East Asian Mathematical Journal
Vol. 28 (2012), No. 3, pp. 281–285

GENERALIZED PRIME IDEALS IN NON-ASSOCIATIVE NEAR-RINGS I

Yong Uk Cho

Abstract. In this paper, the concept of $*$-prime ideals in non-associative near-rings is introduced and then will be studied. For this purpose, first we introduce the notions of $*$-operation, $*$-prime ideal and $*$-system in a near-ring. Next, we will define the $*$-sequence, $*$-strongly nilpotent and $*$-prime radical of near-rings, and then obtain some characterizations of $*$-prime ideal and $*$-prime radical $r_s(I)$ of an ideal I of near-ring N.

1. Introduction

A near-ring N is an algebraic system $(N, +, \cdot)$ with two binary operations, say $+$ and $\cdot$ such that $(N, +)$ is a group (not necessarily abelian) with neutral element 0, $(N, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all a, b, c in N.

In this near-ring, if $(N, \cdot)$ is not a semigroup, then N is a non-associative near-ring. If N has a unity 1, then N is called *unitary*. An element d in N is called *distributive* if $(a + b)d = ad + bd$ for all a and b in N. A near-ring N is called *distributive* if every element in N is distributive.

An *ideal* of N is a subset I of N such that (i) $(I,\ +)$ is a normal subgroup of $(N,\ +)$, (ii) $a(I + b) - ab \subset I$ for all $a,\ b \in N$, (iii) $(I + a)b - ab \subset I$ for all $a,\ b \in N$. If I satisfies (i) and (ii) then it is called a *left ideal* of N. If I satisfies (i) and (iii) then it is called a *right ideal* of N.

On the other hand, an *N-subgroup* of N is any subset H of N such that (i) $(H,\ +)$ is a subgroup of $(N,\ +)$, (ii) $NH \subset H$ and (iii) $HN \subset H$. If H satisfies (i) and (ii) then it is called a *left N-subgroup* of N. If H satisfies (i) and (iii) then it is called a *right N-subgroup* of N. In case, $(H,\ +)$ is normal in above, we say that *normal N-subgroup*, *normal left N-subgroup* and *normal right N-subgroup* instead of N-subgroup, left N-subgroup and right N-subgroup, respectively.

Note that normal N-subgroups of N are not equivalent to ideals of N.

We consider the following notations: Given a near-ring N,

$$N_0 = \{a \in N \mid 0a = 0\}$$

Received June 7, 2011; Revised September 25, 2011; Accepted April 25, 2012.
2000 *Mathematics Subject Classification.* 16Y30.
Key words and phrases. Non-associative near-rings, $*$-prime ideal, $*$-system, $*$-strongly nilpotent and $*$-prime radical.

which is called the *zero symmetric part* of N,

$$N_c = \{a \in N \mid 0a = a\} = \{a \in N \mid ra = a,\ for\ all\ r \in N\}$$

which is called the *constant part* of N.

We note that N_0 and N_c are subnear-rings of N. A near-ring N with the extra axiom $0a = 0$ for all $a \in N$, that is, $N = N_0$ is said to be *zero symmetric,* also, in case $N = N_c$, N is called a *constant* near-ring. From the Pierce decomposition theorem, we get the important fact:

$$N = N_0 \oplus N_c$$

as additive groups. So every element $a \in N$ has a unique representation of the form $a = b + c$, where $b \in N_0$ and $c \in N_c$.

Throughout this paper, by a near-ring, we mean a zero-symmetric non-associative near-ring. For basic definitions and results on near-rings, one may refer Pilz [5].

Let $(G, +)$ be a group (not necessarily abelian). In the set

$$M(G) = \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f + g$ of any two mappings f, g in $M(G)$ by the rule $x(f + g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* of the group G. Also, if we define the set

$$M_0(G) = \{f \in M(G) \mid 0f = 0\},$$

then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

2. Results on $*$-prime ideals and $*$-prime radicals

Groenewald and Potgieter [1] generalized the notion of prime ideals in associative near-rings and introduced the concept of f-prime ideals in associative near-rings. The notion of f-prime ideals in associative near rings actually extends the notion of f-prime ideals in associative rings due to Murata et al. [2]. Myung [3] introduced the notion of $*$-prime ideals in non-associative rings. Corresponding to $*$-prime ideals in non-associative rings, we can introduce in this paper the $*$-prime ideals in non-associative near-rings. For this purpose, first we define the notions of $*$-system and $*$-prime ideal in a near-ring and prove that complement of a $*$-system is a $*$-prime ideal.

In this section, we define $*$-operation for the purpose of $*$-prime ideals, and obtain some characterizations of $*$-prime ideal and $*$-prime radical.

