

AN IMPLICIT ITERATES FOR NON-LIPSCHITZIAN ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN CAT(0) SPACES

G. S. SALUJA

ABSTRACT. The purpose of this paper is to establish strong convergence of an implicit iteration process to a common fixed point for a finite family of asymptotically quasi-nonexpansive type mappings in $\mathbf{CAT}(0)$ spaces. Our results improve and extend the corresponding results of Fukhar-uddin et al. [15] and some others from the current literature.

1. Introduction

A metric space X is a $\mathbf{CAT}(0)$ space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $\mathbf{CAT}(0)$ space. Other examples include Pre-Hilbert spaces (see [3]), \mathbb{R} -trees (see [27]), Euclidean buildings (see [4]), the complex Hilbert ball with a hyperbolic metric (see [16]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [3].

Fixed point theory in $\mathbf{CAT}(0)$ spaces was first studied by Kirk (see [25, 26]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $\mathbf{CAT}(0)$ space always has a fixed point. Since then the fixed point theory for single-valued and multi-valued mappings in $\mathbf{CAT}(0)$ spaces has been rapidly developed and many papers have appeared (see, e.g., [1], [6]-[9], [18], [21], [24], [29]-[31], [33] and references therein). It is worth mentioning that fixed point theorems in $\mathbf{CAT}(0)$ spaces (specially in \mathbb{R} -trees) can be applied to graph theory, biology and computer science (see, e.g., [2, 10, 26, 28, 32]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and let $d(c(t), c(t')) = |t - t'|$ for

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all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y . We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y .

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in \mathbb{R}^2 such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [3]).

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let Δ be a geodesic triangle in X and let $\bar{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ . Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}). \quad (1)$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [20]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the mid point of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \quad (2)$$

The inequality (2) is the (CN) inequality of Bruhat and Titz [5]. The above inequality has been extended in [8] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1 - \alpha)y) &\leq \alpha d^2(z, x) + (1 - \alpha)d^2(z, y) \\ &\quad - \alpha(1 - \alpha)d^2(x, y). \end{aligned} \quad (3)$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [3, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1 - \alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(z, x) + (1 - \alpha)d(z, y), \quad (4)$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1 - \alpha)y : \alpha \in [0, 1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

Let T be a self map on a nonempty subset C of X . Denote the set of fixed points of T by $F(T) = \{x \in C : T(x) = x\}$. We say that T is:

(1) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y), \quad (5)$$

for all $x, y \in C$ and $n \geq 1$.

(2) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$d(T^n x, p) \leq k_n d(x, p), \quad (6)$$

for all $x \in C, p \in F(T)$ and $n \geq 1$.

(3) asymptotically quasi-nonexpansive type if $F(T) \neq \emptyset$ and

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{x \in C, p \in F(T)} \left(d(T^n x, p) - d(x, p) \right) \right\} \leq 0. \quad (7)$$

(4) uniformly L -Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x, y), \quad (8)$$

for all $x, y \in C$ and $n \geq 1$.

(5) semi-compact if for any bounded sequence $\{x_n\}$ in C with $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there is a convergent subsequence of $\{x_n\}$.

Denote the indexing set $\{1, 2, \dots, N\}$ by I . Let $\{T_i : i \in I\}$ be the set of N self mappings of C . Throughout the paper, it is supposed that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. We say condition (A) [15] is satisfied if there exists a nondecreasing function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ and at least one $T \in \{T_i : i \in I\}$ such that $d(x, Tx) \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

Recently, number of papers have appeared on the iterative approximation of fixed points of asymptotically nonexpansive (asymptotically quasi-nonexpansive) mappings through Mann, Ishikawa and implicit iterates in uniformly convex Banach spaces, convex metric spaces and CAT(0) spaces (see, e.g., [11]-[14], [17], [19], [21]-[23], [34], [36]).

