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STRONG CONVERGENCE THEOREMS FOR GENERALIZED VARIATIONAL INEQUALITIES AND RELATIVELY WEAK NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce an iterative sequence by using a hybrid generalized $f-$projection algorithm for finding a common element of the set of fixed points of a relatively weak nonexpansive mapping and the set of solutions of a generalized variational inequality in a Banach space. Our results extend and improve the recent ones announced by Y. Liu [Strong convergence theorems for variational inequalities and relatively weak nonexpansive mappings, J. Glob. Optim. 46 (2010), 319–329], J. Fan, X. Liu and J. Li [Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces, Nonlinear Analysis 70 (2009), 3997–4007], and many others.

1. Introduction

Let B be a Banach space, B^* be the dual space of B. $\langle\cdot,\cdot\rangle$ denotes the duality pairing of B^* and B. We denote by $J : B \to 2^{B^*}$ the normalized duality mapping from B to 2^{B^*}, defined by

$$J(x) := \{v \in B^* : \langle v, x\rangle = \|v\|^2 = \|x\|^2\}, \quad \forall x \in B.$$

The duality mapping J has the following properties:

(i) if B is smooth, then J is single-valued;
(ii) if B is strictly convex, then J is one-to-one;
(iii) if B is reflexive, then J is surjective;
(iv) if B is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of B.

Let B be a reflexive, strictly convex, smooth Banach space and J the duality mapping from B into B^*. Then J^* is also single-valued, one-to-one, surjective, and it is the duality mapping from B^* into B, i.e., $J^*J = I$. Let K be

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a nonempty closed convex subset of B, let $T : K \to 2^{B^*}$ be a set-valued mapping and let $f : K \to R \bigcup \{+\infty\}$ be a functional. We consider the following generalized variational inequality:

$$\text{Find } x \in K, \text{ such that there exists } u \in Tx \text{ satisfying}$$

$$\langle u, y - x\rangle + f(y) - f(x) \geq 0, \text{ for all } y \in K. \tag{1.1}$$

The set of solutions of the generalized variational inequality (1.1) is denoted by Ω.

If $f \equiv 0$ and T is single-valued, then (1.1) reduces to the classical variational inequality which consists in finding $x \in K$ such that

$$\langle Tx, y - x\rangle \geq 0, \text{ for all } y \in K. \tag{1.2}$$

The set of solutions of the classical variational inequality (1.2) is denoted by $VI(K, T)$. The theory of variational inequalities has been intensively considered due to its various applications in operations research, economic equilibrium and engineering design. One of the most interesting and important problems in the theory of the variational inequality is the development of efficient iterative algorithms for approximating its solutions. Many iterative methods for solving the variational inequality (1.2) have been developed, e.g., see [1,6,7,9,11-13,15-17]. Alber[1-3] introduced the generalized projections $\pi_K : B^* \to K$ and $\Pi_K : B \to K$ in uniformly convex and uniformly smooth Banach spaces. Li[16] extended the generalized projections from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces. Many scholars studied various iterative algorithms to solve the variational inequality (1.2) by using the generalized projections, see [6,9,13,15,17]. But, due to the presence of nonlinear terms, the projection methods presented in [6,9,13,15,17] cannot be applied to suggest any iterative scheme for generalized variational inequality (1.1) in Banach spaces. Fortunately, Wu and Huang in [20,21] introduced a new generalized $f-$projection operator. They in [20] proposed an iterative method of approximate solutions for the generalized variational inequality (1.1) when T is single-valued and f is convex lower semi-continuous and positively homogeneous in compact subsets of Banach spaces. Recently, Fan et al. [10] established a Mann type iterative scheme of approximating solutions for generalized variational inequality (1.1) when T is set-valued in noncompact subsets of Banach spaces, without assuming the positive homogeneity of f. More precisely, they proved the following theorem:

Theorem FLL. (Fan, Liu and Li [10], Theorem 3.5) *Let B be a uniformly convex and uniformly smooth Banach space and let K be a closed convex subset of B and $0 \in K$. Let $f : K \to R$ be convex, lower semi-continuous and $f(x) \geq 0$ for all $x \in K$ and $f(0) = 0$. Let $T : K \to 2^{B^*}$ be upper-continuous with closed values; Suppose that there exists a positive number ρ such that*

$$\langle u, J^*(Jx - \rho u)\rangle \geq 0, \ \textit{for all } x \in K, u \in Tx$$

and $J - \rho T : K \to 2^{B^*}$ *is compact. Suppose*

$$\langle \rho u, y\rangle + \rho f(y) \le 0, \text{ for all } x \in K, u \in Tx, y \in \Omega.$$

Then the variational inequality (1.1) has a solution $x^* \in K$ *and the sequence* $\{x_n\}$ *defined by the following iteration scheme:*

$$\begin{cases} u_n \in Tx_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \pi_K^f(Jx_n - \rho u_n), \quad n = 0, 1, 2, \cdots, \end{cases}$$

where $\{\alpha_n\}$ *satisfies: (a)* $0 \le \alpha_n \le 1$ *for all* $n = 0, 1, 2, \cdots$*; (b)* $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, *converges strongly to* $x^* \in K$

