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CONVERGENCE THEOREMS OF IMPLICIT ITERATION PROCESS WITH ERRORS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE IN BANACH SPACES

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Abstract. The aim of this article is to study an implicit iteration process with errors for a finite family of non-Lipschitzian asymptotically nonexpansive mappings in the intermediate sense in Banach spaces. Also we establish some strong convergence theorems and a weak convergence theorem for said scheme to converge to a common fixed point for non-Lipschitzian asymptotically nonexpansive mappings in the intermediate sense. The results presented in this paper extend and improve the corresponding results of [1], [3]-[8], [10]-[11], [13]-[14], [16] and many others.

1. Introduction and preliminaries

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. Recall the following concepts.

(1) T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1}$$

for all $x, y \in K$.

(2) T is asymptotically nonexpansive if there exists a sequence $\{a_n\}$ in $[1, \infty)$ with $a_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq a_n \|x - y\|, \tag{2}$$

for all $x, y \in K$ and $n \geq 1$.

(3) T is uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \tag{3}$$

for all $x, y \in K$ and $n \geq 1$.

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It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive is uniformly Lipschitzian.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] as a generalization of the class of nonexpansive mappings. T is said to be asymptotically nonexpansive mapping in the intermediate sense [2] if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty}\sup_{x,y\in K}\Big(\|T^n x - T^n y\| - \|x-y\|\Big) \quad\le\quad 0. \tag{4}$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense and asymptotically quasi-nonexpansive mapping. But the converse does not hold as the following example:

Example 1. Let $X = \mathbb{R}$ be a normed linear space and $K = [0,1]$. For each $x \in K$, we define

$$T(x) = \begin{cases} \text{kx}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $0 < k < 1$. Then

$$|T^n x - T^n y| = k^n|x-y| \le |x-y|$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ and

$$\begin{aligned}\limsup_{n\to\infty}\{|T^n x - T^n y| - |x-y|\} \quad &= \quad \limsup_{n\to\infty}\{k^n\,\|x-y\| - \|x-y\|\} \\ &\le \quad 0\end{aligned}$$

because $\lim_{n\to\infty} k^n = 0$ as $0 < k < 1$ and for all $x, y \in K$, $n \in \mathbb{N}$. Hence T is an asymptotically nonexpansive mapping in the intermediate sense.

Example 2. Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in K$, define

$$T(x) = \begin{cases} kxsin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive mapping in the intermediate sense but it is not asymptotically nonexpansive mapping.

Recall that E is said to satisfy Opial condition [9] if for any sequence $\{x_n\}$ in E, the condition that the sequence $x_n \to x$ weakly implies that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Let E be a Hilbert space, let K be a nonempty closed convex subset of E and let $\{T_1, T_2, \dots, T_N\}\colon K \to K$ be N nonexpansive mappings. In 2001, Xu and Ori [17] introduced the following implicit iteration process $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{n(mod\ N)}x_n, \quad n \geq 1, \tag{5}$$

where $x_0 \in K$ is an initial point, $\{\alpha_n\}_{n\geq 1}$ is a real sequence in $(0,1)$ and proved the weakly convergence of the sequence $\{x_n\}$ defined by (5) to a common fixed point $p \in F = \bigcap_{i=1}^{N} F(T_i)$.

In 2006, Gu [6] introduced the following implicit iterative sequence $\{x_n\}$ with errors

$$\begin{aligned} x_n &= (1-\alpha_n)x_{n-1} + \alpha_n T^n_{n(mod\ N)}y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n_{n(mod\ N)}x_n + v_n, \quad n \geq 1, \end{aligned} \tag{6}$$

for a finite family of asymptotically nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$ on a closed convex subset K of a real Banach space E with $K + K \subset K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in [0,1], $\{u_n\}$ and $\{v_n\}$ be two sequences in K and proved the weak and strong convergence of the sequence $\{x_n\}$ defined by (6) to a common fixed point $p \in F = \bigcap_{i=1}^{N} F(T_i)$.