The concept of $*$-operation for rings was introduced by Myung [3], [4]. We can extend this concept to near-rings as following:

Definition 1. Let $F(N)$ be the set of all ideals in N. A $*$-operation is a mapping from $F(N) \times F(N)$ into the family of additive subgroups of N satisfying the following conditions.

(i) for A, B, C, D in $F(N)$, if $A \subseteq B$ and $C \subseteq D$, then $A * C \subseteq B * D$.

(ii) $A * B \subseteq A \cap B$ for all A, B in $F(N)$.
(iii) $(A + C) * (B + C) \subseteq (A * B) + C$ for all A, B, C in $F(N)$.

Hereafter, by a near-ring we mean a near-ring N in which a $*$-operation is defined.

Now, we may obtain the following examples of $*$-operations in N.

Example 1. Let N be a near-ring. Define $*$ on $F(N) \times F(N)$ by $A * B$ is a normal subgroup generated by $\{ab | a \in A, b \in B\}$. Then this $*$-operation satisfy the conditions stated in the above Definition 1. For, the conditions (i) and (ii) are trivially true. If $A, B, C \in F(N)$, then $(A+C)(B+C) \subseteq AB+C$. Thus the set of all generators of $(A + C) * (B + C)$ are of the form $ab + c$ for $a \in A, b \in B$ and $c \in C$. Clearly $A * B + C$ is a normal subgroup of $(N, +)$ and it contains all the elements of $AB + C$. Thus $(A + C) * (B + C) \subseteq A * B + C$. Hence for any near-ring N, always $*$-operation exists.

Definition 2. A proper ideal I in a near-ring is said to be $*$-prime if $A * B \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$ for A, B in $F(N)$.

Definition 3. A non-empty subset M of N is said to be $*$-system if $A \cap M \neq \emptyset$ and $B \cap M \neq \emptyset$ implies $A * B \cap M \neq \emptyset$ for $A, B \in F(N)$.

In the following, we give some examples of $*$-prime ideals in N.

Example 2. Consider the near-ring $(N, +, .)$ defined on Dihedral group $(D_8, +)$ according to the scheme (0,9,0,9,1,3,1,3)(p. 415 [5]). This near-ring is non-associative, since $(a+b)((2a+b)(3a+b)) = a+b$ and $((a+b)(2a+b))(3a+b) = 3a + b$. One can check that the proper ideals of the above near-ring are $I_1 = \{0, 2a\}$ and $I_2 = \{0, a, 2a, 3a\}$. This follows from the fact that the above near-ring is distributive and I_1 and I_2 are the only normal subgroups which are closed under left and right multiplications by elements of N. Define $*$ on $F(N) \times F(N)$ as in Example 1. For this $*$-operation, it is easy to observe that I_2 is $*$-prime and I_1 is not a $*$-prime ideal in N.

Now, we can obtain some equivalent conditions of $*$-prime ideals in N.

Proposition 2.1. *Let I be a proper ideal in a near-ring N. Then the following are equivalent:*

(i) If $A * B \subseteq I$ for A, B in $F(N)$, then either $A \subseteq I$ or $B \subseteq I$.
(ii) If $A \cap C(I) \neq \emptyset$ and $B \cap C(I) \neq \emptyset$, then $(A * B) \cap C(I) \neq \emptyset$ for $A, B \in F(N)$. Here $C(I)$ denotes complement of I.
(iii) If a and b are in $C(I)$, then $(< a > * < b >) \cap C(I) \neq \emptyset$, where $< x >$ denotes the ideal generated by $x \in N$.

Proof. $(i) \Rightarrow (ii)$. Assume the condition (i). If $A \cap C(I) \neq \emptyset$ and $B \cap C(I) \neq \emptyset$, then there exist a in A and b in B such that $a \in C(I)$ and $b \in C(I)$, that is, $a \notin I$ and $b \notin I$. These fact implies that $A \nsubseteq I$ and $B \nsubseteq I$. From the condition (i), we see that $A * B \nsubseteq I$, that is, there exists $c \in (A * B)$ such that $c \notin I$, equivalently, there exists $c \in (A * B)$ such that $c \in C(I)$. Hence, $(A * B) \cap C(I) \neq \emptyset$ for $A, B \in F(N)$.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ can be, analogously, proved as $(i) \Rightarrow (ii)$. □

Remark 1. By the above Proposition 2.1, an ideal I is a $*-$prime ideal if and only if $C(I)$ is a $*-$system. Thus in Example 2, the set $M = \{b, a+b, 2a+b, 3a+b\}$ is a $*-$system.

Definition 4. A sequence $a_0, a_1, \ldots, a_n, \ldots$ of elements in N is said to be a $*$-sequence if $a_n \in < a_{n-1} > * < a_{n-1} >$ for all $n \geq 1$.

