East Asian Mathematical Journal
Vol. 28 (2012), No. 1, pp. 37–47

PROXIMAL POINTS METHODS FOR GENERALIZED IMPLICIT VARIATIONAL-LIKE INCLUSIONS IN BANACH SPACES

Xin-feng He, Jian Lou and Zhen He

Abstract. In this paper, we study generalized implicit variational-like inclusions and J^{η}-proximal operator equations in Banach spaces.It is established that generalized implicit variational-like inclusions in real Banach spaces are equivalent to fixed point problems.We also establish a relationship between generalized implicit variational-like inclusions and J^{η}-proximal operator equations. This equivalence is used to suggest an iterative algorithm for solving J^{η}-proximal operator equations.

1. Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science. For the past few years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, please see [1-8] and the references therein.

The resolvent operator techniques for solving variational inequalities and variational inclusions are interesting and important. The resolvent operator technique is used to establish an equivalence between mixed variational inequalities and resolvent equations. The resolvent equation technique is used to develop powerful and efficient numerical techniques for solving mixed variational inequalities and related optimization problems.

In this paper, we generalize the resolvent equations by introducing J^{η}-proximal operator equations in Banach spaces. A relationship between generalized implicit variational-like inclusions and J^{η}- proximal operator equations is established. We propose an iterative algorithm for computing the approximate solutions which converge to the exact solutions of J^{η}-proximal operator equations.

Received February 16, 2011; Accepted September 26, 2011.
2000 *Mathematics Subject Classification.* 47H05, 47H09, 47H10.
Key words and phrases. generalized implicit variational-like inclusions, J^{η}-Proximal operator, Algorithm, J^{η}-proximal operator equations.

2. J^η-proximal-point mapping

Throughout this paper, we assume that E is a real Banach space with dual space E^*; $CB(E)$ (respectively, 2^E) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E; $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$H(A,B) = max\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y)\} \quad A,B \in CB(E);$$

$\langle\cdot,\cdot\rangle$ is the dual pair between E and E^*, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x,f\rangle = \|x\|^2, \|x\| = \|f\|\} \quad x \in E.$$

We observe immediately that if $E \equiv H$, a Hilbert space, then J is the identity map on H.

First, we define the following concepts.

Definition 2.1.([8]) Let $\eta : E \times E \to E$ be a mapping.Then a mapping $P : E \to E^*$ is said to be

(1) η-monotone, if

$$\langle Px - Py, \eta(x,y)\rangle \geq 0, \quad \forall x,y \in E;$$

(2) strictly η-monotone, if

$$\langle Px - Py, \eta(x,y)\rangle \geq 0, \quad \forall x,y \in E$$

and equality holds if and only if $x = y$;

(3) strongly η-monotone, if $\exists r > 0$ such that

$$\langle Px - Py, \eta(x,y)\rangle \geq r\|x-y\|^2, \quad \forall x,y \in E.$$

Definition 2.2.([8]) A mapping $\eta : E \times E \to E$ is said to be τ-Lipschitz continuous, if $\exists\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau\|x-y\|, \quad \forall x,y \in E.$$

Definition 2.3.([7]) A mapping $T : E \to E^*$ is called γ-cocoercive if for all $x,y \in E$, there exists a constant $\gamma > 0$, such that $\langle Tx - Ty, \eta(x,y)\rangle \geq -\gamma\|Tx - Ty\|^2$.

Definition 2.4.([14]) A mapping $T : E \to E$ is called relaxed (γ,r)-cocoercive if for all $x,y \in K$, there exists constants $\gamma > 0, r > 0, j(x-y) \in J(x-y)$, such that

$$\langle Tx - Ty, j(x-y)\rangle \geq -\gamma\|Tx - Ty\|^2 + r\|x-y\|^2.$$

The class of relaxed (γ,r)-cocoercive mappings is more general than the class of strongly accretive mappings.