Very recently, Fukhar-ud-din et al. [15] generalized the Sun's [34] implicit algorithm in CAT(0) space by using the concept of convexity in CAT(0) space. The generalized implicit algorithm is as follows:

$$x_0 \in C,$$

$$\begin{aligned}
x_1 &= \alpha_1 x_0 \oplus (1 - \alpha_1) T_1 x_1, \\
x_2 &= \alpha_2 x_1 \oplus (1 - \alpha_2) T_2 x_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} \oplus (1 - \alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N \oplus (1 - \alpha_{N+1}) T_1^2 x_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} \oplus (1 - \alpha_{2N}) T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} \oplus (1 - \alpha_{2N+1}) T_1^3 x_{2N+1}, \\
&\vdots
\end{aligned} \tag{9}$$

where $0 \leq \alpha_n \leq 1$.

Starting from arbitrary $x_0 \in C$, the above process in the compact form can be written as

$$x_n = \alpha_n x_{n-1} \oplus (1 - \alpha_n) T_{i(n)}^{k(n)} x_n, \quad n \geq 1, \tag{10}$$

where $n = (k-1)N + i$, $i = i(n) \in I$ and $k = k(n) \geq 1$ is a positive integer such that $k(n) \rightarrow \infty$ as $n \rightarrow \infty$. They have proved some strong convergence theorems using implicit iteration scheme (10) for a finite family of generalized asymptotically quasi-nonexpansive mappings in CAT(0) space and also gave the necessary and sufficient condition to converge to common fixed point for said mappings in CAT(0) space.

In a normed space, iteration scheme (10) can be written as

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n, \quad n \geq 1, \tag{11}$$

where $n = (k-1)N + i$, $i = i(n) \in I$ and $k = k(n) \geq 1$ is a positive integer such that $k(n) \rightarrow \infty$ as $n \rightarrow \infty$.

The iteration scheme (10) - (11) exist as follows.

Let X be a CAT(0) space. Then, the following inequality holds:

$$d(\lambda x \oplus (1 - \lambda)z, \lambda y \oplus (1 - \lambda)w) \leq \lambda d(x, y) + (1 - \lambda)d(z, w), \tag{12}$$

for all $x, y, z, w \in X$ (see [30]).

Denote the indexing set $\{1, 2, \dots, N\}$ by I . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive type self-mappings of C . We show that (12) exists. Let $x_0 \in C$ and $x_1 = \alpha_1 x_0 \oplus (1 - \alpha_1) T_1 x_1$. Define $W : C \rightarrow C$ by: $Wx = \alpha_1 x_0 \oplus (1 - \alpha_1) T_1 x$ for all $x \in C$. The existence of x_1 is guaranteed if W has a fixed point. For any $x, y \in C$, we have

$$d(Wx, Wy) \leq (1 - \alpha_1) d(T_1 x, T_1 y) \leq (1 - \alpha_1) L d(x, y). \tag{13}$$

Now, W is a contraction if $(1 - \alpha_1)L < 1$ or $L < 1/(1 - \alpha_1)$. As $\alpha_1 \in (0, 1)$, therefore W is a contraction even if $L > 1$. By the Banach contraction principle, W has a unique fixed point. Thus, the existence of x_1 is established. Similarly,

we can establish the existence of x_2, x_3, x_4, \dots . Thus, the implicit algorithm (10) is well defined. Similarly, we can prove that (11) exists.

The purpose of this paper is to study strong convergence of implicit iteration process (10) for the class of uniformly L -Lipschitzian and asymptotically quasi-nonexpansive type self mappings on a CAT(0) space. Our results extend the corresponding results of Fukhar-ud-din et al. [15] and many others.

We need the following useful lemma to prove our convergence results.

Lemma 1.1. (see [35]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 1. \tag{14}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.

2. Convergence in CAT(0) spaces

We establish some convergence results for the iteration scheme (10) to a common fixed point of a finite family of uniformly L -Lipschitzian and asymptotically quasi-nonexpansive type self mappings in the framework of CAT(0) spaces.