Lemma 2.2. *Every $*$-sequence is a $*-$system in N.*

Proof. Let $S = \{a_0, a_1, \ldots, a_n, \ldots\}$ be a $*-$sequence in N. If $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$, then there exist elements a_k and a_ℓ in S such that $a_k \in A$ and $a_\ell \in B$. If $k \geq \ell$, then $a_{k+1} \in < a_k > * < a_k > \subseteq < a_k > * < a_\ell > \subseteq A * B$ and so $(A * B) \cap S \neq \emptyset$. Thus S is a $*-$system in N. □

Definition 5. An element $a \in N$ is said to be $*$-strongly nilpotent if every $*$-sequence $a_0, a_1, \ldots, a_n, \ldots$ with $a_0 = a$ vanishes. That is, there exists an integer $k \geq 1$ such that $a_s = 0$ for all $s \geq k$.

Definition 6. If I is a proper ideal of N, then the $*$-prime radical $r_S(I)$ of I is the set of all elements $x \in N$ such that every $*$-system that contains x contains an element of I.

Proposition 2.3. *For an ideal I of a near-ring N, $r_S(I)$ is the intersection of all $*$-prime ideals in N containing I.*

Proof. Let $x \in r_S(I)$. Suppose $x \notin \cap P_i$, where P_i is a $*$-prime ideal containing I. By assumption there exists a $*$-prime ideal P such that $x \notin P$ and $I \subseteq P$. Since P is a $*$-prime ideal, $C(P)$ is a $*$-system containing x and $C(P) \cap I = \emptyset$. This is a contradiction. Hence $r_S(I) \subseteq \cap P_i$.

Conversely, if $x \in \cap P_i$ and $x \notin r_S(I)$, then there exists a $*$-system M such that $x \in M$ and $M \cap I = \emptyset$. This implies that $C(M) = P$ is a $*$-prime ideal and $x \notin P$, a contradiction. Thus $\cap P_i \subseteq r_S(I)$ □

Proposition 2.4. *Let N be a near-ring. Then $r_S(N) = \{n \in N/n$ is $*$-strongly nilpotent $\}$.*

Proof. Let $x \in r_S(N)$. If x is not $*$-strongly nilpotent, then there exists a $*$-sequence $S = \{a_0, a_1, \ldots, a_n, \ldots\}$ with $a_0 = x$ and $a_n \neq 0$ for all $n \geq 1$. By Lemma 2.2, S is a $*$-system. Again by Proposition 2.1, $C(S)$ is a $*$-prime ideal and note that $x \notin C(S)$. Thus $x \notin r_S(N)$, a contradiction.

Conversely let x be a $*$-strongly nilpotent. If $x \notin r_S(N)$, then there exists a $*$-prime ideal$P$ such that $x \notin P$. By Proposition 2.1, $C(P)$ is a $*$-system and $x \in C(P)$. Since $a_0 = x \in < x > \cap C(P)$, by the definition of $*$-system we get $(< a_0 > * < a_0 >) \cap C(P) \neq \emptyset$. Let $a_1 \in (< a_0 > * < a_0 >) \cap C(P)$. Since $< a_1 > \cap C(P) \neq \emptyset$ we get an element $a_2 \in (< a_1 > * < a_1 >) \cap C(P)$. Continuing in this way we get a $*$-sequence $S = \{a_0, a_1, \ldots\}$ with $a_0 = x$. Note that $S \subseteq C(P)$. By the assumption, x is $*$-strongly nilpotent, there exists an integer $k \geq 1$ such that $a_s = 0$ for all $s \geq k$. Thus $a_k = 0 \in P$ and so $P \cap C(P) \neq \emptyset$, a contradiction. Thus $x \in r_S(N)$. □

References

[1] N. J. Groenewald and P. C. Potgieter *A generalization of prime ideals in near-rings*, Comm. in Algebra **12**(15) (1984), 1835–1853.

[2] K. Murata, Y. Murata and K. Marubayashi *A generalization of prime radical in rings*, Osaka J. Math. **66** (1969), 291–301.

[3] H. C. Myung *On prime ideals and primary decomposition in a non-associative rings*, Osaka J. Math. **9** (1972), 41–47.

[4] H. C. Myung *A generalization of prime radical in non-associative rings*, Pacific Journal of Mathematics **42** (1) (1972), 187–193.

[5] G. Pilz *Near-rings*, , North Holland Publishing Company, Amsterdam, New York, Oxford 1983.

Yong Uk Cho

Department of Mathematics, College of Education, Silla University, Pusan 617-736, Korea

E-mail address: yucho@silla.ac.kr