East Asian Mathematical Journal
Vol. 28 (2012), No. 1, pp. 25–36

ON PROJECTIVELY FLAT FINSLER SPACE WITH AN APPROXIMATE INFINITE SERIES (α, β)-METRIC

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Abstract. We introduced a Finsler space F^n with an approximate infinite series (α, β)-metric $L(\alpha, \beta) = \beta \sum_{k=0}^{r} \left(\frac{\alpha}{\beta}\right)^k$, where $\alpha < \beta$ and investigated it with respect to Berwald space ([12]) and Douglas space ([13]). The present paper is devoted to finding the condition that is projectively flat on a Finsler space F^n with an approximate infinite series (α, β)-metric above.

1. Introduction

A Finsler metric function L in a differentiable manifold M^n is called an (α, β)-metric, if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = (a_{ij}y^i y^j)^{1/2}$ and a non-vanishing 1-form $\beta = b_i y^i$ on M^n. An infinite sereis (α, β)-metric $L(\alpha, \beta) = \beta^2/(\beta - \alpha)$ is expressed as an infinite series form, where $\alpha < \beta$. We introduced an approximate infinite series (α, β)-metric $L(\alpha, \beta) = \beta \sum_{k=0}^{r} \left(\frac{\alpha}{\beta}\right)^k$ as the r-th finite series (α, β)-metric form and investigated it in [12] and [13].

A change $L \longrightarrow \overline{L}$ of a Finsler metric on a same underlying manifold M^n is called *projective*, if any geodesic in (M^n, L) remains to be a geodesic in $(M^n, \overline{L})$ and vice versa. A Finsler space is called *projective flat* if it is projective to a locally Minkowski space. The condition for a Finsler space with (α, β)-metric to be projectively flat was studied by M. Matsumoto [7]. Aikou, Hashiguchi and Yamauchi [2] give interesting results on the projective flatness of Matsumoto space.

The purpose of the present paper is to find condition that is projectively flat on a Finsler space with an approximate infinite series (α, β)-metric.

Received February 14, 2011; Accepted November 17, 2011.
2000 *Mathematics Subject Classification.* 53B40.
Key words and phrases. Finsler space, projectively flat, infinite series (α, β)-metric, approximate infinite serie (α, β)-metric, homogeneous polynomials in (y^i) of degree r.
This paper was supported by Kyungsung University Research Grant in 2011.

2. Preliminaries

In a Finsler space (M^n, L), the metric

$$L(\alpha,\beta) = \beta\left\{\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k\right\} \tag{2.1}$$

is called an *approximate infinite series* (α,β)*-metric*. The infinite series (α,β)-metric is expressed as

$$\lim_{r\to\infty}\beta\left\{\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k\right\} = \frac{\beta^2}{\beta-\alpha}$$

for $\alpha < \beta$ in (2.1). If $r = 0$, then $L = \beta$ is a non-vanishing 1-form. If $r = 1$, then $L = \alpha + \beta$ is a Randers metric. The condition for a Randers space to be projectively flat was given by Hashiguchi-Ichijō [4], and M. Matsumoto [7]. Therefore in this paper, we suppose that $r > 1$.

Let $\gamma_j{}^i{}_k$ be the Christoffel symbols with respect to α and denote by (;) the covariant differentiation with respect to $\gamma_j{}^i{}_k$. From the differential 1-form $\beta(x,y) = b_i(x)y^i$ we define

$$2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i} = (\partial_j b_i - \partial_i b_j),$$
$$s^i_j = a^{ir}s_{rj}, \quad b^i = a^{ir}b_r, \quad b^2 = a^{rs}b_r b_s.$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity and the subscription 0 means contraction by y^i, for instance, $\mu_0 = \mu_i y^i$. In the following we denote $L_\alpha = \partial_\alpha L$, $L_\beta = \partial_\beta L$, $L_{\alpha\alpha} = \partial_\alpha\partial_\alpha L$.

Now the following Matsumoto's theorem [7] is well-known.

Theorem 2.1. *A Finsler space (M^n, L) with an (α,β)-metric $L(\alpha,\beta)$ is projectively flat if and only if for any point of space M^n there exist local coordinate neighborhoods containing the point such that $\gamma_j{}^i{}_k$ satisfies:*

$$\begin{aligned}&(\gamma_0{}^i{}_0 - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0\\ &+ (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0,\end{aligned} \tag{2.2}$$

where C is given by

$$C + (\alpha^2 L_\beta/\beta L_\alpha)s_0 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0. \tag{2.3}$$