Definition 2.5.([8]) Let $M : E \to 2^{E^*}$ be a multivalued operator. $P : E \to E^*$, $\eta : E \times E \to E$ be single-valued operator. M is said to be:

(1) monotone if

$$\langle u - v, x - y\rangle \geq 0; \qquad \forall x,y \in E, u \in Mx, v \in My;$$

(2) $\eta-$monotone if

$$\langle u-v, \eta(x,y)\rangle \geq 0, \qquad \forall x,y \in E, u \in Mx, v \in My;$$

(3) strongly $\eta-$monotone if there exists some constant $r>0$, such that

$$\langle u-v, \eta(x,y)\rangle \geq r\|x-y\|^2, \qquad \forall x,y \in E, u \in Mx, v \in My;$$

(4) m-relaxed η-monotone if there exists some constant $m>0$, such that

$$\langle u-v, \eta(x,y)\rangle \geq -m\|x-y\|^2, \qquad \forall x,y \in E, u \in Mx, v \in My;$$

(5) maximal monotone if M is monotone and has no a proper monotone extension in E, i.e. for all $u, v_0 \in E$, $u \in Mx$,

$$\langle u-v_0, x-y_0\rangle \geq 0$$

implies $v_0 \in My_0$.

When E is reflexive Banach space, M is maximal monotone if and only if $(J+\lambda M)E = E^*$, for all $\lambda > 0$.

(6) maximal $\eta-$monotone if M is $\eta-$monotone and has no a proper $\eta-$monotone extension in E.

(7) $P-$monotone if M is monotone and $(P+\lambda M)E = E^*$, for all $\lambda > 0$; if M is $\eta-$monotone and $(P+\lambda M)E = E^*$, for all $\lambda > 0$, then M is said to be $(P,\eta)-$monotone operators.

(8) P-η-monotone if M is m-relaxed η-monotone and $(P+\lambda M)E = E^*$, for all $\lambda > 0$.

In [7] we introduced the following results.

Theorem 2.1. *Let E be a real Banach space $\eta : E \times E \to E$ be a τ-Lipschitz continuous operator , Let $P : E \to E^*$ be a strongly $\eta-$monotone operator with constants $r > 0$, $M : E \to 2^{E^*}$ be a multivalued P-η-monotone operator. Then, the mapping $(P+\rho M)^{-1} : E^* \to E$ is single-valued Lipschitz continuous with constant $\frac{\tau}{r-m\rho}$ for $0 < \rho < \frac{r}{m}$, that is,*

$$\|(P+\rho M)^{-1}(u)-(P+\rho M)^{-1}(v)\| \leq \frac{\tau}{r-m\rho}\|u-v\|, \qquad \forall u,v \in X^*.$$

By Theorem 2.1, we can define P-η-proximal point mapping (or the resolvent operator) for a P-η-monotone mapping M as follows:

$$J_\rho^M(z) = (P+\rho M)^{-1}(z), \qquad \forall z \in E,$$

where $0 < \rho < \frac{r}{m}$ is a constant, $\eta : E \times E \to E$ is a mapping and $P : E \to E^*$ be a strongly $\eta-$monotone mapping with constant $r > 0$.

Remark 2.1. P-η-proximal-point mapping generalize the corresponding concepts given by Kazmi [12] and Fang and Huang [11].

3. Multi-valued variational-like inclusion and iterative algorithm

Let $T, A, B : E \to CB(E^*)$ be set-valued mappings. Let $N : E^* \times E^* \times E^* \to E^*, f : E \to E^*$, $\eta : E \times E \to E$, $M : E \times E \to E^*$ and $g : E \to E$ be single-valued mappings. Assume that $g(E) \cap \text{dom}(M(\cdot, x)) \neq \emptyset$. Let $\varphi : E \times E \to R \cup \{+\infty\}$ be such that for each fixed $x \in E$, $\varphi(\cdot, x)$ is a lower semi-continuous, η-subdifferentiable functional on E (may not be convex) satisfying $g(E) \cap \text{dom}(\partial_\eta \varphi(\cdot, x)) \neq \emptyset$, where $\partial_\eta \varphi(\cdot, x)$ is the η-subdifferential of $\varphi(\cdot, x)$.(see [9]) We consider the following generalized implicit variational-like inclusion problems in Banach spaces (for short GIVLIP): find $x \in E$,$u \in T(x), v \in A(x), w \in B(x)$ such that

$$f(x) \in N(u, v, w) + M(g(x), x). \tag{3.1}$$

Special cases of GIVLIP (3.1):

(1) If we given single-valued mappings $P, f, h : E \to E^*, g : E \to E, \eta : E \times E \to E$ and multivalued mappings $M, S, T : E \to CB(E^*)$. Then problem (3.1) reduces to the following mixed variational-like inclusion introduced by R. Ahmad and A.H. Siddiqi [9].