Theorem 2.1. *Let (X, d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian and asymptotically quasi-nonexpansive type self mappings of C . Suppose that F is closed. Let $\{x_n\}$ be the implicit iteration process defined by (10). Put*

$$G_{in} = \max \left\{ 0, \sup_{p \in F, n \geq 1} \left(d(T_i^n x_n, p) - d(x_n, p) \right) : i \in I \right\}, \tag{15}$$

where $n = (k - 1)N + i$ and $i = i(n) \in I$. Assume that $\sum_{n=1}^{\infty} G_{in} < \infty$ and $\{\alpha_n\} \subset [s, 1 - s]$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i : i \in I\}$ if and only if

$$\liminf_{n \rightarrow \infty} d(x_n, F) = 0,$$

where $d(x, F) = \inf_{p \in F} \{d(x, p)\}$.

Proof. The necessity is obvious and so it is omitted. Now, we prove the sufficiency. For any $p \in F = \bigcap_{i=1}^N F(T_i)$ from (10) and (15), where $n \geq 1$, $n = (k - 1)N + i$ and $i = i(n) \in I$, we have

$$\begin{aligned} d(x_n, p) &= d\left(\alpha_n x_{n-1} \oplus (1 - \alpha_n) T_{i(n)}^{k(n)} x_n, p\right) \\ &\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d\left(T_{i(n)}^{k(n)} x_n, p\right) \\ &\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) [d(x_n, p) + G_{ik(n)}] \\ &= \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d(x_n, p) + (1 - \alpha_n) G_{ik(n)}. \end{aligned} \tag{16}$$

Since $\alpha_n \in (s, 1 - s)$, the above inequality gives that

$$\begin{aligned} d(x_n, p) &\leq d(x_{n-1}, p) + \left(\frac{1}{s} - 1\right)G_{ik(n)} \\ &= d(x_{n-1}, p) + Q_{ik(n)}, \end{aligned} \quad (17)$$

where $Q_{ik(n)} = \left(\frac{1}{s} - 1\right)G_{ik(n)}$. Since $\sum_{k(n)=1}^{\infty} G_{ik(n)} < \infty$ for all $i \in I$, it follows that $\sum_{k(n)=1}^{\infty} Q_{ik(n)} < \infty$. Therefore, from Lemma 1.1, we know that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. Since by hypothesis $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, so by Lemma 1.1, we have

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0. \quad (18)$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C . It follows from (17) that for any $m \geq 1$, for all $n \geq n_0$ and for any $p \in F$, we have

$$d(x_{n+m}, p) \leq d(x_n, p) + \sum_{i=1}^N \sum_{k(n)=1}^{\infty} Q_{ik(n)}. \quad (19)$$

So, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p) + d(x_n, p) \\ &\leq d(x_n, p) + \sum_{i=1}^N \sum_{k(n)=1}^{\infty} Q_{ik(n)} + d(x_n, p) \\ &= 2d(x_n, p) + \sum_{i=1}^N \sum_{k(n)=1}^{\infty} Q_{ik(n)}. \end{aligned} \quad (20)$$

Then, we have

$$d(x_{n+m}, x_n) \leq 2d(x_n, F) + \sum_{i=1}^N \sum_{k(n)=1}^{\infty} Q_{ik(n)}. \quad (21)$$

For any given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that for any $n \geq n_1$,

$$d(x_n, F) < \frac{\varepsilon}{4}, \quad (22)$$

and

$$\sum_{i=1}^N \sum_{k(n)=1}^{\infty} Q_{ik(n)} < \frac{\varepsilon}{2}. \quad (23)$$

Thus, from (21)-(23) and $n \geq n_1$, we have

$$d(x_{n+m}, x_n) < 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \quad (24)$$