The equation (2.3) is rewritten in the form

$$\begin{aligned}&(C + \alpha r_{00}/2\beta)\{1 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)\}\\ &-(\alpha/2\beta)\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\} = 0,\end{aligned} \tag{2.4}$$

that is,

$$C + \alpha r_{00}/2\beta = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha L_\beta s_0)}{2\{\beta^2 L_\alpha + \alpha L_{\alpha\alpha}(\alpha^2 b^2 - \beta^2)\}}.$$

Therefore (2.2) leads us to

$$\begin{aligned} \{L_\alpha(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)+2\alpha^3L_\beta s^i{}_0\}\{\beta^2L_\alpha+\alpha L_{\alpha\alpha}(\alpha^2b^2-\beta^2)\} \\ +\alpha^3L_{\alpha\alpha}(r_{00}L_\alpha-2\alpha L_\beta s_0)(\alpha^2b^i-\beta y^i)=0. \end{aligned} \tag{2.5}$$

We shall state the following lemma for later:

Lemma 2.2. ([3]) *If $\alpha^2\equiv 0$ (mod β), that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta=d_i(x)y^i$ satisfying $\alpha^2=\beta\delta$ and $d_ib^i=2$.*

3. Projectively flat space

In the present section, we find the condition that a Finsler space F^n with the r-th approximate infinite series (α,β)-metric (2.1) be projectively flat. In the n-dimensional Finsler space F^n with the approximate infinite series (α,β)-metric (2.1), we have

$$\begin{aligned} L_\alpha=\sum_{k=0}^{r}k\left(\frac{\alpha}{\beta}\right)^{k-1},\quad L_\beta=-\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k}, \\ L_{\alpha\alpha}=\frac{1}{\beta}\sum_{k=0}^{r}k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2}. \end{aligned} \tag{3.1}$$

Here, by means of (2.5) and (3.1) we have

$$\begin{aligned} &\left\{\sum_{k=0}^{r}k\left(\frac{\alpha}{\beta}\right)^{k-1}(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)-2\alpha^3s^i{}_0\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k}\right\} \\ &\times\left\{\beta^2\sum_{k=0}^{r}k\left(\frac{\alpha}{\beta}\right)^{k-1}+(\alpha^2b^2-\beta^2)\sum_{k=0}^{r}k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-1}\right\} \\ &+\alpha^2\sum_{k=0}^{r}k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-1}\left\{r_{00}\sum_{k=0}^{r}k\left(\frac{\alpha}{\beta}\right)^{k-1}\right. \\ &\left.+2\alpha s_0\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k}\right\}\times(\alpha^2b^i-\beta y^i)=0. \end{aligned} \tag{3.2}$$

We shall divide our consideration in two cases of which r is even or odd.

(1) Case of $r=2h$, where h is a positive integer.

When $r = 2h$, we have

$$\begin{aligned}
\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1} &= \frac{\beta}{\beta^{2h}}\sum_{k=0}^{2h} k\alpha^{k-1}\beta^{2h-k},\\
\sum_{k=0}^{r} (k-1)\left(\frac{\alpha}{\beta}\right)^{k} &= \frac{1}{\beta^{2h}}\sum_{k=0}^{2h} (k-1)\alpha^{k}\beta^{2h-k},\\
\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-1} &= \frac{1}{\beta^{2h-1}}\sum_{k=0}^{2h} k(k-1)\alpha^{k-1}\beta^{2h-k}.
\end{aligned} \tag{3.3}$$

Separating the rational and irrational parts in y^i with respect to (3.3), we obtain

$$\begin{aligned}
\sum_{k=0}^{2h} k\alpha^{k-1}\beta^{2h-k} &= \sum_{k=0}^{h-1}(2k+1)\alpha^{2k}\beta^{2h-2k-1} + \alpha\sum_{k=1}^{h} 2k\alpha^{2k-2}\beta^{2h-2k}\\
&= M + \alpha K,\\
\sum_{k=0}^{2h}(k-1)\alpha^{k}\beta^{2h-k} &= \sum_{k=0}^{h}(2k-1)\alpha^{2k}\beta^{2h-2k}\\
&\quad + \alpha^3\sum_{k=1}^{h-1} 2k\alpha^{2k-2}\beta^{2h-2k-1}\\
&= L + \alpha^3 N,\\
\sum_{k=0}^{2h} k(k-1)\alpha^{k-1}\beta^{2h-k} &= \alpha^2\sum_{k=1}^{h-1}(2k+1)2k\alpha^{2k-2}\beta^{2h-2k-1}\\
&\quad + \alpha\sum_{k=1}^{h} 2k(2k-1)\alpha^{2k-2}\beta^{2h-2k}\\
&= \alpha^2 Q + \alpha P,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
K &= \sum_{k=1}^{h} 2k\alpha^{2k-2}\beta^{2h-2k}, & L &= \sum_{k=0}^{h}(2k-1)\alpha^{2k}\beta^{2h-2k},\\
M &= \sum_{k=0}^{h-1}(2k+1)\alpha^{2k}\beta^{2h-2k-1}, & N &= \sum_{k=1}^{h-1} 2k\alpha^{2k-2}\beta^{2h-2k-1},\\
P &= \sum_{k=1}^{h} 2k(2k-1)\alpha^{2k-2}\beta^{2h-2k}, & Q &= \sum_{k=1}^{h-1}(2k+1)2k\alpha^{2k-2}\beta^{2h-2k-1}.
\end{aligned}$$