In addition, Qin et al. [19] introduced a hybrid projection algorithm to find a common element of the set of solutions of an equilibrium problem and the set of common fixed points of two quasi-$\phi-$nonexpansive mappings in Banach spaces. More precisely, they proposed the following iterative sequence:

$$\begin{cases} x_0 \in B \quad \text{chosen arbitrarily}, \\ C_1 = K, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in K \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \ \ \forall y \in K, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases}$$

On the other hand, Kohasaka and Takahashi [14] introduced the definition of the relatively weak nonexpansive mapping. They proved that $J_r = (J + rA)^{-1}J$, for $r > 0$ is relatively weak nonexpansive, where $A \subset B \times B^*$ is a continuous monotone mapping with $A^{-1}0 \ne \emptyset$ and B is a smooth, strictly convex and reflexive Banach space. Very recently, Liu [1] introduced a new iterative sequence by using a hybrid generalized projection algorithm for finding a common element of the set of fixed points of a relatively weak nonexpansive mapping and the set of solutions of the variational inequality (1.2) in a noncompact subsets of Banach spaces without assuming the compactness of the operator $J - \rho T$.

Although Theorem FLL removes the compactness of K, but it is assumed that $J - \rho T$ is compact which is a very strong condition. Motivated by [1], [10], [19] and some other related papers, our purpose in this paper is to establish a new iteration sequence for approximating the common element of the set of solutions of the generalized variational inequality (1.1) and the set of fixed points of a relatively weak nonexpansive mapping in noncompact subsets of Banach spaces without assuming the compactness of the operator $J - \rho T$.

2. Preliminaries

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively.

Let X, Y be two topological spaces. Let $F : X \to 2^Y$ be a set-valued mapping with nonempty values, F is said to be:

(i) upper semi-continuous at $x_0 \in X$ if, for any open set V in Y containing $F(x_0)$, there exists an open neighborhood U of x_0 in X such that $F(U) \subset V$;

(ii) upper semi-continuous in X if it is upper semi-continuous at each point of X;

(iii) closed if it has a closed graph, i.e., $GrF = \{(x, y) : x \in X, y \in F(x)\}$ is closed in $X \times Y$;

(iv) compact if it is continuous and maps the bounded subset of $D(F)$ onto the relatively compact subsets of Y.

When $\{x_n\}$ is a sequence in B, we denote strong convergence of $\{x_n\}$ to $x \in B$ by $x_n \to x$.

Let $U = \{x \in B : \|x\| = 1\}$. A Banach space B is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in U$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in U and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. A Banach space B is said to be smooth provided $\lim_{t\to 0} \dfrac{\|x + ty\| - \|x\|}{t}$ exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$.

In [1], Alber introduced the functional $V : B^* \times B \to R$ defined by

$$V(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2,$$

where $\phi \in B^*$, $x \in B$ and B is a uniformly convex and uniformly smooth Banach space.

It is easy to see that $V(\phi, x) \geq (\|\phi\| - \|x\|)^2$. Thus the functional $V : B^* \times B \to R^+$ is nonnegative.

Definition 1. ([1]) If B is a uniformly convex and uniformly smooth Banach space, the generalized projection $\pi_K : B^* \to K$ is a mapping that assigns an arbitrary point $\phi \in B^*$ to the minimum point of the functional $V(\phi, x)$, i.e., a solution to the minimization problem

$$V(\phi, \pi_K(\phi)) = \inf_{y\in K} V(\phi, y).$$

In [1,16], we can find the following properties of V and π_K:

(i) $V : B^* \times B \to R$ is continuous.

(ii) $V(\phi, x) = 0$ if and only if $\phi = Jx$.

(iii) $V(J\pi_K\phi, x) \leq V(\phi, x)$ for all $\phi \in B^*$ and $x \in B$.

(iv) $V(\phi, x)$ is convex with respect to ϕ when x is fixed and with respect to x when ϕ is fixed.

(v) The operator π_K is J fixed at each point $x \in K$, i.e., $\pi_K(Jx) = x$.

The functional $V_2 : B \times B \to R$ is defined by $V_2(x, y) = V(Jx, y)$, $\forall x, y \in B$. From [1], we know that if B is strictly convex, then $V_2(x, y) = 0$ if and only if $x = y$.

Let $f : K \to R \bigcup \{+\infty\}$ be proper, convex and lower semi-continuous, Wu and Huang [21] introduced the functional $G : B^* \times K \to R \bigcup \{+\infty\}$ defined as follows:

$$G(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2 + 2\rho f(x),$$

where $\phi \in B^*, x \in B$ and $\rho > 0$ is a fixed constant. In fact, $G(\phi, x) = V(\phi, x) + 2\rho f(x)$.

From the definitions of G and f, it is easy to have the following properties:

i) $G(\phi, x)$ is convex and continuous with respect to ϕ when x is fixed;
ii) $G(\phi, x)$ is convex and lower semi-continuous with respect to x when ϕ is fixed;
iii) $(\|\phi\| - \|x\|)^2 + 2\rho f(x) \le G(\phi, x) \le (\|\phi\| + \|x\|)^2 + 2\rho f(x)$.

