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# CHARACTERIZATION OF LIPSCHITZ-TYPE FUNCTIONS BY GARSIA-TYPE NORMS ON THE UPPER HALF SPACE

Jisoo Byun, Hong Rae Cho, and Bong-Hak Im

Abstract. It is well-known that the $BMO$ norm is equivalent to the Garsia norm. In this paper, we characterize mean-Lipschitz spaces by using Garsia-type norms on the upper half space $\mathbb{R}^{n+1}_+$.

## 1. Introduction and statement of results

Let $\mathbb{R}^{n+1}_+$ be the $(n+1)$-dimensional upper-half space. In the coordinate notation, we have

$$\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}.$$

We can consider $\mathbb{R}^n$ as the boundary of $\mathbb{R}^{n+1}_+$. For $t > 0$, we denote the Euclidean ball in $\mathbb{R}^n$ by

$$Q_t(x) = \{y \in \mathbb{R}^n : |x-y| < t\}, \quad x \in \mathbb{R}^n.$$

We define the integral mean $f_{Q_t}$ by

$$f_{Q_t(x)} = \frac{1}{|Q_t(x)|}\int_{Q_t(x)} f(y)dy$$

and the $BMO$ norm as

$$\|f\|_{BMO} = \sup_{\substack{x\in\mathbb{R}^n \\ t>0}} \frac{1}{|Q_t(x)|}\int_{Q_t(x)} |f - f_{Q_t(x)}|dy.$$

Here $|Q_t(x)|$ is the volume of $Q_t(x)$ in $\mathbb{R}^n$. The space $BMO$ of bounded mean oscillation is a set of all $L^1_{loc}$ functions on $\mathbb{R}^n$ with the finite norm $\|f\|_{BMO} < \infty$.

The Poisson kernel in $\mathbb{R}^{n+1}_+$ has an explicit expression;

$$P_t(x) = \frac{c_n t}{(|x|^2+t^2)^{(n+1)/2}}, \quad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}.$$

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The Poisson integral of a function given on $\mathbb{R}^n$ is defined by

$$\mathcal{P}f(x,t) = P_t * f(x) = \int_{y\in\mathbb{R}^n} P_t(x-y)f(y)dy.$$

Garsia has observed that there is another norm for functions in $BMO$ which is easier to use.[3] For $f \in L^1_{loc}(\mathbb{R}^n)$, the Garsia norm $\mathcal{G}(f)$ is defined by

$$\mathcal{G}(f) = \sup_{(x,t)\in\mathbb{R}^{n+1}_+} \int_{y\in\mathbb{R}^n} |f(y) - \mathcal{P}f(x,t)|P_t(x-y)dy.$$

It is well-known that [2]

$$\|f\|_{BMO} \sim \mathcal{G}(f). \tag{1}$$

For the unit ball in $\mathbb{C}^n$ the $BMO$ norm is defined by using the non-isotropic ball on the boundary of the unit ball. The same result as (1) on the unit ball in $\mathbb{C}^n$ was proved by Garsia (see [2], one-dimensional case) and by Axler-Shapiro (see [1], $n$-dimensional case).

Let $1 \le p, q \le \infty$ and $0 < \alpha < 1$. For $f \in L^p(\mathbb{R}^n)$, we denote

$$\Delta^{p,q}_\alpha(f) = \left(\int_{y\in\mathbb{R}^n} \frac{\|f(\cdot+y) - f(\cdot)\|^q_{L^p(\mathbb{R}^n)}}{|y|^{n+\alpha q}} dy\right)^{1/q} \tag{2}$$

and we define the mean-Lipschitz norm by

$$\|f\|_{\Lambda^{p,q}_\alpha} = \|f\|_{L^p(\mathbb{R}^n)} + \Delta^{p,q}_\alpha(f).$$

Then $\Lambda^{p,q}_\alpha$ consists of all functions $f$ in $L^p(\mathbb{R}^n)$ for which the norm $\|f\|_{\Lambda^{p,q}_\alpha}$ is finite. It is called the mean-Lipschitz space. For a measurable function $F$ on $\mathbb{R}^{n+1}_+$ we define

$$L^{p,q}_\alpha(F) = \left(\int_0^\infty \left(t^{1-\alpha}\|F(\cdot,t)\|_{L^p(\mathbb{R}^n)}\right)^q \frac{dt}{t}\right)^{1/q}. \tag{3}$$

We note that [4]

$$\int_0^\infty \left(t^{1-\alpha}\|\nabla\mathcal{P}f(\cdot,t)\|_{L^p(\mathbb{R}^n)}\right)^q \frac{dt}{t} \sim \int_0^\infty \left(t^{1-\alpha}\left\|\frac{\partial}{\partial t}\mathcal{P}f(\cdot,t)\right\|_{L^p(\mathbb{R}^n)}\right)^q \frac{dt}{t}.$$