Substituting (3.3) and (3.4) into (3.2), we have

$$(\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)\beta\big[\beta^2(M^2 + 2\alpha KM + \alpha^2K^2) + \alpha(\alpha^2b^2 - \beta^2)\{MP + \alpha(KP + MQ) + \alpha^2KQ\}\big] - 2\alpha^3s^i{}_0\{\beta^2(LM + \alpha KL + \alpha^3MN + \alpha^4KN) + \alpha(\alpha^2b^2 - \beta^2)(LP + \alpha LQ + \alpha^3NP + \alpha^4NQ\} + (\alpha^2b^i - \beta y^i)\alpha^2\big[\beta\alpha r_{00}\{MP + \alpha(KP + MQ) + \alpha^2KQ\} + 2\alpha^2s_0(LP + \alpha LQ + \alpha^3NP + \alpha^4NQ)\big] = 0.$$

The above is rewritten in the form

$$A + \alpha B = 0,$$

where

$$\begin{aligned} A = {} & (\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)\{\beta^3(M^2 + \alpha^2K^2) + \beta\alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP)\} \\ & - 2\alpha^4s^i{}_0\{\beta^2(KL + \alpha^2MN) + (\alpha^2b^2 - \beta^2)(LP + \alpha^4NQ)\} \\ & + \alpha^2(\alpha^2b^i - \beta y^i)\{\beta\alpha^2r_{00}(MQ + KP) + 2\alpha^2s_0(LP + \alpha^4NQ)\}, \\ B = {} & (\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)\{2\beta^3KM + \beta(\alpha^2b^2 - \beta^2)(MP + \alpha^2KQ)\} \\ & - 2\alpha^2s^i{}_0\{\beta^2(LM + \alpha^4KN) + \alpha^2(\alpha^2b^2 - \beta^2)(LQ + \alpha^2NP)\} \\ & + \alpha^2(\alpha^2b^i - \beta y^i)\{\beta r_{00}(MP + \alpha^2KQ) + 2\alpha^2s_0(LQ + \alpha^2NP)\}. \end{aligned}$$

Since A, B are rational parts and α is an irrational part in y^i, we have $A = 0$ and $B = 0$, that is,

$$\begin{aligned} & (\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)\{\beta^3(M^2 + \alpha^2K^2) + \beta\alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP)\} \\ & - 2\alpha^4s^i{}_0\{\beta^2(KL + \alpha^2MN) + (\alpha^2b^2 - \beta^2)(LP + \alpha^4NQ)\} \\ & + \alpha^2(\alpha^2b^i - \beta y^i)\{\beta\alpha^2r_{00}(MQ + KP) + 2\alpha^2s_0(LP + \alpha^4NQ)\} = 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} & (\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)\{2\beta^3KM + \beta(\alpha^2b^2 - \beta^2)(MP + \alpha^2KQ)\} \\ & - 2\alpha^2s^i{}_0\{\beta^2(LM + \alpha^4KN) + \alpha^2(\alpha^2b^2 - \beta^2)(LQ + \alpha^2NP)\} \\ & + \alpha^2(\alpha^2b^i - \beta y^i)\{\beta r_{00}(MP + \alpha^2KQ) + 2\alpha^2s_0(LQ + \alpha^2NP)\} = 0. \end{aligned} \tag{3.6}$$

Eliminating $(\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)$ from (3.5) and (3.6), we have

$$\begin{aligned} & 2s^i{}_0\big[\alpha^2\{2\beta^2KM + (\alpha^2b^2 - \beta^2)(MP + \alpha^2KQ)\} \\ & \times\{\beta^2(KL + \alpha^2MN) + (\alpha^2b^2 - \beta^2)(LP + \alpha^4NQ)\} \\ & - \{\beta^2(M^2 + \alpha^2K^2) + \alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP)\} \\ & \times\{\beta^2(LM + \alpha^4KN) + \alpha^2(\alpha^2b^2 - \beta^2)(LQ + \alpha^2NP)\}\big] \\ & - (\alpha^2b^i - \beta y^i)\big[\{2\beta^2KM + (\alpha^2b^2 - \beta^2)(MP + \alpha^2KQ)\} \\ & \times\{\beta\alpha^2r_{00}(MQ + KP) + 2\alpha^2s_0(LP + \alpha^4NQ)\} \\ & - \{\beta^2(M^2 + \alpha^2K^2) + \alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP)\} \\ & \times\{\beta r_{00}(MP + \alpha^2KQ) + 2\alpha^2s_0(LQ + \alpha^2NP)\}\big] = 0. \end{aligned} \tag{3.7}$$