The functional $G_2 : B \times B \to R$ is defined by

$$G_2(x, y) = G(Jx, y), \quad \forall x, y \in B.$$

From the definitions of V_2 and G_2, we have $G_2(x, y) = V_2(x, y) + 2\rho f(y)$ for all $x, y \in B$.

Remark 1. If $f \equiv 0$, then $G(\phi, x) = V(\phi, x)$, $\forall \phi \in B^*, x \in K$.

Definition 2. ([21]) Let B be a Banach space with dual space B^* and K be a nonempty, closed and convex subset of B. We say that $\pi_K^f : B^* \to 2^K$ is a generalized f -projection operator if

$$\pi_K^f \phi = \{u \in K : G(\phi, u) = \inf_{y \in K} G(\phi, y)\} \forall \phi \in B^*.$$

Remark 2. If $f \equiv 0$, then the generalized f-projection operator reduces to the generalized projection operator $\pi_K : B^* \to K$ defined by Alber [1]and Li [16].

Definition 3. We say that a Banach space B has the property (h) if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ implies $x_n \to x$.

Remark 3. It is well known that any locally uniformly convex space has the property (h).

Let B be a reflexive and strictly convex Banach space with dual space B^* and K be a nonempty closed convex subset of B. Let $f : K \to R \bigcup \{+\infty\}$ is proper, convex, lower semi-continuous, then the following properties of the operator π_K^f can be found in [10, 20]:

(f1) For any given $\phi \in B^*$, $\pi_K^f \phi$ is a nonempty, closed and convex subset of K;
(f2) π_K^f is monotone, i.e., for any $\phi_1, \phi_2 \in B^*$, $x_1 \in \pi_K^f \phi_1$ and $x_2 \in \pi_K^f \phi_2$,

$$\langle x_1 - x_2, \phi_1 - \phi_2\rangle \ge 0;$$

(f3) If B is smooth, then for any given $\phi \in B^*$, $x \in \pi_K^f \phi$ if and only if

$$\langle \phi - Jx, x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \forall y \in K;$$

(f4) $\pi_K^f : B^* \to K$ is single-valued and norm to weak continuous;

(f5) moreover, if B has the property (h), then $\pi_K^f : B^* \to K$ is continuous.

Using the property (f3) of the generalized $f-$projection operator π_K^f, Fan et al. obtained the following lemma in [10].

Lemma 2.1. *Let B be a smooth reflexive Banach space with dual space B^*. Let $T : K \to 2^{B^*}$ be a set-valued mapping, $\rho > 0$. Then the point $x^* \in K$ solves the generalized variational inequality (1.1) if and only if x^* solves the following inclusion:*

$$x \in \pi_K^f (Jx - \rho Tx).$$

From the proof of lemma 4.1 in [20], we can obtain the following lemma which plays an important role in the proof of our main result.

Lemma 2.2. *If B is smooth, then for any given $\phi \in B^*$, $x \in \pi_K^f \phi$, the following inequality holds*

$$G(Jx, y) \leq G(\phi, y) - G(\phi, x) + 2\rho f(y), \quad \forall y \in K.$$

Proof. From the property (f3) of the generalized $f-$projection operator π_K^f, we know that

$$\langle \phi - Jx, x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in K$$

and so

$$-2\langle \phi, y \rangle + 2\rho f(y) \geq 2\|x\|^2 - 2\langle \phi, x \rangle - 2\langle Jx, y \rangle + 2\rho f(x).$$

It follows that

$$\|\phi\|^2 - 2\langle \phi, y \rangle + \|y\|^2 + 2\rho f(y) \geq \|\phi\|^2 - 2\langle \phi, x \rangle + \|x\|^2 + 2\rho f(x) + \|Jx\|^2 - 2\langle Jx, y \rangle + \|y\|^2.$$

This implies that

$$G(\phi, y) \geq G(Jx, y) + G(\phi, x) - 2\rho f(y),$$

and so

$$G(Jx, y) \leq G(\phi, y) - G(\phi, x) + 2\rho f(y).$$

Furthermore, from lemma 4.1 of [20], we know that if for all $x \in K$, $f(x) \geq 0$, then

$$G(Jx, y) \leq G(\phi, y) + 2\rho f(y), \quad \forall y \in K.$$

□

Lemma 2.3. ([8]) *Let B be a uniformly convex and uniformly smooth Banach space. We have*

$$\|\phi + \Phi\|^2 \leq \|\phi\|^2 + 2\langle \Phi, J^*(\phi + \Phi) \rangle, \quad \forall \phi, \Phi \in B^*.$$

Lemma 2.4. ([22]) *Let B be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

for all $x, y \in B_r$ and $t \in [0,1]$, where $B_r = \{z \in B : \|z\| \leq r\}$.