Recently concerning the convergence problems of an implicit (or non-implicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been considered by several authors (see, e.g., Bauschke [1], Chang and Cho [3], Goebel and Kirk [4], Gornicki [5], Gu [6], Halpern [7], Lions [8], Osilike [10], Reich [11], Schu [12], Sun [13], Tan and Xu [14], Wittmann [16], Xu and Ori [17] and Zhou and Chang [18]).

The purpose of this article is to study an implicit iterative sequence defined by (6) for a finite family of asymptotically nonexpansive mappings in the intermediate sense in Banach spaces and establish some strong convergence theorems and a weak convergence theorem for said iteration scheme and mappings.

In the sequel we need the following lemmas to prove our main results.

Lemma 1.1. (see [15]): *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{T_i\}_{i=1}^{N} \colon K \to K$ be N asymptotically nonexpansive in the intermediate sense mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Put*

$$A_n = \max\Big\{0, \sup_{p\in F,\ n\geq 1} \Big(\Big\| T^n_{n(mod\ N)}x - T^n_{n(mod\ N)}y \Big\| - \|x-y\| \Big)\Big\}, \tag{7}$$

such that $\sum_{n=1}^{\infty} A_n < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in K and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:

(i) $\rho = \sup\{\alpha_n : n \geq 1\} < 1$;
(ii) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.
If $\{x_n\}$ *is the implicit iterative sequence defined by (6), then for each* $p \in F = \bigcap_{i=1}^{N} F(T_i)$ *the limit* $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. Since $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, for any given $p \in F$, it follows from (6) and (7) that

$$\begin{aligned}
\|x_n - p\| &= \left\|(1-\alpha_n)x_{n-1} + \alpha_n T^n_{n(mod\ N)} y_n + u_n - p\right\| \\
&\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \left\|T^n_{n(mod\ N)} y_n - p\right\| \\
&\quad + \|u_n\| \\
&= (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \left\|T^n_{n(mod\ N)} y_n - T^n_{n(mod\ N)} p\right\| \\
&\quad + \|u_n\|, \\
&\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n[\|y_n - p\| + A_n] \\
&\quad + \|u_n\| \\
&\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \|y_n - p\| + A_n \\
&\quad + \|u_n\|. \qquad (8)
\end{aligned}$$

Again it follows from (6) and (7) that

$$\begin{aligned}
\|y_n - p\| &\leq (1-\beta_n)\|x_n - p\| + \beta_n \left\|T^n_{n(mod\ N)} x_n - p\right\| \\
&\quad + \|v_n\| \\
&= (1-\beta_n)\|x_n - p\| + \beta_n \left\|T^n_{n(mod\ N)} x_n - T^n_{n(mod\ N)} p\right\| \\
&\quad + \|v_n\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n[\|x_n - p\| + A_n] \\
&\quad + \|v_n\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \|x_n - p\| + A_n \\
&\quad + \|v_n\| \\
&\leq \|x_n - p\| + A_n + \|v_n\|. \qquad (9)
\end{aligned}$$

Substituting (9) into (8), we obtain that

$$\begin{aligned}
\|x_n - p\| &\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \|x_n - p\| + (\alpha_n + 1)A_n \\
&\quad + \alpha_n \|v_n\| + \|u_n\|, \\
&\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \|x_n - p\| + 2A_n \\
&\quad + \alpha_n \|v_n\| + \|u_n\|,
\end{aligned}$$

which implies that

$$\begin{aligned}(1-\alpha_n)\|x_n-p\| &\leq (1-\alpha_n)\|x_{n-1}-p\| \\ &\quad +\sigma_n, \end{aligned} \tag{10}$$

where $\sigma_n = 2A_n+\alpha_n\|v_n\|+\|u_n\|$. By the assumption $\sum_{n=1}^{\infty} A_n < \infty$, condition (ii) and boundedness of the sequences $\{\alpha_n\}$, we know that $\sum_{n=1}^{\infty}\sigma_n < \infty$. From condition (i) we have