$$\begin{cases} \text{Find } \ x \in E, u \in M(x), v \in S(x), w \in T(x) \ \text{ such that } \ g(x) \in \text{dom}(\partial_\eta \varphi) \\ \text{and} \\ \langle P(u) - (f(v) - h(w)), \eta(y, g(x)) \rangle \geq \varphi(g(x)) - \varphi(y), \ \text{ for all } \ y \in E. \end{cases} \tag{3.2}$$

(2) If $\eta(x.y) + \eta(y, x) = 0$,$M(\cdot, x) = \partial_\eta \varphi(\cdot, x)$ such that $R(P + \lambda \partial_\eta \varphi(\cdot, x)) = E^*$,then problem (3.1) is equivalent to the following

$$\begin{cases} \text{Find } \ x \in E, u \in T(x), v \in A(x), w \in B(x) \ \text{ such that } \ g(x) \in \text{dom}(\partial_\eta \varphi(\cdot, x)) \\ \text{and} \\ \langle f(x) - N(u, v, w), \eta(y, g(x)) \rangle \geq \varphi(g(x), x) - \varphi(y, x), \ \text{ for all } \ y \in E. \end{cases} \tag{3.3}$$

(3) If $N(u, v, w) = N(u, v)$, for all $u, v, w \in E$, then problem (3.1) is equivalent to the following generalized multivalued nonlinear quasi-variational-like inclusions in Banach spaces:

$$\begin{cases} \text{Find } \ x \in E, u \in T(x), v \in A(x), \ \text{ such that } \ g(x) \in \text{dom}(\partial_\eta \varphi(\cdot, x)) \\ \text{and} \\ \langle f(x) - N(u, v), \eta(y, g(x)) \rangle \geq \varphi(g(x), x) - \varphi(y, x), \ \text{ for all } \ y \in E. \end{cases} \tag{3.4}$$

Problem (3.4) is introduced and studied by Ahmad et al.[3].

(4) If $E = H$, is a Hilbert space, $f(x) \equiv 0, N(u, v, w) = u - v$, for all $u, v, w \in H$ and $T, A : H \to H$ are both single-valued mappings, then problem (3.1) reduces to the following general quasi-variational-like inclusion problem introduced by Ding and Lou [6].

$$\begin{cases} \text{Find } \ x \in H, \ \text{ such that } \ g(x) \in \text{dom}(\partial_\eta \varphi(\cdot, x)) \\ \text{and} \\ \langle T(x) - A(x), \eta(y, g(x)) \rangle \geq \varphi(g(x), x) - \varphi(y, x), \ \text{ for all } \ y \in E. \end{cases} \tag{3.5}$$

(5) If $E = H$ is a Hilbert space, $\eta(y,x) = y - x$ for all $y, x \in H$, and $f(x) \equiv 0, N(u,v,w) = f(u) - P(v)$ for all $u, v, w \in H$, where $f, P : H \to H$ are single-valued mappings and $\varphi(x,y) = \varphi(x)$ for all $x, y \in H$, then problem (3.1) reduces to the following problem:

$$\begin{cases} \text{Find } \ x \in H, u \in T(x), v \in A(x) \ \text{ such that} \\ \text{and} \\ \langle f(u) - P(v), (y - g(x)) \rangle \geq \varphi(g(x), x) - \varphi(y, x), \ \text{ for all } \ y \in E. \end{cases} \tag{3.6}$$

Problem (3.6) is called the set-valued nonlinear generalized variational inclusion problem which was introduced by Huang [10]. From the above special cases, it is clear that for a suitable choice of the maps involved in the formulation of (GIVLIP), we can drive many known variational inclusions considered and studied in the literature.

In connection with (GIVLIP), we consider the following $J_\rho^{M(\cdot,x)}$ -proximal operator equation problem ($J_\rho^{M(\cdot,x)}$-POEP):

$$(J_\rho^{M(\cdot,x)}\text{−POEP}) \quad \begin{cases} \text{Find } \ z \in E^*, u \in T(x), v \in A(x), w \in B(x) \ \text{ such that} \\ N(u,v,w) - f(x) + \rho^{-1} R_\rho^{M(\cdot,x)}(z) = 0, \end{cases} \tag{3.7}$$

where $\rho > 0$ is a constant,$J_\rho^{M(\cdot,x)} = (P + \rho M(\cdot,x))^{-1}$, $R_\rho^{M(\cdot,x)}(z) = [I - PJ_\rho^{M(\cdot,x)}](z)$, where $P[J_\rho^{M(\cdot,x)}(z)] = [P(J_\rho^{M(\cdot,x)})](z)$ and I is the identity mapping in E^*. Equation (3.7) is called $J_\rho^{M(\cdot,x)}$-proximal operator equation.

