_n - Bp\| \\
&\quad + 2r_n(1-\alpha_n)\|y_n - u_n\|\|Ay_n - Ap\|,
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
(1-a)\|y_n - u_n\|^2 &\le (1-\alpha_n)\|y_n - u_n\|^2 \\
&\le \|x_n - p\|^2 - \|v_n - p\|^2 \\
&\quad + 2\lambda_n(1-\alpha_n)\|x_n - y_n\|\|Bx_n - Bp\| \\
&\quad + 2r_n(1-\alpha_n)\|y_n - u_n\|\|Ay_n - Ap\|.
\end{aligned} \tag{3.18}$$

It follows from (3.6), (3.14), (3.17) and (3.18) that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.19}$$

From $\|u_n - x_n\| \le \|u_n - y_n\| + \|y_n - x_n\|$, we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.20}$$

From $\|z_n - u_n\| \le \|z_n - x_n\| + \|x_n - u_n\|$, (3.7) and (3.20), we have

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{3.21}$$

Step 5. Show that $w \in F$.

Observing that $\|T_{[n]}u_n - u_n\| = \frac{1}{1-\gamma_n}\|z_n - u_n\| \le \frac{1}{1-\gamma}\|z_n - u_n\|$, from (3.21), we obtain

$$\lim_{n\to\infty} \|T_{[n]}u_n - u_n\| = 0. \tag{3.22}$$

Since $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $x_n \to w, \quad n \to \infty$, we have

$$u_n \to w, \quad n \to \infty. \tag{3.23}$$

Take a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $n_i(modN) = l, \quad l \in \{0, 1, 2, .., N-1\}$, we deduce that

$$\lim_{i\to\infty} \|T_{[l+j]}u_{n_i+j} - u_{n_i+j}\| = \lim_{i\to\infty} \|T_{[n_i+j]}u_{n_i+j} - u_{n_i+j}\| = 0.$$

Then, by Lemma 2.4, we have $w \in T_{[l+j]}$ for all $j \in \{0, 1, 2, \cdots, N-1\}$. This ensures that $w \in \bigcap_{i=0}^{N-1} F(T_i)$. Now we show that $w \in \Omega$. Since $u_n = T_{r_n}(y_n - r_n Ay_n)$, for any $y \in H$, we have

$$f(u_n, y) + \langle Ay_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0,$$

From (A2), we have

$$\langle Ay_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq f(y, u_n). \tag{3.24}$$

Put $z_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in H$. From (3.24), we have

$$\begin{aligned}
\langle z_t - u_n, Az_t\rangle &\geq \langle z_t - u_n, Az_t\rangle - \langle z_t - u_n, Ay_n\rangle \\
&\quad - \langle z_t - u_n, \frac{u_n - y_n}{r_n}\rangle + f(z_t, u_n) \\
&= \langle z_t - u_n, Az_t - Au_n\rangle + \langle z_t - u_n, Au_n - Ay_n\rangle \\
&\quad - \langle z_t - u_n, \frac{u_n - y_n}{r_n}\rangle + f(z_t, u_n).
\end{aligned}$$

Since A is a relaxed $(\mu_1, \nu_1)-$cocoercive, s_1-Lipschitz continuous mapping, from condition (B2), for any $x, y \in H$, we have

$$\begin{aligned}
\langle Ax - Ay, x - y\rangle &\geq (-\mu_1)\|Ax - Ay\|^2 + \nu_1\|x - y\|^2 \\
&\geq (-\mu_1 s_1^2 + \nu_1)\|x - y\|^2 \geq 0,
\end{aligned}$$

which yields that A is monotone, so is B. Further, from (3.19) and the continuity of A, we have $\|Au_n - Ay_n\| \to 0$, as $n \to \infty$. So, from (A4), we have

$$\langle z_t - w, Az_t\rangle \geq f(z_t, w). \tag{3.25}$$

From (A1), (A4) and (3.25), we also have

$$\begin{aligned}
0 &= f(z_t, z_t) \\
&\leq tf(z_t, y) + (1-t)f(z_t, w) \\
&\leq tf(z_t, y) + (1-t)\langle z_t - w, Az_t\rangle \\
&= tf(z_t, y) + (1-t)t\langle y - w, Az_t\rangle,
\end{aligned}$$

and hence,

$$0 \leq f(z_t, y) + (1-t)\langle y - w, Az_t\rangle.$$

Letting $t \to 0$, from (A3), we have, for each $y \in H$,

$$0 \leq f(w, y) + \langle y - w, Aw\rangle.$$

This implies $w \in \Omega$. Next, we show that $w \in VI(H,B,M)$. Since B is a s_2-Lipschitz continuous monotone mapping and $D(B) = H$, by Lemma 2.3, $M + B$ is a maximal monotone mapping. Let $\langle v, f\rangle \in G(M + B)$. Since $f - Bv \in Mv$ and $\frac{1}{\lambda_n}(x_n - y_n - \lambda_n Bx_n) \in My_n$, we have

$$\langle v - y_n, (f - Bv) - \frac{1}{\lambda_n}(x_n - y_n - \lambda_n Bx_n)\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - y_n, f\rangle &\geq \langle v - y_n, Bv + \frac{1}{\lambda_n}(x_n - y_n - \lambda_n Bx_n)\rangle \\
&= \langle v - y_n, Bv - Bx_n\rangle + \langle v - y_n, \frac{1}{\lambda_n}(x_n - y_n)\rangle \\
&= \langle v - y_n, Bv - By_n\rangle + \langle v - y_n, By_n - Bx_n\rangle \\
&\quad + \langle v - y_n, \frac{1}{\lambda_n}(x_n - y_n)\rangle \\
&\geq \langle v - y_n, By_n - Bx_n\rangle + \langle v - y_n, \frac{1}{\lambda_n}(x_n - y_n)\rangle.
\end{aligned}$$

Let $n \to \infty$, we obtain $\langle v - w, f\rangle \geq 0$. Since $B + M$ is maximal monotone, we have $0 \in Bw + Mw$ and hence $w \in VI(H,B,M)$. Therefore, we obtain $w \in F$.

Step 6. Show that $w = P_F x_0$.

By taking limit in (3.4), we have

$$\|w - x_0\| \leq \|q - x_0\|, \quad \text{where } q = P_F x_0.$$

Since $w \in F$, it follows from the uniqueness of $P_F x_0$ that $w = P_F x_0$. □

Remark 1. In theorem 3.1, taking $M = \partial\phi$ or $M = \partial\delta_C$, $f \equiv 0$, $A \equiv 0$, $k = 0$, then, we can obtain a common element of the set of solutions of problem (1.2) or (1.3) and the set of common fixed points of a finite family of nonexpansive mappings in Hilbert spaces.

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