Transvecting (3.7) by b_i, we get

$$\begin{aligned}&2s_0\big[\alpha^2\{2\beta^2KM+(\alpha^2b^2-\beta^2)(MP+\alpha^2KQ)\}(KL+\alpha^2MN)\\&-\{\beta^2(M^2+\alpha^2K^2)+\alpha^2(\alpha^2b^2-\beta^2)(MQ+KP)\}(LM+\alpha^4KN)\big]\\&-\beta r_{00}(\alpha^2b^2-\beta^2)\{2\alpha^2KM(MQ+KP)-(M^2+\alpha^2K^2)(MP+\alpha^2KQ)\}\\&=0.\end{aligned}\tag{3.8}$$

Thus the term of (3.8) which seemingly does not contain α^2 is $2(\beta s_0-r_{00})\beta^{8h-2}$. Therefore there exists $hp(8h-2)$: V_{8h-2} such that

$$2(\beta s_0-r_{00})\beta^{8h-2}=\alpha^2V_{8h-2}.\tag{3.9}$$

We suppose that $\alpha^2\not\equiv 0$ (mod. β) due to Lemma 2.2. From (3.9) there exists a function $k=k(x)$ satisfying $V_{8h-2}=k\beta^{8h-2}$, which leads to

$$2(\beta s_0-r_{00})=k\alpha^2.\tag{3.10}$$

Substituting (3.10) into (3.8), we have

$$\begin{aligned}&k(x)\big[\alpha^2\{2\beta^2KM+(\alpha^2b^2-\beta^2)(MP+\alpha^2KQ)\}(KL+\alpha^2MN)\\&-\{\beta^2(M^2+\alpha^2K^2)+\alpha^2(\alpha^2b^2-\beta^2)(MQ+KP)\}(LM+\alpha^4KN)\big]\\&+r_{00}\Big\{2\big[\{2\beta^2KM+(\alpha^2b^2-\beta^2)(MP+\alpha^2KQ)\}(KL+\alpha^2MN)\\&-\beta^2(K^2LM+\alpha^2KM^2N+\alpha^4K^3N)-(\alpha^2b^2-\beta^2)(MQ+KP)\\&(LM+\alpha^4KN)\big]-2\beta^2(\alpha^2b^2-\beta^2)KM(MQ+KP)+\beta^2b^2\big[M^3P\\&+\alpha^2\{KM(MQ+KP)+\alpha^2K^3Q\}\big]-\beta^4\{KM(MQ+KP)\\&+\alpha^2K^3Q\}-\beta^2M^3(2L_1+\beta^2P_1)\Big\}=0,\end{aligned}\tag{3.11}$$

where

$$L_1=\sum_{k=1}^{h}(2k-1)\alpha^{2k-2}\beta^{2h-2k},$$
$$P_1=\sum_{k=2}^{h}2k(2k-1)\alpha^{2k-4}\beta^{2h-2k}.$$

Here the term of (3.11) which seemingly does not contain α^2 is $\beta^{8h-3}\{k\beta^2+2(b^2-7)r_{00}\}$. Thus there exists $hp(8h-3)$: V_{8h-3} such that

$$\beta^{8h-3}\{k\beta^2+2(b^2-7)r_{00}\}=\alpha^2V_{8h-3}.\tag{3.12}$$

From (3.12) there exists a function $h=h(x)$ satisfying $V_{8h-3}=h(x)\beta^{8h-3}$, and hence

$$k(x)\beta^2+2(b^2-7)r_{00}=h(x)\alpha^2.\tag{3.13}$$

Since $\alpha^2 \not\equiv 0$ (mod. β), we obtain $k(x) = 0$, which leads to

$$r_{00} = \frac{h(x)}{2(b^2-7)}\alpha^2, \tag{3.14}$$

where we assume $b^2 \neq 7$.

Substituting $k(x) = 0$ and (3.14) into (3.10), we have

$$\beta s_0 = \frac{h(x)}{2(b^2-7)}\alpha^2,$$

which leads to $s_0 = 0$ by virtue of $h(x) = 0$, and hence $r_{00} = 0$ from (3.14), that is, $s_j = 0$ and $r_{ij} = 0$.