By Hardy-Littlewood lemma [4], we get

$$\Delta^{p,q}_\alpha(f) \sim L^{p,q}_\alpha(\nabla\mathcal{P}f).$$

Now, we define the Garsia-type $(p,q)$-norm by

$$\begin{aligned} &\mathcal{G}^{p,q}_\alpha(f) \qquad\qquad (4)\\ &= \left(\int_0^\infty \frac{1}{t^{1+\alpha q}}\left(\int_{x\in\mathbb{R}^n}\left(\int_{y\in\mathbb{R}^n}|f(y)-\mathcal{P}f(x,t)|P_t(x-y)dy\right)^p dx\right)^{q/p} dt\right)^{1/q}. \end{aligned}$$

When $q = \infty$, the expressions (2), (3), and (4) are interpreted in the normal limiting way, namely

$$\Delta_\alpha^{p,\infty}(f) = \sup_{|y|>0} \frac{\|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{R}^n)}}{|y|^\alpha},$$

$$L_\alpha^{p,\infty}(\nabla \mathcal{P} f) = \sup_{t>0} t^{1-\alpha} \|\nabla \mathcal{P} f(\cdot, t)\|_{L^p(\mathbb{R}^n)},$$

and

$$\mathcal{G}_\alpha^{p,\infty}(f) = \sup_{t>0} \frac{1}{t^\alpha} \left( \int_{x\in\mathbb{R}^n} \left( \int_{y\in\mathbb{R}^n} |f(y) - \mathcal{P}f(x,t)| P_t(x-y) dy \right)^p dx \right)^{1/p}.$$

**Theorem 1.1.** *Let $1 \le p, q \le \infty$ and $0 < \alpha < 1$. For $f \in L^p(\mathbb{R}^n)$ we have*

$$\Delta_\alpha^{p,q}(f) \sim \mathcal{G}_\alpha^{p,q}(f).$$

Recall that the Poisson kernel for the upper half space is given by

$$P_t(x-y) = \frac{c_n t}{(|x-y|^2 + t^2)^{(n+1)/2}}.$$

**Lemma 1.2.** ([4]) *Let $0 < \alpha < 1$. Then*

(i) $\int_{\mathbb{R}^n} |x-y|^\alpha P_t(x-y) dy \lesssim t^\alpha$;

(ii) $|\nabla P_t(x-y)| \lesssim t^{-1} P_t(x-y)$ *for all* $(x,t) \in \mathbb{R}^{n+1}_+$ *and for all* $y \in \mathbb{R}^n$.

**Lemma 1.3.** (Hardy's inequalities) *Let $h$ is a non-negative function and $p \ge 1, r > 0$. Then we have*

(i) $\left[ \int_0^\infty \left( \int_0^x h(y) dy \right)^p x^{-r-1} dx \right]^{1/p} \le \frac{p}{r} \left( \int_0^\infty (yh(y))^p y^{-r-1} dy \right)^{1/p}$;

(ii) $\left[ \int_0^\infty \left( \int_x^\infty h(y) dy \right)^p x^{r-1} dx \right]^{1/p} \le \frac{p}{r} \left( \int_0^\infty (yh(y))^p y^{r-1} dy \right)^{1/p}$.

## 2. Proof of Theorem 1.1

First we consider the case $p = q = \infty$.

For $(x,t) \in \mathbb{R}^{n+1}_+$ and $y \in \mathbb{R}^n$ we have

$$|f(y) - \mathcal{P}f(x,t)| \lesssim |f(y) - f(x)| + |f(x) - \mathcal{P}f(x,t)|$$

and

$$\begin{aligned} |f(x) - \mathcal{P}f(x,t)| &= \left| \int_{y\in\mathbb{R}^n} (f(x) - f(y)) P_t(x-y) dy \right| \\ &\lesssim \Delta_\alpha^{\infty,\infty}(f) \int_{y\in\mathbb{R}^n} |x-y|^\alpha P_t(x-y) dy \\ &\lesssim \Delta_\alpha^{\infty,\infty}(f) t^\alpha, \end{aligned}$$

by (i) of Lemma 1.2. Thus we have

$$|f(y)-\mathcal{P}f(x,t)| \lesssim \Delta_\alpha^{\infty,\infty}(f)(|x-y|^\alpha + t^\alpha).$$

By (i) of Lemma 1.2 again, we have

$$\begin{aligned}
&\int_{y\in\mathbb{R}^n} |f(y)-\mathcal{P}f(x,t)|P_t(x-y)dy \\
&\lesssim \Delta_\alpha^{\infty,\infty}(f)\left(\int_{y\in\mathbb{R}^n} |x-y|^\alpha P_t(x-y)dy + t^\alpha\right) \\
&\lesssim \Delta_\alpha^{\infty,\infty}(f)t^\alpha.
\end{aligned}$$