Assume that $\mathrm{dom}(P) \cap g(E) \neq \emptyset$. The following lemma which will be used in the sequel, is an immediate consequence of the definition of $J_\rho^{M(\cdot,x)}$.

Lemma 3.1. *Let (x,u,v,w), where $x \in E, u \in T(x), v \in A(x)$ and $w \in B(x)$, is a solution of (GIVLIP) if and only if it is a solution of the following equation:*

$$g(x) = J_\rho^{M(\cdot,x)}\{P(g(x)) - \rho[N(u,v,w) - f(x)]\}. \tag{3.8}$$

Now, we show that the (GIVLIP) is equivalent to the ($J_\rho^{M(\cdot,x)}$-POEP).

Lemma 3.2. *The (GIVLIP) has a solution (x,u,v,w) with $x \in E, u \in T(x), v \in A(x)$ and $w \in B(x)$ if and only if ($J_\rho^{M(\cdot,x)}$-POEP) has a solution (z,x,u,v,w) with $z \in E^*$, $x \in E, u \in T(x), v \in A(x)$ and $w \in B(x)$, where*

$$g(x) = J_\rho^{M(\cdot,x)}(z) \tag{3.9}$$

and

$$z = P(g(x)) - \rho[N(u,v,w) - f(x)].$$

Proof. Let (x,u,v,w) be a solution of (GIVLIP). Then by Lemma 3.1, it is a solution of the following equation:

$$g(x) = J_\rho^{M(\cdot,x)}\{P(g(x)) - \rho[N(u,v,w) - f(x)]\},$$

using the fact $R_\rho^{M(\cdot,x)} = I - PJ_\rho^{M(\cdot,x)}$, and Eq. (3.8), we have

$$\begin{aligned}
&R_\rho^{M(\cdot,x)}\{P(g(x)) - \rho[N(u,v,w) - f(x)]\} \\
&= P(g(x)) - \rho[N(u,v,w) - f(x)] \\
&\quad - P[J_\rho^{M(\cdot,x)}]\{P(g(x)) - \rho[N(u,v,w) - f(x)]\} \\
&= P(g(x)) - \rho[N(u,v,w) - f(x)] - P(g(x)) \\
&= -\rho[N(u,v,w) - f(x)],
\end{aligned}$$

which implies that

$$N(u,v,w) - f(x) + \rho^{-1}R_\rho^{M(\cdot,x)}(z) = 0,$$

with $z = P(g(x)) - \rho[N(u,v,w) - f(x)]$, i.e. (z,u,v,w) is a solution of ($J_\rho^{M(\cdot,x)}$-POEP).

Conversely, let (z,x,u,v,w) be a solution of ($J_\rho^{M(\cdot,x)}$-POEP), then

$$\rho[N(u,v,w) - f(x)] = -R_\rho^{M(\cdot,x)}(z) = P[J_\rho^{M(\cdot,x)}(z)] - z. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned}
\rho[N(u,v,w) - f(x)] \quad = \quad & P[J_\rho^{M(\cdot,x)}\{P(g(x)) - \rho[N(u,v,w) - f(x)]\}] \\
& -P(g(x)) + \rho[N(u,v,w) - f(x)]
\end{aligned}$$

which implies that

$$P(g(x)) = P[J_\rho^{M(\cdot,x)}\{P(g(x)) - \rho[N(u,v,w) - f(x)]\}]$$

and thus

$$g(x) = J_\rho^{M(\cdot,x)}\{P(g(x)) - \rho[N(u,v,w) - f(x)]\},$$

i.e. (x,u,v,w) is a solution of (GIVLIP). □

4. An iterative algorithm and convergence analysis

First, we define the following concepts.

Definition 4.1. Let $\eta : E \times E \to E$ be a single-valued mapping and let $T, A, B : E \to CB(E^*)$ be three multi-valued mappings. A mapping $N : E^* \times E^* \times E^* \to E^*$ is said to be (α,β,γ)-Lipschitz continuous, if $\exists\ \alpha,\beta,\gamma > 0$ such that

$$\|N(x_1,y_1,z_1) - N(x_2,y_2,z_2)\| \le \alpha\|x_1 - x_2\| + \beta\|y_1 - y_2\| + \gamma\|z_1 - z_2\|,$$

$\forall x_1, x_2, y_1, y_2, z_1, z_2 \in E$.