Lemma 2.5. ([10]) *Let B be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of B. If $V_2(y_n, z_n) \to 0$, and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.6. *Let B be a uniformly convex and uniformly smooth Banach space and K be a nonempty, closed convex subset of B. Let $f : K \to R$ be a convex lower semi-continuous functional and $f(x) \geq 0$ for all $x \in K$. Suppose that there exists a positive number ρ such that*

$$\langle u, J^*(Jx - \rho u)\rangle \geq 0, \quad \text{for all} \quad x \in K, u \in Tx \tag{2.1}$$

and

$$\langle u, y\rangle + f(y) \leq 0, \quad \forall x \in K, u \in Tx, y \in \Omega. \tag{2.2}$$

Then the set of solutions of the generalized variational inequality (1.1) Ω is closed and convex.

Proof. We first show that Ω is closed. Let $\{x_n\}$ be a sequence of Ω such that $x_n \to \hat{x} \in K$. Let $\hat{u} \in T\hat{x}$, then it follows from the definitions of V_2, G_2, G, lemma 2.2, the property (ii) of G and (2.1), (2.2) that

$$\begin{aligned}
0 &\leq V_2(\pi_K^f(J\hat{x} - \rho\hat{u}), \hat{x}) \\
&= G_2(\pi_K^f(J\hat{x} - \rho\hat{u}), \hat{x}) - 2\rho f(\hat{x}) \\
&= G(J\pi_K^f(J\hat{x} - \rho\hat{u}), \hat{x}) - 2\rho f(\hat{x}) \\
&\leq G(J\hat{x} - \rho\hat{u}, \hat{x}) + 2\rho f(\hat{x}) - 2\rho f(\hat{x}) \\
&= G(J\hat{x} - \rho\hat{u}, \hat{x}) \leq \liminf_{n\to\infty} G(J\hat{x} - \rho\hat{u}, x_n) \\
&= \liminf_{n\to\infty}(\|J\hat{x} - \rho\hat{u}\|^2 - 2\langle J\hat{x} - \rho\hat{u}, x_n\rangle + \|x_n\|^2 + 2\rho f(x_n)) \\
&\leq \liminf_{n\to\infty}\Big(\|J\hat{x}\|^2 - 2\rho\langle\hat{u}, J^*(J\hat{x} - \rho\hat{u})\rangle - 2\langle J\hat{x}, x_n\rangle + 2\rho\langle\hat{u}, x_n\rangle \\
&\quad + 2\rho f(x_n) + \|x_n\|^2\Big) \\
&\leq \liminf_{n\to\infty} V_2(\hat{x}, x_n) \\
&= V_2(\hat{x}, \hat{x}) \\
&= 0,
\end{aligned}$$

which implies that $\hat{x} = \pi_K^f(J\hat{x} - \rho\hat{u})$, i.e., $\hat{x} \in \pi_K^f(J\hat{x} - \rho T\hat{x})$. So, we have $\hat{x} \in \Omega$. Next, we show that Ω is convex. For $x, y \in \Omega$ and $t \in (0,1)$, put

$z = tx + (1-t)y$. It is sufficient to show $z \in \pi_K^f(Jz - \rho Tz)$. Let $u_z \in Tz$, in fact, we have

$$\begin{aligned}
0 &\le V_2(\pi_K^f(Jz - \rho u_z), z) \\
&= G(J\pi_K^f(Jz - \rho u_z), z) - 2\rho f(z) \\
&\le G(Jz - \rho u_z, z) + 2\rho f(z) - 2\rho f(z) \\
&= G(Jz - \rho u_z, z) \\
&= \|Jz - \rho u_z\|^2 - 2\langle Jz - \rho u_z, z\rangle + \|z\|^2 + 2\rho f(z) \\
&\le \|Jz\|^2 - 2\rho\langle u_z, J^*(Jz - \rho u_z)\rangle - 2\langle Jz, z\rangle + 2\rho\langle u_z, z\rangle + 2\rho f(z) + \|z\|^2 \\
&\le 2\rho(\langle u_z, z\rangle + f(z)) \\
&= 2\rho(\langle u_z, tx + (1-t)y\rangle + f(tx + (1-t)y)) \\
&\le 2\rho(t\langle u_z, x\rangle + (1-t)\langle u_z, y\rangle + tf(x) + (1-t)f(y)) \\
&\le 0.
\end{aligned}$$

This implies that $z \in \pi_K^f(Jz - \rho Tz)$. Therefore, Ω is closed and convex. □

Remark 4. If $f \equiv 0$ and T is single-valued, then (2.1) and (2.2) reduces to (2.3) and (2.4) as follows respectively:

$$\langle Tx, J^*(Jx - \rho Tx)\rangle \ge 0, \quad \text{for all} x \in K, \tag{2.3}$$