This implies that $\{x_n\}$ is a Cauchy sequence in C . Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n \rightarrow \infty} x_n = z$. Since C is closed, therefore $z \in C$. Next, we show that $z \in F$. Now, the following two inequalities:

$$\begin{aligned} d(z, p) &\leq d(z, x_n) + d(x_n, p) \quad \forall p \in F, \quad n \geq 1, \\ d(z, x_n) &\leq d(z, p) + d(x_n, p) \quad \forall p \in F, \quad n \geq 1 \end{aligned}$$

give that

$$-d(z, x_n) \leq d(z, F) - d(x_n, F) \leq d(z, x_n), \quad n \geq 1. \quad (25)$$

That is,

$$|d(z, F) - d(x_n, F)| \leq d(z, x_n), \quad n \geq 1. \quad (26)$$

As $\lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, we conclude that $z \in F$. \square

Theorem 2.2. *Let (X, d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian and asymptotically quasi-nonexpansive type self mappings of C . Suppose that F is closed. Let $\{x_n\}$ be the implicit iteration process defined by (10). Put*

$$G_{in} = \max \left\{ 0, \sup_{p \in F, n \geq 1} \left(d(T_i^n x_n, p) - d(x_n, p) \right) : i \in I \right\},$$

where $n = (k-1)N + i$ and $i = i(n) \in I$. Assume that $\sum_{n=1}^{\infty} G_{in} < \infty$ and $\{\alpha_n\} \subset [s, 1-s]$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i : i \in I\}$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $p \in F$.

Proof. The proof of Theorem 2.2 follows from Lemma 1.1 and Theorem 2.1. \square

We prove a lemma which plays an important role in establishing strong convergence of the implicit iteration process with errors in a CAT(0) space.

Lemma 2.3. *Let (X, d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian and asymptotically quasi-nonexpansive type self mappings of C . Suppose that F is closed. Let $\{x_n\}$ be the implicit iteration process defined by (10). Put*

$$G_{in} = \max \left\{ 0, \sup_{p \in F, n \geq 1} \left(d(T_i^n x_n, p) - d(x_n, p) \right) : i \in I \right\},$$

where $n = (k-1)N + i$ and $i = i(n) \in I$. Assume that $\sum_{n=1}^{\infty} G_{in} < \infty$ and $\{\alpha_n\} \subset [s, 1-s]$ for some $s \in (0, \frac{1}{2})$. Then $\lim_{n \rightarrow \infty} d(x_n, T_l x_n) = 0$ for all $l \in I$.

Proof. Note that $\{x_n\}$ is bounded as $\lim_{n \rightarrow \infty} d(x_n, p)$ exists (proved in Theorem 2.1). So, there exists $R > 0$ and $x_0 \in X$ such that $x_n \in B_R(x_0) = \{x : d(x, x_0) < R\}$ for all $n \geq 1$. Let $\sigma_n = d(x_{n-1}, T_{i(n)}^{k(n)})$.

We claim that $\lim_{n \rightarrow \infty} \sigma_n = 0$.

For any $p \in F$, using (3) and (10), we get

$$\begin{aligned}
d^2(x_n, p) &= d^2\left(\alpha_n x_{n-1} \oplus (1 - \alpha_n) T_{i(n)}^{k(n)} x_n, p\right) \\
&\leq \alpha_n d^2(x_{n-1}, p) + (1 - \alpha_n) d^2\left(T_{i(n)}^{k(n)} x_n, p\right) \\
&\quad - \alpha_n (1 - \alpha_n) d^2\left(T_{i(n)}^{k(n)} x_n, x_{n-1}\right) \\
&\leq \alpha_n d^2(x_{n-1}, p) + (1 - \alpha_n) [d(x_n, p) + G_{ik(n)}]^2 \\
&\quad - \alpha_n (1 - \alpha_n) d^2\left(T_{i(n)}^{k(n)} x_n, x_{n-1}\right) \\
&\leq \alpha_n d^2(x_{n-1}, p) + (1 - \alpha_n) [d^2(x_n, p) + \delta_{ik(n)}] \\
&\quad - \alpha_n (1 - \alpha_n) d^2\left(T_{i(n)}^{k(n)} x_n, x_{n-1}\right), \tag{27}
\end{aligned}$$

where $\delta_{ik(n)} = G_{ik(n)}^2 + 2G_{ik(n)}d(x_n, p)$. Since $\sum_{k(n)=1}^{\infty} G_{ik(n)} < \infty$, it follows that $\sum_{k(n)=1}^{\infty} \delta_{ik(n)} < \infty$.