Definition 4.2. A multi-valued mapping $T : E \to CB(E^*)$ is said to be α-H-Lipschitz continuous, if $\exists\ \alpha > 0$ such that

$$H(Tx,Ty) \le \alpha\|x - y\|, \quad \forall x, y \in E.$$

Lemma 4.1. ([13]) *Let E be a complete metric space, $T : E \to CB(E)$ be a set-valued mapping. Then for any given $\varepsilon > 0$ and any given $x, y \in E, u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \leq (1 + \varepsilon)H(Tx, Ty).$$

Lemma 4.2. ([9]) *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

for all $j(x + y) \in J(x + y)$.

Using Lemma 3.1 and Nadler's above results , we develop an iterative algorithm for finding the approximate solution of (GIVLIP) as follows.

Iterative Algorithm 4.1. For any $z_0 \in E^*, x_0 \in E, u_0 \in T(x_0), v_0 \in A(x_0)$ and $w_0 \in B(x_0)$, from (2.9), let

$$z_1 = P(g(x_0) - \rho[N(u_0, v_0, w_0) - f(x_0)].$$

Take $z_1 \in E^*, x_1 \in E$ such that $g(x_1) = J_\rho^{M(\cdot,x_1)}(z_1)$.

Since $u_0 \in T(x_0), v_0 \in A(x_0)$ and $w_0 \in B(x_0)$, by Nadlers Lemma 4.1 , there exists $u_1 \in T(x_1), v_1 \in A(x_1)$ and $w_1 \in B(x_1)$ such that

$$\|u_0 - u_1\| \leq (1 + 1)H(T(x_0), T(x_1)),$$
$$\|v_0 - v_1\| \leq (1 + 1)H(A(x_0), A(x_1)),$$
$$\|w_0 - w_1\| \leq (1 + 1)H(B(x_0), B(x_1)),$$

where H is Hausdorff metric on $CB(E)$. Let

$$z_2 = P(g(x_1) - \rho[N(u_1, v_1, w_1) - f(x_1)]$$

and take any $x_2 \in E$ such that

$$g(x_2) = J_\rho^{M(\cdot,x_2)}(z_2).$$

Continuing the above process inductively, we can obtain the following:

For any $z_0 \in E^*, x_0 \in E, u_0 \in T(x_0), v_0 \in A(x_0)$ and $w_0 \in B(x_0)$, compute the sequences $\{z_n\}, \{x_n\}, \{u_n\}, \{v_n\}$ and $\{w_n\}$ by iterative schemes such that

(i) $g(x_n) = J_\rho^{M(\cdot,x_n)}(z_n)$; (4.1)

(ii) $u_n \in T(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + \frac{1}{n+1})H(T(x_n), T(x_{n+1}))$, (4.2)

(iii) $v_n \in A(x_n), \quad \|v_n - v_{n+1}\| \leq (1 + \frac{1}{n+1})H(A(x_n), A(x_{n+1}))$, (4.3)

(iv) $w_n \in B(x_n), \quad \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n+1})H(B(x_n), B(x_{n+1}))$, (4.4)

(v) $z_{n+1} = P(g(x_n) - \rho[N(u_n, v_n, w_n) - f(x_n)], \quad n = 0, 1, 2, \cdots$ (4.5)

and $\rho > 0$ is a constant.

Theorem 4.1. *Let E be a reflexive Banach space. Let $T, A, B : E \to CB(E^*)$ be H-Lipschitz continuous mappings with Lipschitz constants λ_T, λ_A and λ_B, respectively. Let $N : E^* \times E^* \times E^* \to E^*$ be (α, β, γ)-Lipschitz continuous mapping. Suppose that $P : X \to X^*$ is Lipschitz continuous with Lipschitz* *constant λ_P and strongly $\eta-$monotone operator with constants $r > 0$. Let $f : E \to E^*$ be a λ_f-Lipschitz continuous mapping , $\eta : E \times E \to E$ be τ-Lipschitz continuous mapping, $M : E\times E \to E^*$ be such that for each fixed $x \in E, M(\cdot, x)$ be P-η -monotone mapping, and $g : E \to E$ be (δ, σ)-relaxed cocoercive and λ_g-Lipschitz continuous mapping. Suppose that there exists $\lambda > 0$ such that, for each $x_1, x_2, z \in E$*