and

$$\langle Tx, y\rangle \le 0, \quad \forall x \in K, y \in VI(K,T). \tag{2.4}$$

Let S be a mapping from K into itself. We denote by $F(S)$ the set of fixed points of S. A point p in K is said to be an asymptotic fixed point of S [4] if K contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed point of S will be denoted by $\hat{F}(S)$. A mapping S from K into itself is called relatively nonxpansive (see, eg., [4,5,18]) if $\hat{F}(S) = F(S)$ and $V_2(Sx,p) \le V_2(x,p)$ for all $x \in K$ and $p \in F(S)$. The asymptotic behavior of relatively nonexpansive mappings were studied in [4,5]. A point p in K is said to be a strong asymptotic fixed point of S if K contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of strong asymptotic fixed points of S will be denoted by $\tilde{F}(S)$. A mapping S from K into itself is called relatively weak nonexpansive [14] if $\tilde{F}(S) = F(S)$ and $V_2(Sx,p) \le V_2(x,p)$ for all $x \in K$ and $p \in F(S)$. If B is smooth, strictly convex and reflexive Banach space, and $A \subset B \times B^*$ is a continuous monotone mapping with $A^{-1}0 \ne \emptyset$, then it is proved in [14] that $J_r = (J + rA)^{-1}J$, for $r > 0$ is relatively weak nonexpansive. Moreover, if $S : K \to K$ is relatively weak nonexpansive, then using the definition of V_2(i.e.the same argument as in the proof of [18, p.260]), we can show that $F(S)$ is closed and convex.

It is obvious that relatively nonexpansive mapping is relatively weak nonexpansive mapping. In fact, for any mapping $S : K \to K$ we have $F(S) \subset \tilde{F}(S) \subset \hat{F}(S)$. Therefore, if S is a relatively nonexpansive mapping, then $F(S) = \tilde{F}(S) = \hat{F}(S)$.

3. Main results

For any $x_0 \in K$, we define the iteration process $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in K \quad \text{chosen arbitrarily,} \\ z_n = \pi_K(\beta_n Jx_n + (1-\beta_n)JSx_n), \\ u_n \in Tz_n, \\ y_n = J^*(\alpha_n Jx_n + (1-\alpha_n)J\pi_K^f(Jz_n - \rho u_n)), \\ C_0 = K, \\ C_{n+1} = \{u \in C_n : G_2(y_n, u) \le G_2(x_n, u)\}, \\ x_{n+1} = \pi_{C_{n+1}}^f Jx_0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy:

$$0 \le \alpha_n < 1, \text{ and } \limsup_{n\to\infty} \alpha_n < 1; \ 0 < \beta_n < 1 \text{ and } \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Theorem 3.1. *Let B be a uniformly convex and uniformly smooth Banach space and let K be a nonempty, closed convex subset of B. Let $f : K \to R$ be convex, lower semi-continuous and $f(x) \ge 0$ for all $x \in K$. Assume that $T : K \to 2^{B^*}$ is a set-valued mapping that satisfies conditions (2.1) and (2.2) and $S : K \to K$ is a relatively weak nonexpansive mapping with $\Omega \bigcap F(S) \ne \emptyset$. If $T : K \to 2^{B^*}$ is upper semi-continuous with closed values, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $\pi^f_{\Omega\bigcap F(S)} Jx_0$.*

Proof. We first show that C_n is closed and convex for each $n \in N \bigcup \{0\}$. It is obvious that $C_0 = K$ is closed and convex. Suppose that C_k is closed and convex for some $k \in N$. For $u \in C_k$, we obtain that

$$G_2(y_k, u) \le G_2(x_k, u)$$

is equivalent to

$$2\langle Jx_k - Jy_k, u\rangle \le \|x_k\|^2 - \|y_k\|^2.$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \ge 0$, C_n is closed and convex. This shows that $\pi^f_{C_{n+1}} Jx_0$ is well defined. Next, we show that $\Omega \bigcap F(S) \subset C_n$ for all $n \in N \bigcup \{0\}$. $\Omega \bigcap F(S) \subset C_0 = K$ is obvious. Suppose $\Omega \bigcap F(S) \subset C_k$ for some $k \in N$. Then, for $\forall p \in \Omega \bigcap F(S) \subset C_k$, it follows from the definitions of G_2, V, V_2, S and properties (iii), (iv) of V that

$$\begin{aligned} G_2(z_k,p) &= V_2(z_k,p) + 2\rho f(p) \\ &= V(Jz_k,p) + 2\rho f(p) \\ &\le V(\beta_k Jx_k + (1-\beta_k)JSx_k, p) + 2\rho f(p) \\ &\le \beta_k V(Jx_k,p) + (1-\beta_k)V(JSx_k,p) + 2\rho f(p) \\ &= \beta_k V_2(x_k,p) + (1-\beta_k)V_2(Sx_k,p) + 2\rho f(p) \\ &\le \beta_k V_2(x_k,p) + (1-\beta_k)V_2(x_k,p) + 2\rho f(p) \\ &= G_2(x_k,p). \end{aligned} \tag{3.2}$$