Substituting $s_0 = 0$ and $r_{00} = 0$ into (3.7), we have

$$\begin{aligned} & {s^i}_0\big[\alpha^2\{2\beta^2 KM + (\alpha^2 b^2 - \beta^2)(MP + \alpha^2 KQ)\} \\ & \times \{\beta^2(KL + \alpha^2 MN) + (\alpha^2 b^2 - \beta^2)(LP + \alpha^4 NQ)\} \\ & - \{\beta^2(M^2 + \alpha^2 K^2) + \alpha^2(\alpha^2 b^2 - \beta^2)(MQ + KP)\} \\ & \times \{\beta^2(LM + \alpha^4 KN) + \alpha^2(\alpha^2 b^2 - \beta^2)(LQ + \alpha^2 NP)\}\big] = 0. \end{aligned} \tag{3.15}$$

Hence the term of (3.15) which seemingly does not contain α^2 is $\beta^{8h+1}{s^i}_0$. Then there exists $hp(8h) : V_{8h}$ such that

$$\beta^{8h+1}{s^i}_0 = \alpha^2 V_{8h}. \tag{3.16}$$

From $\alpha^2 \not\equiv 0$ (mod. β), there exists from (3.16) a function $\rho = \rho(x)$ satisfying $V_{8h} = \rho(x)\beta^{8h}$, and hence

$$\beta {s^i}_0 = \rho(x)\alpha^2,$$

which leads to ${s^i}_0 = 0$ by virtue of $\rho(x) = 0$, that is, $s_{ij} = 0$.

Consequently we have $r_{ij} = 0$ and $s_{ij} = 0$, that is, $b_{i:j} = 0$ is obtained.

Next, substituting $s_0 = 0$, $r_{00} = 0$ and ${s^i}_0 = 0$ into (3.5) we have

$$(\alpha^2{\gamma_0}^i{}_0 - \gamma_{000}y^i)\{\beta^2(M^2 + \alpha^2 K^2) + \alpha^2(a^2 b^2 - \beta^2)(MQ + KP)\} = 0. \tag{3.17}$$

Thus the term of (3.17) which seemingly does not contain α^2 is $-\gamma_{000}y^i\beta^{4h}$. Therefore there exists $hp(1) : \mu_0 = \mu_i(x)y^i$ such that

$$\gamma_{000} = \mu_0\alpha^2. \tag{3.18}$$

Substituting (3.18) into (3.17), we have

$$({\gamma_0}^i{}_0 - \mu_0 y^i)D = 0,$$

where

$$D = \beta^2(M^2 + \alpha^2 K^2) + \alpha^2(\alpha^2 b^2 - \beta^2)(MQ + KP). \tag{3.19}$$

From (3.19) if $D = 0$, then the term of $D = 0$ which seemingly does not contain α^2 is β^{4h}. In this case, there exists $hp(4h-2) : V_{4h-2}$ such that $\beta^{4h} = \alpha^2 V_{4h-2}$. Hence we have $V_{4h-2} = 0$, which leads to a contradiction, that is, $D \neq 0$. Therefore we obtain ${\gamma_0}^i{}_0 = \mu_0 y^i$, that is,

$$2{\gamma_j}^i{}_k = \mu_j\delta^i_k + \mu_k\delta^i_j, \tag{3.20}$$

which shows that the associated Riemannian space is projectively flat.

Conversely it is easy to see that (3.2) is a consequence of $b_{i;j} = 0$ and (3.20).

(2) Case of $r = 2h+1$, where h is a positive integer.
When $r = 2h+1$, we have

$$\begin{aligned}
\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1} &= \frac{1}{\beta^{2h}}\sum_{k=0}^{2h+1} k\alpha^{k-1}\beta^{2h-k+1},\\
\sum_{k=0}^{r} (k-1)\left(\frac{\alpha}{\beta}\right)^{k} &= \frac{1}{\beta^{2h+1}}\sum_{k=0}^{2h+1} (k-1)\alpha^{k}\beta^{2h-k+1},\\
\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-1} &= \frac{1}{\beta^{2h}}\sum_{k=0}^{2h+1} k(k-1)\alpha^{k-1}\beta^{2h-k+1}.
\end{aligned} \tag{3.21}$$