$$\alpha_n \leq \rho < 1,$$

and so

$$1-\alpha_n \geq 1-\rho > 0, \tag{11}$$

hence using (11) in (10), we have

$$\begin{aligned}\|x_n-p\| &\leq \|x_{n-1}-p\| + \frac{\sigma_n}{1-\alpha_n} \\ &\leq \|x_{n-1}-p\| + \frac{\sigma_n}{1-\rho} \\ &= \|x_{n-1}-p\| + \lambda_n, \end{aligned} \tag{12}$$

where

$$\lambda_n = \frac{\sigma_n}{1-\rho}.$$

By assumption of the theorem and condition (ii) we have that

$$\sum_{n=1}^{\infty}\lambda_n = \sum_{n=1}^{\infty}\frac{\sigma_n}{1-\rho} < \infty.$$

Taking $A_n = \|x_{n-1}-p\|$ in inequality (12), we have

$$A_{n+1} \leq A_n + \lambda_n, \quad \forall n \geq 1,$$

and satisfied all conditions in Lemma 1.1. Therefore the limit $\lim_{n\to\infty}\|x_n-p\|$ exists. This completes the proof of Lemma 1.2. □

2. Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K+K \subset K$. Let $\{T_i\}_{i=1}^{N}: K \to K$ be N asymptotically nonexpansive mappings in the intermediate sense with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Put*

$$A_n = \max\Big\{0, \sup_{p\in F,\ n\geq 1}\Big(\Big\|T^n_{n(mod\ N)}x - T^n_{n(mod\ N)}y\Big\| - \|x-y\|\Big)\Big\},$$

such that $\sum_{n=1}^{\infty} A_n < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in K and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:

(i) $\rho = \sup\{\alpha_n : n \geq 1\} < 1$;

(ii) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Then the implicit iterative sequence $\{x_n\}$ *defined by (6) converges strongly to a common fixed point* $p \in F = \bigcap_{i=1}^{N} F(T_i)$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{13}$$

Proof. The necessity of condition (13) is obvious.

Next we prove the sufficiency of Theorem 2.1. For any given $p \in F$, it follows from equation (12) in Lemma 1.2 that

$$\|x_n - p\| \quad \le \quad \|x_{n-1} - p\| + \lambda_n \quad \forall n \ge 1, \tag{14}$$

where

$$\lambda_n = \frac{\sigma_n}{1-\rho}.$$

with $\sum_{n=1}^{\infty} \lambda_n < \infty$. Hence, we have

$$d(x_n, F) \quad \le \quad d(x_{n-1}, p) + \lambda_n \quad \forall n \ge 1, \tag{15}$$

It follows from (15) and Lemma 1.1 that the limit $\lim_{n\to\infty} d(x_n, F)$ exists. By the condition (13), we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. For any integer $m \ge 1$, we have from (14) that

$$\begin{aligned} \|x_{n+m} - p\| &\le \|x_{n+m-1} - p\| + \lambda_{n+m-1} \\ &\le \|x_{n+m-2} - p\| + \lambda_{n+m-2} + \lambda_{n+m-1} \\ &\le \dots \\ &\le \|x_n - p\| + \sum_{k=n}^{n+m-1} \lambda_k. \end{aligned} \tag{16}$$

Since $\liminf_{n\to\infty} d(x_n, F) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset F$ such that $\|x_{n_k} - p_{n_k}\| \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$\|x_{n_k} - p_{n_k}\| < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{k=n_{k_\varepsilon}}^{\infty} \lambda_k < \frac{\varepsilon}{4}, \tag{17}$$

for all $k \ge k_\varepsilon$.

