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PYTHAGOREAN-HODOGRAPH CUBICS AND GEOMETRIC HERMITE INTERPOLATION

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Abstract. In this paper, we present the geometric Hermite interpolation for planar Pythagorean-hodograph cubics for some general Hermite data.

1. Introduction

To find an offset curve $\mathbf{r}_o(t) = \mathbf{r}(t) + d\mathbf{n}(t)$ at a fixed distance d from a given polynomial curve $\mathbf{r}(t)$, in the direction of its unit normal vector $\mathbf{n}(t)$, is an important problem in computer aided geometric design. The offset $\mathbf{r}_o(t)$ is not, in general, a rational curve. Thus we need some approximation schemes to deal with offsets of a polynomial curves.

However, if the speed function of a given polynomial curve is a polynomial, then an offset curve can be expressed in a rational parametrization. For this reason, the Pythagorean-hodograph (PH) curves, whose speed function is a polynomial, were introduced by Farouki and Sakkalis ([6]). Since then, there have been vast researches on this class of curves by themselves and others ([1], [2], [3], [4], [5], and [11]). A PH curve $\mathbf{r}(t) = (x(t), y(t))$ means a special polynomial curve which is characterized by the algebraic property that its hodograph $\mathbf{r}'(t) = (x'(t), y'(t))$ satisfies the Pythagorean condition, that is,

$$x'^2(t) + y'^2(t) = \sigma^2(t)$$

for some polynomial $\sigma(t)$. So, the arc-length function and the offsets of the PH curve are rational.

Meek and Walton ([9]) tried to solve the geometric Hermite interpolation problem with Tschirnhausen cubics (T-cubics). However, we found a counter example in Theorem 1 of [9]. Our goal is to present the proper geometric Hermite interpolants with PH cubics. To reach the goal, we invoke the characterization of the planar PH curves and Theorem about the planar PH cubics, given by Farouki and Sakkalis ([6]). For spatial PH cubics, Pelosi et al. [10]

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partially gave the geometric Hermite interpolants. For Minkowski PH cubics, Kosinka and Jütter [7] solved the geometric Hermite interpolation problem.

This paper is organized as follows: In Section 2, we introduce some fundamental definitions and properties of PH curves. In Section 3, we solve the geometric Hermite interpolation problem with planar PH cubics. We, in Section 4, see some examples of the geometric Hermite interpolants. Here we will compare the bending energies of a piecewise PH cubic and a quadric Bézier curve.

2. Preliminaries

In this section, we review some basic properties of the planar Pythagorean-hodograph (PH) curves of degree three.

Let $\mathbb{R}$ and $\mathbb{R}^2$ be the field of real numbers and the plane respectively. Let $\mathbb{R}[t]$ be the set of real polynomials in real variable t. A *planar polynomial curve* $\mathbf{r}(\mathrm{t})$ is a function $\mathbf{r} : \mathbf{I} \to \mathbb{R}^2$, which is defined on an open interval $\mathbf{I}$ into the plane $\mathbb{R}^2$, and whose component functions of $\mathbf{r}(t) = (x(t), y(t))$ are members of $\mathbb{R}[t]$. The *hodograph* of $\mathbf{r}(\mathrm{t})$ is the planar polynomial curve, which defined by $\mathbf{r}'(t) = (x'(t), y'(t))$ where $x'(t)$, $y'(t)$ are derivatives of $x(t)$, $y(t)$ with respect to t, respectively.

The hodograph $\mathbf{r}'(t) = (x'(t), y'(t))$ is said to be *Pythagorean* if there is a polynomial $\sigma(t)$ such that

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2.$$

A planar polynomial curve $\mathbf{r}(t)$ is called a *Pythagorean-hodograph* (PH) curve if its hodograph $\mathbf{r}'(t)$ is Pythagorean (See [6]).

A planar polynomial curve $\mathbf{r}(t) = (x(t), y(t))$ is said to be *primitive* if $x(t)$ and $y(t)$ are relatively prime, i. e., $\gcd(x(t), y(t)) = 1$. For relatively prime polynomials $u(t)$ and $v(t)$, we can find the property:

Lemma 2.1. *Suppose that polynomials $u(t)$ and $v(t)$ are relatively prime. Then the polynomials $u(t)^2 - v(t)^2$ and $u(t)v(t)$ are relatively prime.*

Proof. Let $x(t) = u(t)^2 - v(t)^2$ and $y(t) = u(t)v(t)$. Suppose that $x(t)$ and $y(t)$ are not relatively prime. Then there exists an irreducible polynomial $d(t)$ such that $d(t) \mid x(t)$ and $d(t) \mid y(t)$. From $d(t) \mid u(t)v(t)$, it is true that $d(t) \mid u(t)$ or $d(t) \mid v(t)$. If $d(t) \mid u(t)$, then since $d(t) \mid [u(t)^2 - v(t)^2]$, $d(t) \mid v(t)$, which implies that $u(t)$ and $v(t)$ are not relatively prime. If $d(t) \mid v(t)$, then since $d(t) \mid [u(t)^2 - v(t)^2]$, $d(t) \mid u(t)$, which implies that $u(t)$ and $v(t)$ are not relatively prime. □