Since $s \leq \alpha_n \leq (1 - s)$, from (27), we obtain

$$\begin{aligned}
s^2 \sigma_n^2 &\leq \alpha_n d^2(x_{n-1}, p) - d^2(x_n, p) + (1 - \alpha_n) d^2(x_n, p) \\
&\quad + (1 - \alpha_n) \delta_{ik(n)} \\
&= \alpha_n d^2(x_{n-1}, p) - \alpha_n d^2(x_n, p) + (1 - \alpha_n) \delta_{ik(n)}, \tag{28}
\end{aligned}$$

further, using (17), we obtain

$$\begin{aligned}
s^2 \sigma_n^2 &\leq \alpha_n d^2(x_{n-1}, p) - \alpha_n [d(x_{n-1}, p) + Q_{ik(n)}]^2 \\
&\quad + (1 - \alpha_n) \delta_{ik(n)} \\
&\leq \alpha_n d^2(x_{n-1}, p) - \alpha_n [d^2(x_{n-1}, p) + \theta_{ik(n)}] \\
&\quad + (1 - \alpha_n) \delta_{ik(n)}, \tag{29}
\end{aligned}$$

where $\theta_{ik(n)} = Q_{ik(n)}^2 + Q_{ik(n)}d(x_{n-1}, p)$. Since $\sum_{k(n)=1}^{\infty} Q_{ik(n)} < \infty$, it follows that $\sum_{k(n)=1}^{\infty} \theta_{ik(n)} < \infty$. The inequality (29) gives that

$$\sigma_n^2 \leq \left(\frac{1-s}{s^2}\right) \delta_{ik(n)} - \left(\frac{1}{s}\right) \theta_{ik(n)}. \tag{30}$$

For $m \geq 1$, we have that

$$\sum_{n=1}^m \sigma_n^2 \leq \left(\frac{1-s}{s^2}\right) \sum_{k(n)=1}^m \delta_{ik(n)} - \left(\frac{1}{s}\right) \sum_{k(n)=1}^m \theta_{ik(n)}. \tag{31}$$

When $m \rightarrow \infty$, we have that $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$ as $\sum_{k(n)=1}^{\infty} \delta_{ik(n)} < \infty$ and $\sum_{k(n)=1}^{\infty} \theta_{ik(n)} < \infty$.

Hence,

$$\lim_{n \rightarrow \infty} \sigma_n = 0. \tag{32}$$

Further,

$$\begin{aligned} d(x_n, x_{n-1}) &\leq (1 - \alpha_n)d\left(T_{i(n)}^{k(n)}x_n, x_{n-1}\right) \\ &= (1 - \alpha_n)\sigma_n \leq (1 - s)\sigma_n, \end{aligned} \quad (33)$$

implies that $\lim_{n \rightarrow \infty} d(x_n, x_{n-1}) = 0$.

For a fixed $j \in I$, we have $d(x_{n+j}, x_n) \leq d(x_{n+j}, x_{n+j-1}) + \cdots + d(x_n, x_{n-1})$ and hence

$$\lim_{n \rightarrow \infty} d(x_{n+j}, x_n) = 0 \quad \forall j \in I. \quad (34)$$

For $n > N$, $n = (n - N)(\text{mod } N)$. Also, $n = (k(n) - 1)N + i(n)$. Hence, $n - N = ((k(n) - 1) - 1)N + i(n) = k(n - N)N + i(n - N)$. That is, $k(n - N) = k(n) - 1$ and $i(n - N) = i(n)$.