For any $m \geq 1$ and for all $n \geq n_{k_\varepsilon}$, by (17), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_{n_k}\| + \|x_n - p_{n_k}\| \\ &\leq \|x_{n_k} - p_{n_k}\| + \sum_{k=n_{k_\varepsilon}}^{\infty} \lambda_k \\ &\quad + \|x_{n_k} - p_{n_k}\| + \sum_{k=n_{k_\varepsilon}}^{\infty} \lambda_k \\ &= 2\|x_{n_k} - p_{n_k}\| + 2\sum_{k=n_{k_\varepsilon}}^{\infty} \lambda_k \\ &< 2.\frac{\varepsilon}{4} + 2.\frac{\varepsilon}{4} = \varepsilon. \end{aligned} \tag{18}$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. By the completeness of K, we can assume that $\lim_{n\to\infty} x_n = p^*$. Since the set of fixed points of an asymptotically nonexpansive mapping in the intermediate sense is closed, hence F is closed. This implies that $p^* \in F$ and so p^* is a common fixed point of the mappings $\{T_i\}_{i=1}^N$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K+K \subset K$. Let $\{T_i\}_{i=1}^N\colon K \to K$ be N asymptotically nonexpansive mappings in the intermediate sense with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Put*

$$A_n = \max\Big\{0, \sup_{p\in F,\ n\geq 1} \Big(\Big\|T^n_{n(mod\ N)}x - T^n_{n(mod\ N)}y\Big\| - \|x-y\| \Big)\Big\},$$

such that $\sum_{n=1}^\infty A_n < \infty$. Let the implicit iterative sequence $\{x_n\}$ defined by (6) with the restrictions $\rho = \sup\{\alpha_n : n \geq 1\} < 1$, $\sum_{n=1}^\infty \|u_n\| < \infty$ and $\sum_{n=1}^\infty \|v_n\| < \infty$. Suppose that the mapping T_i for all $i \in I = \{1,2,\ldots,N\}$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$ for all $i \in I = \{1,2,\ldots,N\}$;

(ii) there exists a constant $A > 0$ such that $\|x_n - T_ix_n\| \geq Ad(x_n, F)$, $\forall n \geq 1$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. From condition (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 2.1 that $\{x_n\}$ must converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$. This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let E be a real Banach space satisfying Opial's condition and K be a weakly compact subset of E with $K + K \subset K$. Let $T_i\colon K \to K$ be*

N asymptotically nonexpansive mappings in the intermediate sense with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Put

$$A_n = \max\Big\{0, \sup_{x,y\in K,\ n\geq 1} \Big(\Big\|T^n_{n(mod\ N)}x - T^n_{n(mod\ N)}y\Big\| - \|x-y\| \Big)\Big\},$$

such that $\sum_{n=1}^{\infty} A_n < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in K and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ with the restrictions $\rho = \sup\{\alpha_n : n \geq 1\} < 1$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$. Suppose that $\{T_i : i \in I\}$ has a common fixed point, $I - T_i$ for all $i \in I = \{1, 2, \ldots, N\}$ is demiclosed at zero and $\{x_n\}$ is an approximating common fixed point sequence for T_i, that is, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I = \{1, 2, \ldots, N\}$. Then the implicit iterative sequence $\{x_n\}$ defined by (6) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$.

Proof. First, we show that $\omega_w(x_n) \subset F$. Let $x_{n_k} \to x$ weakly. By assumption, we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$. Since $I - T_i$ for all $i \in I$ is demiclosed at zero, $x \in F$. By Opial's condition, $\{x_n\}$ possesses only one weak limit point, that is, $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{N}$. This completes the proof of Theorem 2.3. □

Remark 1. Theorem 2.1 extends the corresponding results of of Chang and Cho [3] to the case of more general class of asymptotically nonexpansive mapping considered in this paper.

Remark 2. Our results also improve and extend the corresponding results of [1, 4, 5, 7, 8, 10, 11, 13, 14, 16] to the case of more general class of spaces, mappings and iteration schemes considered in this paper.

Remark 3. Our results also extend the corresponding results of Gu [6] to the case of more general class of asymptotically nonexpansive mapping considered in this paper.

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