Therefore, from the properties of G, lemma 2.2, lemma 2.3, inequalities (2.1), (2.2) and (3.2), we obtain

$$\begin{aligned} G_2(y_k,p) &= G(Jy_k,p) \\ &= G(\alpha_k Jx_k + (1-\alpha_k)J\pi_K^f(Jz_k - \rho u_k), p) \\ &\le \alpha_k G(Jx_k,p) + (1-\alpha_k)G(J\pi_K^f(Jz_k-\rho u_k),p) \\ &\le \alpha_k G_2(x_k,p) + (1-\alpha_k)G(Jz_k-\rho u_k,p) + 2(1-\alpha_k)\rho f(p) \\ &= \alpha_k G_2(x_k,p) + (1-\alpha_k)\{\|Jz_k-\rho u_k\|^2 - 2\langle Jz_k-\rho u_k, p\rangle + \|p\|^2 \\ &\quad + 2\rho f(p)\} + 2(1-\alpha_k)\rho f(p) \\ &\le \alpha_k G_2(x_k,p) + (1-\alpha_k)\{\|Jz_k\|^2 - 2\rho\langle u_k, J^*(Jz_k-\rho u_k)\rangle \\ &\quad - 2\langle Jz_k,p\rangle + 2\rho\langle u_k,p\rangle + \|p\|^2 + 2\rho f(p)\} + 2(1-\alpha_k)\rho f(p) \\ &\le \alpha_k G_2(x_k,p) + (1-\alpha_k)G_2(z_k,p) \\ &\le G_2(x_k,p), \end{aligned}$$

which shows that $p \in C_{k+1}$. This implies that $\Omega \bigcap F(S) \subset C_n$ for all $n \in N \bigcup \{0\}$. From $x_n = \pi_{C_n}^f Jx_0$, we have

$$\langle Jx_0 - Jx_n, x_n - y\rangle + \rho f(y) - \rho f(x_n) \ge 0, \quad \forall y \in C_n. \tag{3.3}$$

Since $\Omega \bigcap F(S) \subset C_n$ for all $n \in N \bigcup \{0\}$, we arrive at

$$\langle Jx_0 - Jx_n, x_n - p\rangle + \rho f(p) - \rho f(x_n) \ge 0, \quad \forall p \in \Omega \bigcap F(S). \tag{3.4}$$

Using $x_n = \pi_{C_n}^f Jx_0$ and $\Omega \bigcap F(S) \subset C_n$, we have $G(Jx_0,x_n) \le G(Jx_0,p)$ for each $p \in \Omega \bigcap F(S)$. Therefore, $\{G(Jx_0,x_n)\}$ is bounded. Moreover, it follows from lemma 2.2 that $G(Jx_n,p) \le G(Jx_0,p) - G(Jx_0,x_n) + 2\rho f(p), \forall p \in \Omega \bigcap F(S)$. From the definitions of G and V, we have $V(Jx_n,p) \le G(Jx_0,p) - G(Jx_0,x_n)$. Since $\{G(Jx_0,x_n)\}$ is bounded, we can obtain that $(\|Jx_n\| - \|p\|)^2 \le V(Jx_n,p) \le M$, where $M = \sup\{|G(Jx_0,p)| + |G(Jx_0,x_n)|\}$. This implies that $\{x_n\}$ is also bounded. On the other hand, noticing that $x_n = \pi_{C_n}^f Jx_0$ and $x_{n+1} = \pi_{C_{n+1}}^f Jx_0 \in C_{n+1} \subset C_n$, we have $G(Jx_0,x_n) \le G(Jx_0,x_{n+1})$ for each $n \in N \bigcup \{0\}$. Therefore, $\{G(Jx_0,x_n)\}$ is nondecreasing. So there exists the limit of $G(Jx_0,x_n)$. By the construction of C_n, we have that $C_m \subset C_n$ and

$x_m = \pi_{C_m}^f Jx_0 \in C_n$ for any positive integer $m \geq n$. It follows from lemma 2.2 that $G(Jx_n, x_m) \leq G(Jx_0, x_m) - G(Jx_0, x_n) + 2\rho f(x_m)$, which implies that

$$V_2(x_n, x_m) \leq G(Jx_0, x_m) - G(Jx_0, x_n). \tag{3.5}$$

Letting $m, n \to \infty$ in (3.5), we have $V_2(x_n, x_m) \to 0$. It follows from lemma 2.5 that $x_n - x_m \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since B is a Banach space and K is closed and convex, we can assume that $x_n \to x^* \in K$ as $n \to \infty$.

Next, we show that $x^* \in \Omega \bigcap F(S)$. By taking $m = n + 1$ in (3.5), we arrive at

$$\lim_{n\to\infty} V_2(x_n, x_{n+1}) = 0. \tag{3.6}$$

Noticing that $x_{n+1} \in C_{n+1}$, we obtain $G_2(y_n, x_{n+1}) \leq G_2(x_n, x_{n+1})$. It is equivalent to $V_2(y_n, x_{n+1}) \leq V_2(x_n, x_{n+1})$. It follows from (3.6) that $\lim_{n\to\infty} V_2(y_n, x_{n+1}) = 0$. From lemma 2.5, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.7}$$

Since $x_n \to x^*$, then from $\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$ and (3.7), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.8}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.9}$$

Since $\|Jy_n - Jx_n\| = (1-\alpha_n)\|J\pi_K^f(Jz_n - \rho u_n) - Jx_n\|$ and $\limsup_{n\to\infty} \alpha_n < 1$, we have

$$\lim_{n\to\infty} \|J\pi_K^f(Jz_n - \rho u_n) - Jx_n\| = 0.$$

Since J^* is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|\pi_K^f(Jz_n - \rho u_n) - x_n\| = 0. \tag{3.10}$$