This implies that

$$\mathcal{G}_\alpha^{\infty,\infty}(f) \lesssim \Delta_\alpha^{\infty,\infty}(f).$$

Recall that

$$\mathcal{P}f(x,t) = \int_{y\in\mathbb{R}^n} P_t(x-y)f(y)dy.$$

Differentiating the both side, we get

$$\nabla_x \mathcal{P}f(x,t) = \int_{y\in\mathbb{R}^n} (f(y)-\mathcal{P}f(x,t))\nabla_x P_t(x-y)dy.$$

By (ii) of Lemma 1.2, we have

$$|\nabla_x \mathcal{P}f(x,t)| \lesssim \frac{1}{t}\int_{y\in\mathbb{R}^n} |f(y)-\mathcal{P}f(x,t)|P_t(x-y)dy.$$

It follows that

$$t^{1-\alpha}|\nabla_x \mathcal{P}f(x,t)| \lesssim t^{-\alpha}\int_{y\in\mathbb{R}^n} |f(y)-\mathcal{P}f(x,t)|P_t(x-y)dy.$$

This implies that $L_\alpha^{\infty,\infty}(\nabla\mathcal{P}f) \lesssim \mathcal{G}_\alpha^{\infty,\infty}(f)$.

Now we state the proof for the case of $1 < p, q < \infty$. By (ii) of Lemma 1.2, we have

$$|\nabla_x \mathcal{P}f(x,t)| \lesssim \int_{y\in\mathbb{R}^n} \frac{1}{t}P_t(x-y)|f(y)-\mathcal{P}f(x,t)|dy.$$

Thus it follows that

$$\begin{aligned}
&L_\alpha^{p,q}(\nabla\mathcal{P}f) \\
&\leq \left(\int_0^\infty \left(t^{1-\alpha}\left(\int_{x\in\mathbb{R}^n}\left(\int_{y\in\mathbb{R}^n}\frac{1}{t}P_t(x-y)|f(y)-\mathcal{P}f(x,t)|dy\right)^p dx\right)^{1/p}\right)^q \frac{dt}{t}\right)^{1/q} \\
&= \mathcal{G}_\alpha^{p,q}(f).
\end{aligned}$$

For the converse we let

$$\begin{aligned}\Omega(t) =& \left(\int_{x\in\mathbb{R}^n}\left(\int_{y\in\mathbb{R}^n}|f(y)-\mathcal{P}f(x,t)|P_t(x-y)dy\right)^p dx\right)^{1/p}\\ \lesssim& \left(\int_{x\in\mathbb{R}^n}\left(\int_{y\in\mathbb{R}^n}|f(y)-f(x)|P_t(x-y)dy\right)^p dx\right)^{1/p}\\ &+\left(\int_{x\in\mathbb{R}^n}\left(\int_{y\in\mathbb{R}^n}|f(x)-\mathcal{P}f(x,t)|P_t(x-y)dy\right)^p dx\right)^{1/p}\\ =&I_1(t)+I_2(t).\end{aligned}$$

By Minkowski's inequality, we have

$$\begin{aligned}I_2(t) &= \left(\int_{x\in\mathbb{R}^n}|f(x)-\mathcal{P}f(x,t)|^p dx\right)^{1/p}\\ &= \left(\int_{x\in\mathbb{R}^n}\left|\int_0^t \frac{\partial}{\partial s}\mathcal{P}f(x,s)ds\right|^p dx\right)^{1/p}\\ &\le \int_0^t \left\|\frac{\partial}{\partial s}\mathcal{P}f(\cdot,s)\right\|_{L^p} ds.\end{aligned}$$

By Hardy's inequality (i), we have

$$\begin{aligned}\left(\int_0^\infty I_2(t)^q t^{-\alpha q-1}dt\right)^{1/q} &\lesssim \left(\int_0^\infty\left(\int_0^t\left\|\frac{\partial}{\partial s}\mathcal{P}f(\cdot,s)\right\|_{L^p}ds\right)^q t^{-\alpha q-1}dt\right)^{1/q}\\ &\lesssim \left(\int_0^\infty\left(s\left\|\frac{\partial}{\partial s}\mathcal{P}f(\cdot,s)\right\|_{L^p}\right)^q s^{-\alpha q-1}ds\right)^{1/q}\\ &\lesssim L_\alpha^{p,q}(\nabla\mathcal{P}f).\end{aligned}$$

Now we estimate the first term $I_1(t)$. Replacing $y$ by $x+\xi$ we have

$$\int_{y\in\mathbb{R}^n}|f(y)-f(x)|P_t(x-y)dy = \int_{\xi\in\mathbb{R}^n}|f(x+\xi)-f(x)|P_t(\xi)d\xi.$$