We recall the characterization of the planar PH curves (See [6]). A polynomial curve $\mathbf{r}(t) = (x(t), y(t))$ is PH with a polynomial $\sigma(t)$ such that

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2$$

if and only if there are a monic polynomial $w(t)$ and relatively prime polynomials $u(t)$ and $v(t)$ such that

$$\begin{aligned} x'(t) &= w(t)[u(t)^2 - v(t)^2], \\ y'(t) &= w(t)[2u(t)v(t)], \\ \sigma(t) &= w(t)[u(t)^2 + v(t)^2]. \end{aligned}$$

Now we consider the Bézier expression of a planar polynomial curve. For a planar cubic $\mathbf{r}(t) = (x(t), y(t))$, there are the points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ such that

$$\mathbf{r}(t) = \mathbf{p}_0 B_{0,3}(t) + \mathbf{p}_1 B_{1,3}(t) + \mathbf{p}_2 B_{2,3}(t) + \mathbf{p}_3 B_{3,3}(t),$$

where

$$B_{i,3}(t) = \frac{3!}{(3-i)!i!}(1-t)^{3-i}t^i$$

for $i = 0, \ldots, 3$. The Bézier expression is invariant under affine transformation (See [8]). That is, if T be an affine transformation for example, a rotation, reflection, translation, or scaling, then

$$T\left(\sum_{i=0}^{3} \mathbf{p}_i B_{i,3}(t)\right) = \sum_{i=0}^{3} T(\mathbf{p}_i) B_{i,3}(t).$$

Consider two linear polynomials $u(t)$ and $v(t)$ given in Bernstein-Bézier form as

$$u(t) = u_0 B_0^1(t) + u_1 B_1^1(t), \quad v(t) = v_0 B_0^1(t) + v_1 B_1^1(t),$$

where we assume that the ratios $u_0 : u_1$ and $v_0 : v_1$ are unequal. From the Pythagorean hodograph $\mathbf{r}'(t) = (x'(t), y'(t))$, given by

$$\begin{aligned} x'(t) &= u(t)^2 - v(t)^2 \\ &= (u_0^2 - v_0^2) B_0^2(t) + (u_0 u_1 - v_0 v_1) B_1^2(t) + (u_1^2 - v_1^2) B_2^2(t) \end{aligned}$$

and

$$y'(t) = 2u(t)v(t) = 2u_0 v_0 B_0^2(t) + (u_0 v_1 + u_1 v_0) B_1^2(t) + 2u_1 v_1 B_2^2(t),$$

we obtain the PH curve $\mathbf{r}(t) = (x(t), y(t))$, which may be expressed as

$$\begin{aligned}
x(t) &= x_0 + \int_0^t u(s)^2 - v(s)^2\, ds \\
&= x_0 \left(B_0^3(t) + B_1^3(t) + B_2^3(t) + B_3^3(t)\right) \\
&\quad + (u_0^2 - v_0^2)\frac{B_1^3(t) + B_2^3(t) + B_3^3(t)}{3} \\
&\quad + (u_0u_1 - v_0v_1)\frac{B_2^3(t) + B_3^3(t)}{3} + (u_1^2 - v_1^2)\frac{B_3^3(t)}{3} \\
&= x_0 B_0^3(t) + \left(x_0 + \frac{u_0^2 - v_0^2}{3}\right) B_1^3(t) \\
&\quad + \left(x_0 + \frac{u_0^2 - v_0^2}{3} + \frac{u_0u_1 - v_0v_1}{3}\right) B_2^3(t) \\
&\quad + \left(x_0 + \frac{u_0^3 - v_0^3}{3} + \frac{u_0u_1 - v_0v_1}{3} + \frac{u_1^2 - v_1^2}{3}\right) B_3^3(t),
\end{aligned}$$

and

$$\begin{aligned}
y(t) &= y_0 + \int_0^t 2u(s)v(s)\, ds \\
&= y_0(B_0^3(t) + B_1^3(t) + B_2^3(t) + B_3^3(t)) \\
&\quad + (2u_0v_0)\frac{B_1^3(t) + B_2^3(t) + B_3^3(t)}{3} \\
&\quad + (u_0v_1 + v_0u_1)\frac{B_2^3(t) + B_3^3(t)}{3} + (2u_1v_1)\frac{B_3^3(t)}{3} \\
&= y_0 B_0^3(t) + \left(y_0 + \frac{2u_0v_0}{3}\right) B_1^3(t) \\
&\quad + \left(y_0 + \frac{2u_0v_0}{3} + \frac{u_0v_1 + v_0u_1}{3}\right) B_2^3(t) \\
&\quad + \left(y_0 + \frac{2u_0v_0}{3} + \frac{u_0v_1 + v_0u_1}{3} + \frac{2u_1v_1}{3}\right) B_3^3(t).
\end{aligned}$$