Separating the rational and irrational parts in y^i with respect to (3.21), we have

$$\begin{aligned}
\sum_{k=0}^{2h+1} k\alpha^{k-1}\beta^{2h-k+1} &= \sum_{k=0}^{h}(2k+1)\alpha^{2k}\beta^{2h-2k}\\
&\quad + \alpha\sum_{k=1}^{h} 2k\alpha^{2k-2}\beta^{2h-2k+1}\\
&= O + \alpha\beta K,\\
\sum_{k=0}^{2h+1} (k-1)\alpha^{k}\beta^{2h-k+1} &= \sum_{k=0}^{h}(2k-1)\alpha^{2k}\beta^{2h-2k+1}\\
&\quad + \alpha^3\sum_{k=1}^{h} 2k\alpha^{2k-2}\beta^{2h-2k}\\
&= \beta L + \alpha^3 K,\\
\sum_{k=0}^{2h+1} k(k-1)\alpha^{k-1}\beta^{2h-k+1} &= \alpha^2\sum_{k=1}^{h}(2k+1)2k\alpha^{2k-2}\beta^{2h-2k}\\
&\quad + \alpha\Big(\sum_{k=1}^{h} 2k(2k-1)\alpha^{2k-2}\beta^{2h-2k+1}\Big)\\
&= \alpha^2 R + \alpha\beta P,
\end{aligned} \tag{3.22}$$

where

$$O = \sum_{k=0}^{h}(2k+1)\alpha^{2k}\beta^{2h-2k},$$

$$R = \sum_{k=1}^{h}(2k+1)2k\alpha^{2k-2}\beta^{2h-2k}.$$

Substituting (3.21) and (3.22) into (3.2), we have

$$\begin{aligned}&\{\beta(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)(O+\alpha\beta K)-2\alpha^3 s^i{}_0(BL+\alpha^3K)\}\\&\times\{\beta^2(O+\alpha\beta K)+\alpha(\alpha^2b^2-\beta^2)(\alpha R+\beta P)\}\\&+\alpha^3(\alpha R+\beta P)\{\beta r_{00}(O+\alpha\beta K)+2\alpha s_0(\beta L+\alpha^3K)\}\\&\times(\alpha^2b^i-\beta y^i)=0.\end{aligned}\tag{3.23}$$

Separating the rational and irrational parts in y^i, we obtain

$$A'+\alpha B'=0,$$

that is, $A'=0$ and $B'=0$ because α is an irrational part in y^i,
where

$$\begin{aligned}A'=\ &\beta(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{\beta^2(O^2+\alpha^2\beta^2K^2)+\alpha^2(\alpha^2b^2-\beta^2)(OR+\beta^2KP)\}\\&-2\alpha^4 s^i{}_0\{\beta^2(\beta^2LK+\alpha^2KO)+(\alpha^2b^2-\beta^2)(\beta^2LP+\alpha^4KR)\}\\&+\alpha^4\{\beta r_{00}(OR+\beta^2KP)+2s_0(\alpha^4KR+\beta^2LP)\}(\alpha^2b^i-\beta y^i)=0,\end{aligned}\tag{3.24}$$

$$\begin{aligned}B'=\ &\beta(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{2\beta^2KO+(\alpha^2b^2-\beta^2)(OP+\alpha^2KR)\}\\&-2\alpha^2 s^i{}_0\{\beta^2(LO+\alpha^4K^2)+\alpha^2(\alpha^2b^2-\beta^2)(LR+\alpha^2KP)\}\\&+\alpha^2\{\beta r_{00}(\alpha^2KR+OP)+2\alpha^2s_0(LR+\alpha^2KP)\}(\alpha^2b^i-\beta y^i)=0.\end{aligned}\tag{3.25}$$

From (3.24) we have $-\gamma_{000}y^i\beta^{4h+3}=\alpha^2W_{4h+5}$, where W_{4h+5} is a $hp(4h+5)$. Therefore there exists $hp(1):v_0$ satisfying

$$\gamma_{000}=v_0\alpha^2.\tag{3.26}$$

Next, eliminating $(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)$ from (3.24) and (3.25), we have

$$\begin{aligned}&2s^i{}_0\big[\alpha^2\{\beta^2(\beta^2LK+\alpha^2KO)+(\alpha^2b^2-\beta^2)(\beta^2LP+\alpha^4KR)\}\\&\times\{2\beta^2KO+(\alpha^2b^2-\beta^2)(OP+\alpha^2KR)\}\\&-\{\beta^2(LO+\alpha^4K^2)+\alpha^2(\alpha^2b^2-\beta^2)(LR+\alpha^2KP)\}\\&\times\{\beta^2(O^2+\alpha^2\beta^2K^2)+\alpha^2(\alpha^2b^2-\beta^2)(OR+\beta^2KP)\}\big]\\&-(\alpha^2b^i-\beta y^i)\big[\alpha^2\{\beta r_{00}(OR+\beta^2KP)+2s_0(\alpha^4KR+\beta^2LP)\}\\&\times\{2\beta^2KO+(\alpha^2b^2-\beta^2)(OP+\alpha^2KR)\}\\&-\{\beta r_{00}(\alpha^2KR+OP)+2\alpha^2s_0(LR+\alpha^2KP)\}\\&\times\{\beta^2(O^2+\alpha^2\beta^2K^2)+\alpha^2(\alpha^2b^2-\beta^2)(OR+\beta^2KP)\}\big]=0.\end{aligned}\tag{3.27}$$