Therefore, we have

$$\begin{aligned} d(x_{n-1}, T_n x_n) &\leq d\left(x_{n-1}, T_{i(n)}^{k(n)}x_n\right) + d\left(T_{i(n)}^{k(n)}x_n, T_n x_n\right) \\ &\leq \sigma_n + Ld\left(T_{i(n)}^{k(n)-1}x_n, x_n\right) \\ &\leq \sigma_n + L^2d(x_n, x_{n-N}) + Ld\left(T_{i(n-N)}^{k(n-N)}x_{n-N}, x_{(n-N)-1}\right) \\ &\quad + Ld(x_{(n-N)-1}, x_n) \\ &= \sigma_n + L^2d(x_n, x_{n-N}) + L\sigma_{n-N} \\ &\quad + Ld(x_{(n-N)-1}, x_n), \end{aligned} \quad (35)$$

using (32) and (34) in (35) yields that $\lim_{n \rightarrow \infty} d(x_{n-1}, T_n x_n) = 0$.

Since,

$$d(x_n, T_n x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n x_n), \quad (36)$$

we have

$$\lim_{n \rightarrow \infty} d(x_n, T_n x_n) = 0. \quad (37)$$

Hence, for all $l \in I$, we have

$$\begin{aligned} d(x_n, T_{n+l}x_n) &\leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l}x_{n+l}) \\ &\quad + d(T_{n+l}x_{n+l}, T_{n+l}x_n) \\ &\leq (1 + L)d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l}x_{n+l}), \end{aligned} \quad (38)$$

using (34) and (37) in (38) implies that

$$\lim_{n \rightarrow \infty} d(x_n, T_{n+l}x_n) = 0 \quad \forall l \in I. \quad (39)$$

Thus, $\lim_{n \rightarrow \infty} d(x_n, T_l x_n) = 0$ for all $l \in I$. \square

Theorem 2.4. *Let (X, d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian and asymptotically quasi-nonexpansive type self mappings of C . Suppose that*

F is closed, and there exists one member T in $\{T_i : i \in I\}$ which is either semi-compact or satisfies condition (A). Let $\{x_n\}$ be the implicit iteration process defined by (10). Put

$$G_{in} = \max \left\{ 0, \sup_{p \in F, n \geq 1} \left(d(T_i^n x_n, p) - d(x_n, p) \right) : i \in I \right\},$$

where $n = (k-1)N + i$ and $i = i(n) \in I$. Assume that $\sum_{n=1}^{\infty} G_{in} < \infty$ and $\{\alpha_n\} \subset [s, 1-s]$ for some $s \in (0, \frac{1}{2})$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings in $\{T_i : i \in I\}$.

Proof. Without loss of generality, we may assume that T_1 is semi-compact or satisfies condition (A). If T_1 is semi-compact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow p^* \in C$ as $j \rightarrow \infty$. Now, Lemma 2.3 guarantees that $\lim_{n \rightarrow \infty} d(x_{n_j}, T_l x_{n_j}) = 0$ for all $l \in I$ and so $d(p^*, T_l p^*) = 0$ for all $l \in I$. This implies that $p^* \in F$. Therefore, $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. If T_1 satisfies condition (A), then we also have $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. Now, Theorem 2.1 guarantees that $\{x_n\}$ converges strongly to a point in F . \square

Remark 1. Our results extend the corresponding results of Fukhar-ud-din et al. [15] to the case of more general class of generalized asymptotically quasi-nonexpansive mappings considered in this paper.

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G. S. SALUJA

DEPARTMENT OF MATHEMATICS AND INFORMATION TECHNOLOGY, GOVT. NAGARJUNA
P.G. COLLEGE OF SCIENCE, RAIPUR-492010 (C.G.), INDIA
E-mail address: saluja.1963@rediffmail.com, saluja1963@gmail.com