$$\|J_\rho^{M(\cdot,x_1)}(z) - J_\rho^{M(\cdot,x_2)}(z)\| \le \lambda\|x_1 - x_2\|, \tag{4.6}$$

and,for $\rho > 0$ the following condition is satisfied:

$$\begin{cases} h = \sqrt{1 - 2(\delta{\lambda_g}^2 - \sigma + \lambda^2 + 1)} > r^{-1}\sqrt{2}\lambda_P\lambda_g\tau \\ \rho < \max\{\frac{rh-\sqrt{2}\lambda_P\lambda_g\tau}{mh+\sqrt{2}\tau(\alpha\lambda_T+\beta\lambda_A+\gamma\lambda_B+\lambda_f)}, \frac{r}{m}\} \\ 2\sigma - 2\delta{\lambda_g}^2 - 2\lambda^2 - 1 > 0, \end{cases} \tag{4.7}$$

then there exist $z \in E^*$, $x \in E, u \in T(x), v \in A(x)$ and $w \in B(x)$ satisfying ($J_\rho^{M(\cdot,x)}$-POEP) and the iterative sequences $\{z_n\}, \{x_n\}, \{u_n\}, \{v_n\}$ and $\{w_n\}$ generated by Algorithm 4.1 converge strongly to z, x, u, v and w, respectively.

Proof. From Algorithm 4.1, we have

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|P(g(x_n) - \rho[N(u_n, v_n, w_n) - f(x_n)] \\ &\qquad - P(g(x_{n-1}) - \rho[N(u_{n-1}, v_{n-1}, w_{n-1}) - f(x_{n-1})]\| \\ &\le \|P(g(x_n) - P(g(x_{n-1})\| + \rho\|N(u_n, v_n, w_n) \\ &\qquad - N(u_{n-1}, v_{n-1}, w_{n-1})\| + \rho\|f(x_n) - f(x_{n-1})\|. \end{aligned} \tag{4.8}$$

By the Lipschitz continuity of P and g, we have

$$\|P(g(x_n)) - P(g(x_{n-1})\| \le \lambda_P\|g(x_n) - g(x_{n-1})\| \le \lambda_P\lambda_g\|x_n - x_{n-1}\|. \tag{4.9}$$

By the (α, β, γ)-Lipschitz continuity of N and H-Lipschitz continuity of T, A and B,we have

$$\begin{aligned} &\|N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| \\ &\le \alpha\|u_n - u_{n-1}\| + \beta\|v_n - v_{n-1}\| + \gamma\|w_n - w_{n-1}\| \\ &\le \alpha(1 + \frac{1}{n+1})H(T(x_n), T(x_{n-1})) + \beta(1 + \frac{1}{n+1})H(A(x_n), A(x_{n-1})) \\ &\quad + \gamma(1 + \frac{1}{n+1})H(B(x_n), B(x_{n-1})) \\ &\le [\alpha\lambda_T(1 + \frac{1}{n+1}) + \beta\lambda_A(1 + \frac{1}{n+1}) + \gamma\lambda_B(1 + \frac{1}{n+1})]\|x_n - x_{n-1}\|. \end{aligned} \tag{4.10}$$

Combining (4.9)-(4.10) with (4.8), we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| \le \Big[&\lambda_P\lambda_g + \alpha\lambda_T\rho(1 + \frac{1}{n+1}) + \beta\lambda_A\rho(1 + \frac{1}{n+1}) \\ &+ \gamma\lambda_B\rho(1 + \frac{1}{n+1}) + \rho\lambda_f\Big] \times \|x_n - x_{n-1}\|. \end{aligned} \tag{4.11}$$