Transvecting (3.27) by b_i, we have

$$\begin{aligned}&2s_0\big[\alpha^2(\beta^2LK+\alpha^2KO)\{2\beta^2KO+(\alpha^2b^2-\beta^2)(OP+\alpha^2KR)\}\\&-(LO+\alpha^4K^2)\{\beta^2(O^2+\alpha^2\beta^2K^2)+\alpha^2(\alpha^2b^2-\beta^2)(OR+\beta^2KP)\}\big]\\&-\beta(\alpha^2b^2-\beta^2)r_{00}\{2\alpha^2KO(OR+\beta^2KP)-(\alpha^2KR+OP)(O^2\\&+\alpha^2\beta^2K^2)\}=0.\end{aligned}\tag{3.28}$$

The terms of (3.28) which does not contain α^2 are found in $2\beta^{8h+1}(\beta s_0 - r_{00})$. Thus there exists $hp(8h+1) : W_{8h+1}$ such that

$$2\beta^{8h+1}(\beta s_0 - r_{00}) = \alpha^2 W_{8h+1}. \tag{3.29}$$

We suppose that $\alpha^2 \not\equiv 0 \pmod{\beta}$ owing to Lemma 2.2. Therefore there exists from (3.29) a function $f = f(x)$ satisfying $W_{8h+1} = f\beta^{8h+1}$, which leads to

$$2(\beta s_0 - r_{00}) = f(x)\alpha^2. \tag{3.30}$$

Substituting (3.30) into (3.28), we obtain

$$\begin{aligned}
&f(x)\alpha^2\big[\alpha^2(\beta^2 LK + \alpha^2 KO)\{2\beta^2 KO + (\alpha^2 b^2 - \beta^2)(OP + \alpha^2 KR)\}\\
&- (LO + \alpha^4 K^2)\{\beta^2(O^2 + \alpha^2\beta^2 K^2) + \alpha^2(\alpha^2 b^2 - \beta^2)(OR + \beta^2 KP)\}\big]\\
&+ r_{00}\big[2\alpha^2(\beta^2 LK + \alpha^2 KO)\{2\beta^2 KO + (\alpha^2 b^2 - \beta^2)(OP + \alpha^2 KR)\}\\
&- 2\alpha^2\beta^2(\alpha^2 K^2 O^2 + \beta^2 K^2 LO + \alpha^4\beta^2 K^4) - 2\alpha^2(\alpha^2 b^2 - \beta^2)(LO\\
&+ \alpha^4 K^2)(OR + \beta^2 KP) - 2\alpha^2\beta^2(\alpha^2 b^2 - \beta^2)KO(OR + \beta^2 KP)\\
&+ \alpha^2\beta^2 b^2 O^3 P + \alpha^4\beta^2 b^2(KO^2 R + \alpha^2\beta^2 K^3 R + \beta^2 K^2 OP)\\
&- \alpha^2\beta^4(KO^2 R + \alpha^2\beta^2 K^3 R + \beta^2 K^2 OP) - \beta^2 O^3(2L + \beta^2 P)\big]\\
&= 0.
\end{aligned} \tag{3.31}$$

The term of (3.31) which does not contain α^2 is $-\beta^2 O^3(2L + \beta^2 P)$, but the above term can find α^2, that is,

$$-\beta^2 O^3(2L + \beta^2 P) = -\alpha^2\beta^2 O^3(2L_1 + \beta^2 P_1), \tag{3.32}$$