By Minkowski's inequality, we have

$$\begin{aligned}I_1(t) &= \left(\int_{x\in\mathbb{R}^n}\left(\int_{\xi\in\mathbb{R}^n}|f(x+\xi)-f(x)|P_t(\xi)d\xi\right)^p dx\right)^{1/p}\\ &\lesssim \int_{\xi\in\mathbb{R}^n}\|f(\cdot+\xi)-f(\cdot)\|_{L^p}P_t(\xi)d\xi\\ &= \int_{|\xi|\le t}\|f(\cdot+\xi)-f(\cdot)\|_{L^p}P_t(\xi)d\xi + \int_{|\xi|>t}\|f(\cdot+\xi)-f(\cdot)\|_{L^p}P_t(\xi)d\xi\\ &= I_{11}(t)+I_{12}(t).\end{aligned}$$

Since $P_t(\xi) \lesssim 1/t^n$, we have

$$I_{11}(t) = \int_{|\xi|\le t} \|f(\cdot+\xi) - f(\cdot)\|_{L^p} P_t(\xi) d\xi \lesssim \frac{1}{t^n} \int_{|\xi|\le t} \|f(\cdot+\xi) - f(\cdot)\|_{L^p} d\xi.$$

Let $\xi = rz$ where $r = |\xi|$ and $z \in \mathbb{R}^n$ with $|z| = 1$. Then $d\xi = r^{n-1} dr dz$. Let $S^{n-1}$ be the unit sphere on $\mathbb{R}^n$. Let

$$\omega(r) = \int_{S^{n-1}} \|f(\cdot + rz) - f(\cdot)\|_{L^p} dz.$$

Then we have

$$I_{11}(t) \lesssim \frac{1}{t^n} \int_0^t \omega(r) r^{n-1} dr.$$

Thus by Hardy's inequality (i), we have

$$\begin{aligned}
\int_0^\infty \frac{1}{t^{1+\alpha q}} I_{11}(t)^q dt &\lesssim \int_0^\infty \frac{1}{t^{1+\alpha q}} \left( \frac{1}{t^n} \int_0^t \omega(r) r^{n-1} dr \right)^q dt \\
&= \int_0^\infty \left( \int_0^t \omega(r) r^{n-1} dr \right)^q t^{-1-\alpha q - nq} dt \\
&\lesssim \int_0^\infty \omega(r)^q t^{-\alpha q - 1} dr \\
&= \int_0^\infty \left( \int_{S^{n-1}} \|f(\cdot + rz) - f(\cdot)\|_{L^p} dz \right)^q r^{-\alpha q - 1} dr \\
&\lesssim \int_{\mathbb{R}^n} \frac{\|f(\cdot+\xi) - f(\cdot)\|_{L^p}^q}{|\xi|^{n+\alpha q}} d\xi.
\end{aligned}$$

Since $P_t(\xi) \lesssim t/|\xi|^{n+1}$, we have

$$I_{12}(t) = \int_{|\xi|>t} \|f(\cdot+\xi) - f(\cdot)\|_{L^p} P_t(\xi) d\xi \lesssim t \int_{|\xi|>t} \|f(\cdot+\xi) - f(\cdot)\|_{L^p} \frac{d\xi}{|\xi|^{n+1}}.$$

By Hardy's inequality (ii), we have

$$\begin{aligned}
\int_0^\infty \frac{1}{t^{1+\alpha q}} I_{12}(t)^q dt &\lesssim \int_0^\infty \frac{1}{t^{1+\alpha q}} \left( t \int_t^\infty \omega(r) r^{-2} dt \right)^q dt \\
&= \int_0^\infty \left( \int_t^\infty \omega(r) r^{-2} dr \right)^q t^{(1-\alpha)q-1} dt \\
&\lesssim \int_0^\infty (\omega(r) r^{-1})^q r^{(1-\alpha)q-1} dr \\
&\lesssim \int_{\mathbb{R}^n} \frac{\|f(\cdot+\xi) - f(\cdot)\|_{L^p}^q}{|\xi|^{n+\alpha q}} d\xi.
\end{aligned}$$

Therefore

$$\mathcal{G}_\alpha^{p,q}(f) = \left( \int_0^\infty \frac{1}{t^{1+\alpha q}} \Omega(t)^q dt \right)^{1/q} \lesssim \Delta_\alpha^{p,q}(f).$$

The other cases are similar.

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Department of Mathematics Education, Kyungnam University, Masan, South Korea
*E-mail address*: jisoobyun@gmail.com

Department of Mathematics, Pusan National University, Pusan 609-735, South Korea
*E-mail address*: chohr@pusan.ac.kr

Department of Mathematics, Pusan National University, Pusan 609-735, South Korea
*E-mail address*: ibk0504@pusan.ac.kr