Since $\{x_n\}$ is bounded, then $\{Jx_n\}, \{JSx_n\}$ are also bounded. Moreover, since B is a uniformly smooth Banach space, we know that B^* is a uniformly convex Banach space. Therefore, lemma 2.4 is applicable. By the definitions of

$G_2,\ V_2,\ V,\ S$ and the properties of V, for $\forall p\in\Omega\bigcap F(S)$, we have

$$\begin{aligned}
G_2(z_n,p) &= V_2(z_n,p)+2\rho f(p)\\
&= V(Jz_n,p)+2\rho f(p)\\
&\le V(\beta_n Jx_n+(1-\beta_n)JSx_n,p)+2\rho f(p)\\
&= \|\beta_n Jx_n+(1-\beta_n)JSx_n\|^2-2\langle\beta_n Jx_n+(1-\beta_n)JSx_n,p\rangle\\
&\quad +\|p\|^2+2\rho f(p)\\
&\le \beta_n\|Jx_n\|^2+(1-\beta_n)\|JSx_n\|^2-\beta_n(1-\beta_n)g(\|Jx_n-JSx_n\|)\\
&\quad -2\beta_n\langle Jx_n,p\rangle-2(1-\beta_n)\langle JSx_n,p\rangle+\|p\|^2+2\rho f(p)\\
&= \beta_n V_2(x_n,p)+(1-\beta_n)V_2(Sx_n,p)-\beta_n(1-\beta_n)g(\|Jx_n-JSx_n\|)\\
&\quad +2\rho f(p)\\
&\le \beta_n V_2(x_n,p)+(1-\beta_n)V_2(x_n,p)+2\rho f(p)\\
&\quad -\beta_n(1-\beta_n)g(\|Jx_n-JSx_n\|)\\
&= G_2(x_n,p)-\beta_n(1-\beta_n)g(\|Jx_n-JSx_n\|).
\end{aligned}\tag{3.11}$$

From lemma 2.2, lemma 2.3, (2.1) and (2.2), we have

$$\begin{aligned}
&G(J\pi_K^f(Jz_n-\rho u_n),p)\\
&\le G(Jz_n-\rho u_n,p)+2\rho f(p)\\
&= \|Jz_n-\rho u_n\|^2-2\langle Jz_n-\rho u_n,p\rangle+\|p\|^2+2\rho f(p)+2\rho f(p)\\
&\le \|Jz_n\|^2-2\rho\langle u_n,J^*(Jz_n-\rho u_n)\rangle-2\langle Jz_n,p\rangle+2\rho\langle u_n,p\rangle+2\rho f(p)\\
&\quad +\|p\|^2+2\rho f(p)\\
&\le G_2(z_n,p).
\end{aligned}\tag{3.12}$$

Combining (3.12) with (3.11), we obtain that

$$\begin{aligned}
&G_2(y_n,p)\\
&= G(Jy_n,p)\\
&\le \alpha_n G(Jx_n,p)+(1-\alpha_n)G(J\pi_K^f(Jz_n-\rho u_n),p)\\
&\le \alpha_n G_2(x_n,p)+(1-\alpha_n)G_2(z_n,p)\\
&\le \alpha_n G_2(x_n,p)+(1-\alpha_n)[G_2(x_n,p)-\beta_n(1-\beta_n)g(\|Jx_n-JSx_n\|)]\\
&= G_2(x_n,p)-(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n-JSx_n\|).
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n-JSx_n\|) &\le G_2(x_n,p)-G_2(y_n,p)\\
&= \|x_n\|^2-\|y_n\|^2+2\langle Jy_n-Jx_n,p\rangle.
\end{aligned}$$

By (3.8), (3.9) and $\limsup_{n\to\infty}\alpha_n<1$, $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, we have $\lim_{n\to\infty}g(\|Jx_n-JSx_n\|)=0$. By the property of the function g, we obtain $\lim_{n\to\infty}\|Jx_n-JSx_n\|=$

0. Since J^* is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_n - Sx_n\| = \lim_{n\to\infty}\|J^*Jx_n - J^*JSx_n\| = 0. \tag{3.13}$$

Since $x_n \to x^*$, we have $x^* \in \tilde{F}(S) = F(S)$. Moreover, $Sx_n \to x^*$ and $JSx_n \to Jx^*$. Noting the properties of V, we derive that

$$\begin{aligned} V_2(z_n, x_n) &= V(Jz_n, x_n) \\ &\le V(\beta_n Jx_n + (1-\beta_n)JSx_n, x_n) \\ &\le \beta_n V(Jx_n, x_n) + (1-\beta_n)V(JSx_n, x_n) \\ &= (1-\beta_n)V(JSx_n, x_n). \end{aligned}$$

By the continuity of the operator V, we have $\lim_{n\to\infty} V(JSx_n, x_n) = V(Jx^*, x^*) = 0$ and hence $\lim_{n\to\infty}(1-\beta_n)V(JSx_n, x_n) = 0$. Therefore, $\lim_{n\to\infty} V_2(z_n, x_n) = 0$. From lemma 2.5, we have

$$\|x_n - z_n\| \to 0, \ \text{as} \quad n \to \infty. \tag{3.14}$$

Using inequalities (3.10) and (3.14), we obtain

$$\|\pi_K^f(Jz_n - \rho u_n) - z_n\| \le \|\pi_K^f(Jz_n - \rho u_n) - x_n\| + \|x_n - z_n\| \to 0. \tag{3.15}$$

Since $x_n \to x^*$, we have

$$z_n \to x^* \text{ as } n \to \infty. \tag{3.16}$$

Since π_K^f is continuous and $J - \rho T$ is upper semi-continuous with closed values, then $\pi_K^f(J-\rho T)$ is upper semi-continuous with closed values. From (3.15) and (3.16), we have $x^* \in \pi_K^f(Jx^* - \rho Tx^*)$. By lemma 2.1, we have $x^* \in \Omega$. This shows that $x^* \in \Omega \bigcap F(S)$.