By using Theorem 2.1 and (δ,σ)-relaxed cocoercive of g, we have

$$\begin{aligned}
&\|x_n - x_{n-1}\|^2 \\
&= \|J_\rho^{M(\cdot,x_n)}(z_n) - J_\rho^{M(\cdot,x_{n-1})}(z_{n-1}) - [g(x_n) - x_n - (g(x_{n-1}) - x_{n-1})]\|^2 \\
&\le \|J_\rho^{M(\cdot,x_n)}(z_n) - J_\rho^{M(\cdot,x_{n-1})}(z_{n-1})\|^2 \\
&\quad - 2\langle g(x_n) - x_n - (g(x_{n-1}) - x_{n-1}), j(x_n - x_{n-1})\rangle \\
&\le 2\|J_\rho^{M(\cdot,x_n)}(z_n) - J_\rho^{M(\cdot,x_{n-1})}(z_n)\|^2 \\
&\quad + 2\|J_\rho^{M(\cdot,x_{n-1})}(z_n) - J_\rho^{M(\cdot,x_{n-1})}(z_{n-1})\|^2 \\
&\quad - 2\langle g(x_n) - x_n - (g(x_{n-1}) - x_{n-1}), j(x_n - x_{n-1})\rangle \\
&\le 2\left(\frac{\tau}{r - m\rho}\right)^2 \|z_n - z_{n-1}\|^2 + 2(\delta{\lambda_g}^2 - \sigma + \lambda^2 + 1)\|x_n - x_{n-1}\|^2,
\end{aligned}$$

which implies that

$$\|x_n - x_{n-1}\|^2 \le \frac{2}{1 - 2(\delta{\lambda_g}^2 - \sigma + \lambda^2 + 1)} \left(\frac{\tau}{r - m\rho}\right)^2 \|z_n - z_{n-1}\|^2, \quad (4.12)$$

using (4.11), (4.12) becomes

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\le \frac{\sqrt{2}\tau\left[\lambda_P\lambda_g + \alpha\lambda_T\rho(1+\frac{1}{n+1}) + \beta\lambda_A\rho(1+\frac{1}{n+1}) + \gamma\lambda_B\rho(1+\frac{1}{n+1}) + \rho\lambda_f\right]}{(r - m\rho)\sqrt{1 - 2(\delta{\lambda_g}^2 - \sigma + \lambda^2 + 1)}} \\
&\quad \times \|z_n - z_{n-1}\|,
\end{aligned}$$

i.e.,

$$\|z_{n+1} - z_n\| \le \theta_n \|z_n - z_{n-1}\|,$$

where

$$\theta_n = \frac{\sqrt{2}\tau\left[\lambda_P\lambda_g + \alpha\lambda_T\rho(1+\frac{1}{n+1}) + \beta\lambda_A\rho(1+\frac{1}{n+1}) + \gamma\lambda_B\rho(1+\frac{1}{n+1}) + \rho\lambda_f\right]}{(r - m\rho)\sqrt{1 - 2(\delta{\lambda_g}^2 - \sigma + \lambda^2 + 1)}}.$$

Letting $\theta = \frac{\sqrt{2}\tau[\lambda_P\lambda_g + \rho(\alpha\lambda_T + \beta\lambda_A + \gamma\lambda_B + \lambda_f)]}{(r-m\rho)\sqrt{1-2(\delta{\lambda_g}^2 - \sigma + \lambda^2 + 1)}}$, it follows that $\theta_n \to \theta$ as $n \to \infty$. From (4.7), we have $\theta < 1$, and consequently $\{z_n\}$ is a Cauchy sequence in E^*. Since E^* is a Banach space, there exists $z \in E^*$ such that $z_n \to z$ as $n \to \infty$.From (4.12), we know that the sequence $\{x_n\}$ is also a Cauchy sequence in E. Therefore, there exists $x \in E$ such that $x_n \to x$ as $n \to \infty$. Since the mappings T, A and B are H-Lipschitz continuous, it follows from (4.2)-(4.4) that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences, we can assume that $u_n \to u$, $v_n \to v$ and $w_n \to w$. Since P, g, N, f are continuous and by (v) of

Algorithm 4.1, it follows that

$$\begin{aligned} z_{n+1} &= P(g(x_n) - \rho[N(u_n, v_n, w_n) - f(x_n)] \\ &\to z = P(g(x) - \rho[N(u, v, w) - f(x)], \qquad (n \to \infty) \end{aligned} \tag{4.13}$$

$$J_\rho^{M(\cdot,x)}(z_n) = g(x_n) \to g(x) = J_\rho^{M(\cdot,x)}(z) \qquad (n \to \infty). \tag{4.14}$$