where

$$L_1 = \sum_{k=1}^{h}(2k-1)\alpha^{2k-2}\beta^{2h-2k},$$

$$P_1 = \sum_{k=2}^{h}2k(2k-1)\alpha^{2k-4}\beta^{2h-2k}.$$

Substituting (3.32) into (3.31), we get

$$\begin{aligned}
&f(x)\big[\alpha^2(\beta^2 LK + \alpha^2 KO)\{2\beta^2 KO + (\alpha^2 b^2 - \beta^2)(OP + \alpha^2 KR)\}\\
&- (LO + \alpha^4 K^2)\{\beta^2(O^2 + \alpha^2\beta^2 K^2) + \alpha^2(\alpha^2 b^2 - \beta^2)(OR + \beta^2 KP)\}\big]\\
&+ r_{00}\big[2(\beta^2 LK + \alpha^2 KO)\{2\beta^2 KO + (\alpha^2 b^2 - \beta^2)(OP + \alpha^2 KR)\}\\
&- 2\beta^2(\alpha^2 K^2 O^2 + \beta^2 K^2 LO + \alpha^4\beta^2 K^4) - 2(\alpha^2 b^2 - \beta^2)(LO\\
&+ \alpha^4 K^2)(OR + \beta^2 KP) - 2\beta^2(\alpha^2 b^2 - \beta^2)KO(OR + \beta^2 KP)\\
&+ \beta^2 b^2 O^3 P + \alpha^2\beta^2 b^2(KO^2 R + \alpha^2\beta^2 K^3 R + \beta^2 K^2 OP)\\
&- \beta^4(KO^2 R + \alpha^2\beta^2 K^3 R + \beta^2 K^2 OP) - \beta^2 O^3(2L_1 + \beta^2 P_1)\big]\\
&= 0.
\end{aligned} \tag{3.33}$$

Thus the term of (3.33) which seemingly does not contain α^2 is included in the form: $\beta^{8h}\{f(x)\beta^2+2(b^2-7)r_{00}\}$. Therefore there exists $hp(8h):W_{8h}$ such that

$$\beta^{8h}\{f(x)\beta^2+2(b^2-7)r_{00}\}=\alpha^2W_{8h}. \tag{3.34}$$

In this case, there exists from (3.34) a function $g=g(x)$ satisfying $W_{8h}=g(x)\beta^{8h}$, which takes the follow of form

$$f(x)\beta^2+2(b^2-7)r_{00}=g(x)\alpha^2.$$

From $\alpha^2\not\equiv 0$ (mod. β), it follows that $f(x)$ must vanish and hence we have

$$r_{00}=\frac{g(x)}{2(b^2-7)}\alpha^2, \tag{3.35}$$

where we assume $b^2\neq 7$. Substituting $f(x)=0$ and (3.35) into (3.30), we have

$$\beta s_0=\frac{g(x)}{2(b^2-7)}\alpha^2,$$

which leads to $s_0=0$ and $r_{00}=0$, that is, $s_j=0$ and $r_{ij}=0$. Substituting $s_0=0$ and $r_{00}=0$ into (3.27), we obtain

$$\begin{aligned}&{s^i}_0\big[\alpha^2\{\beta^2(\beta^2LK+\alpha^2KO)+(\alpha^2b^2-\beta^2)(\beta^2LP+\alpha^4KR)\}\\&\{2\beta^2KO+(\alpha^2b^2-\beta^2)(OP+\alpha^2KR)\}-\{\beta^2(LO+\alpha^4K^2)\\&+\alpha^2(\alpha^2b^2-\beta^2)(LR+\alpha^2KP)\}\{\beta^2(O^2+\alpha^2\beta^2K^2)\\&+\alpha^2(\alpha^2b^2-\beta^2)(OR+\beta^2KP)\}\big]=0.\end{aligned} \tag{3.36}$$

Thus the term of (3.36) which seemingly does not contain α^2 is $\beta^{8h+4}{s^i}_0$. Then there exists $hp(8h+3):W_{8h+3}$ such that

$${s^i}_0\beta^{8h+4}=\alpha^2W_{8h+3}. \tag{3.37}$$

From $\alpha^2\not\equiv 0$ (mod. β), there exists from (3.37) a function $h=h(x)$ satisfying $W_{8h+3}=h\beta^{8h+3}$, and hence

$$\beta{s^i}_0=h(x)\alpha^2,$$

which leads to ${s^i}_0=0$, that is, $s_{ij}=0$ by virtue of $h(x)=0$.

Consequently we obtain $r_{ij}=0$ and $s_{ij}=0$, that is, $b_{i;j}=0$ is obtained. Substituting $s_0=0$, $r_{00}=0$, ${s^i}_0=0$ and (3.26) into (3.23), we have

$${\gamma_0}^i{}_0=\mu_0y^i,$$

which leads to

$$2{\gamma_j}^i{}_k=\mu_j\delta^i_k+\mu_k\delta^i_j, \tag{3.38}$$

which shows that the associated Riemannian space is projectively flat.

Conversely, it is easy to see that (3.2) is a consequence of $b_{i;j}=0$ and (3.38).

Consequently we obtain the same results from both case of $r=2h$ and case of $r=2h+1$.

Hence we have the following

Theorem 3.1. *A Finsler space F^n $(n > 2)$ with an approximate infinite (α, β)-metric (2.1) provided $b^2 \neq 7$ is projectively flat if and only if $b_{i;j} = 0$ is satisfied, and the associated Riemannian space (M^n, α) is projectively flat if and only if $2\gamma_j{}^i{}_k = \mu_j \delta^i_k + \mu_k \delta^i_j$ is obtained. Then F^n is a Berwald space.*

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