Finally, we prove $x^* = \pi^f_{\Omega \bigcap F(S)} Jx_0$. By taking lower-limit in (3.4), we have

$$\langle Jx_0 - Jx^*, x^* - p\rangle + \rho f(p) - \rho f(x^*) \ge 0, \qquad \forall p \in \Omega \bigcap F(S).$$

At this point, in view of the property (f3) of π_K^f, we see that $x^* = \pi^f_{\Omega \bigcap F(S)} Jx_0$. □

If $S = I$, then (3.1) reduces to the modified Mann iteration for generalized variational inequality (1.1) and so we obtain the following result:

Corollary 3.2. *Let B be a uniformly convex and uniformly smooth Banach space and let K be a nonempty, closed convex subset of B. Let $f : K \to R$ be convex, lower semi-continuous and $f(x) \ge 0$ for all $x \in K$. Assume that $T : K \to 2^{B^*}$ is a set-valued mapping that satisfies conditions (2.1) and (2.2) such that $\Omega \ne \emptyset$. If $T : K \to 2^{B^*}$ is upper semi-continuous with closed values* *and the sequence $\{x_n\}$ is defined by the following modified Mann iteration*

$$\begin{cases} x_0 \in K \quad \textit{chosen arbitrarily}, \\ u_n \in Tx_n, \\ y_n = J^*(\alpha_n Jx_n + (1-\alpha_n)J\pi_K^f(Jx_n - \rho u_n)), \\ C_0 = K, \\ C_{n+1} = \{u \in C_n : G_2(y_n, u) \le G_2(x_n, u)\}, \\ x_{n+1} = \pi_{C_{n+1}}^f Jx_0, \end{cases} \tag{3.17}$$

where $\{\alpha_n\}$ satisfies:

$$0 \le \alpha_n < 1 \quad \textit{and} \quad \limsup_{n\to\infty} \alpha_n < 1,$$

then the sequence $\{x_n\}$ converges strongly to $\pi_\Omega^f Jx_0$.

Proof. Taking $S = I$ in theorem 3.1, by $x_n \in K$ and property(v) of the operator π_K, we have $z_n = \pi_K Jx_n = x_n$. Thus, we can obtain the desired conclusion. □

Remark 5. Corollary 3.1 improves theorem 3.5 of [10] in the following senses:

(1) the condition in theorem 3.5 of [10] that $J - \rho T : K \to 2^{B^*}$ is compact is removed, we only require that $T : K \to 2^{B^*}$ is upper semi-continuous with closed values;

(2) we obtain that the convergence point of $\{x_n\}$ is $\pi_\Omega^f Jx_0$, which is more concrete than related conclusions of [10] and [20].

(3) we remove the condition that $0 \in K$ and $f(0) = 0$.

If $f \equiv 0$ and T is single-valued, then (3.1) reduces to the following iteration sequence $\{x_n\}$:

$$\begin{cases} x_0 \in K \quad \text{chosen arbitrarily}, \\ z_n = \pi_K(\beta_n Jx_n + (1-\beta_n)JSx_n), \\ y_n = J^*(\alpha_n Jx_n + (1-\alpha_n)J\pi_K(Jz_n - \rho Tz_n)), \\ C_0 = K, \\ C_{n+1} = \{u \in C_n : G_2(y_n, u) \le G_2(x_n, u)\}, \\ x_{n+1} = \pi_{C_{n+1}} Jx_0, \end{cases} \tag{3.18}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy:

$$0 \le \alpha_n < 1, \text{ and } \limsup_{n\to\infty} \alpha_n < 1; \quad 0 < \beta_n < 1 \text{ and } \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

So we obtain the following result:

Corollary 3.3. *Let B be a uniformly convex and uniformly smooth Banach space and let K be a nonempty, closed convex subset of B. Assume that $T : K \to B^*$ is a single-valued mapping that satisfies conditions (2.3) and (2.4) and $S : K \to K$ is a relatively weak nonexpansive mapping with $VI(K,T) \bigcap F(S) \ne$*

$\emptyset$, where $VI(K,T)$ denotes the set of solutions of the classical variational inequality (1.2). If T is continuous, then the sequence $\{x_n\}$ defined by (3.18) converges strongly to $\pi_{VI(K,T)\bigcap F(S)}Jx_0$.

Remark 6. The algorithm in corollary 3.2 is more simple and convenient to compute than the one given by Liu [17].

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