By (4.13), (4.14), and Lemma 3.2, we have

$$N(u, v, w) - f(x) + \rho^{-1}[I - P(J_\rho^{M(\cdot,x)}(z))] = 0.$$

Finally, we prove that $u \in T(x), v \in A(x)$ and $w \in B(x)$. In fact, since $u_n \in T(x_n)$ and

$$\begin{aligned} d(u_n, T(x)) &\le \max\left\{ d(u_n, T(x)), \sup_{q_1 \in T(x)} d(T(x_n), q_1) \right\} \\ &\le \max\left\{ \sup_{q_2 \in T(x_n)} d(q_2, T(x)), \sup_{q_1 \in T(x)} d(T(x_n), q_1) \right\} \\ &= H(T(x_n), T(x)). \end{aligned}$$

We have

$$\begin{aligned} d(u, T(x)) &\le \|u - u_n\| + d(u_n, T(x)) \\ &\le \|u - u_n\| + H(T(x_n), T(x)) \\ &\le \|u - u_n\| + \lambda_T \|x_n - x\| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which implies that $d(u, T(x)) = 0$. Since $T(x) \in CB(E^*)$, it follows that $u \in T(x)$. Similarly, we can prove that $v \in A(x)$ and $w \in B(x)$. By Lemma 3.2, the required result follows. □

Remark 4.2. The Theorem 4.1 extends and improves some results from [3], [6], [10], [9], [14] in several aspects.

Acknowledgments

The authors thank the referees for their valuable suggestions which helped to improve this manuscript.

References

[1] R. P. Agarwal, Y. J. Cho and N. J. Huang, *Sensitivity analysis for strongly nonlinear quasi-variational inclusions*, Appl. Math. Lett. **13(6)** (2000), 19–24.

[2] R. . Agarwal, N. J. Huang and Y. J. Cho, *Generalized nonlinear mixed implicit quasi-variational inclusions with setvalued mappings*, J. Inequal. Appl. **7(6)** (2002), 807–828.

[3] R. Ahmad, A. H. Siddiqi and Z. Khan, *Proximal point algorithm for generalized multivalued nonlinear quasivariational- like inclusions in Banach spaces*, Appl. Math. Comput. **163** (2005), 295–308.

[4] S. S. Chang, Y. J. Cho and H. Y. Zhou, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces*, Nova Sci. New York, 2002.

[5] J. Y. Chen, N. C. Wong and J. C. Yao, *Algorithm for generalized co-complementarity problems in Banach spaces*, Comput. Math. Appl. **43(1)** (2002), 49–54.

[6] X. P. Ding and C. L. Lou, *Perturbed proximal point algorithms for general quasi-variational-like inclusions*, J. Comput. Appl. Math. **210** (2000), 153–165.

[7] J. Lou, X. F. He and Z. He, *Iterative methods for solving a system of variational inclusions involving H-η-monotone operators in Banach spaces*, Computers and Mathematics with Applications, Computers and Mathematics with Applications **55** (2008), 1832–1841.

[8] X. F. He, J. Lou and Z. He, *Iterative methods for solving variational inclusions in Banach spaces*, Journal of Computational and Applied Mathematics **203(1)** (2007), 80–86.

[9] R. Ahmad and A. H. Siddiqi, *Mixed variational-like inclusions and J^η-proximal operator equations in Banach spaces*, J. Math. Anal. Appl. **327** (2007), 515–524.

[10] N. J. Huang, *Generlaized nonlinear variational inclusions with non-compact valued mappings*, Appl. Math. Lett. **9(3)** (1996), 25–29.

[11] Y. P. Fang and N. J.Huang, *H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces*, Appl. Math. Lett. **17** (2004), 647–653.

[12] K. R. Kazmi and F. A. Khan, *Sensitivity analysis for parametric generalized implicit quasi-variational-like inclusions involving P-η-accretive mappings*, J. Math. Anal. Appl. **337** (2008), 1198–1210.

[13] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[14] H. Y. Lan, *(A,η)-Accretive mappings and set-valued variational inclusions with relaxed cocoercive mappings in Banach spaces*, Appl. Math. Lett. **20** (2007), 571–577.

Xin-feng He
College of Mathematics and Computer, Hebei University, Baoding 071002, China
E-mail address: hxf@mail.hbu.cn

Jian Lou
Computers Center, Hebei University, Baoding 071002, China
E-mail address: loujianww@163.com

Zhen He
College of Mathematics and Computer, Hebei University, Baoding 071002, China
E-mail address: zhen_he@163.com