From these, we have the control points $\mathbf{p}_k = (x_k, y_k)$ of the form

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \frac{1}{3}(u_0^2 - v_0^2, 2u_0v_0), \\
\mathbf{p}_2 &= \mathbf{p}_1 + \frac{1}{3}(u_0u_1 - v_0v_1, u_0v_1 + u_1v_0), \\
\mathbf{p}_3 &= \mathbf{p}_2 + \frac{1}{3}(u_1^2 - v_1^2, 2u_1v_1),
\end{aligned}$$

where $\mathbf{p}_0$ is arbitrary. We note that the parametric speed $\sigma(t)$ of the PH curve $\mathbf{r}(t)$ is

$$\begin{aligned}\sigma(t) &= u(t)^2 + v(t)^2 \\ &= (u_0^2 + v_0^2)B_0^2(t) + (u_0u_1 + v_0v_1)B_1^2(t) + (u_1^2 + v_1^2)B_2^2(t).\end{aligned}$$

Now we consider the characterization of nonlinear planar PH cubics:

Theorem 2.2. (Farouki and Sakkalis, [6]) *Let*

$$\mathbf{r}(t) = \mathbf{p}_0(1-t)^3 + \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3$$

be a nonlinear *planar cubic in the Bernstein-Bézier form. Let $\mathbf{L}_k = \mathbf{p}_k - \mathbf{p}_{k-1}$ for $k = 1, 2, 3$ be the direction legs of the Bézier control polygon and $L_k = \|\mathbf{L}_k\|$ for $k = 1, 2, 3$. Let θ_1 and θ_2 are control-polygon angles at the interior vertices $\mathbf{p}_1$ and $\mathbf{p}_2$. Then the conditions*

$$L_2^2 = L_1 \cdot L_3 \quad \textit{and} \quad \theta_1 = \theta_2$$

are sufficient and necessary to ensure that $\mathbf{r}(t)$ is a Pythagorean-hodograph curve.

We set $\theta = \theta_1 = \theta_2$, then we can see the parametric speed of the PH curve $\mathbf{r}(t)$ is given

$$\sigma(t) = 3[L_1 B_0^2(t) - L_2 \cos\theta B_1^2(t) + L_3 B_2^3(t)].$$

Recall that Pythagorean hodograph curves of degree n have just $n-2$ *shape freedoms*. Although we expect the Pythagorean-hodograph cubics to exhibit only one shape freedom, there are apparently three freedom associated with the corresponding Bézier control polygons. Two of three lengths L_1, L_2, L_3 can be freely chosen, as can the angel θ $(= \theta_1 = \theta_2)$. However, two of these freedoms are not essential shape freedoms, being expanded by the possibility of reparameterization.

Corollary 2.3. (Farouki and Sakkalis, [6]) *Nonlinear Pythagorean-hodograph cubics have no real inflection points.*

3. Main results

In this section, we will solve the geometric Hermite interpolation problem with planar PH cubics.

Let $\mathbf{r}(t) = (x(t), y(t))$ be a PH cubic. Then we have a monic polynomial $w(t)$ and relatively polynomials $u(t)$ and $v(t)$, which satisfy

$$x'(t) = w(t)[u(t)^2 - v(t)^2], \qquad y'(t) = w(t)[2u(t)v(t)].$$

Note that $\gcd(u(t), v(t)) = 1$ implies $\max\{\deg(u(t)), \deg(v(t))\} = 0$ or 1.

First, we easily characterize the PH cubics, which are lines:

Lemma 3.1. *Let* $\mathbf{r}(t) = (x(t), y(t))$ *be a PH cubic such that*

$$\begin{aligned} x'(t) &= w(t)[u(t)^2 - v(t)^2], \\ y'(t) &= w(t)[2u(t)v(t)], \end{aligned} \tag{1}$$

for a monic polynomial $w(t)$ *and relatively prime polynomials* $u(t)$ *and* $v(t)$. *Then the following are equivalent:*

(a) $\mathbf{r}(t)$ *is a line;*
(b) $\max\{\deg(u(t)), \deg(v(t))\} = 0$*;*
(c) $\deg(w(t)) = 2$*.*

Proof. It is easy to prove $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$. Here we just prove $(a) \Rightarrow (b)$. Suppose that $\mathbf{r}(t) = (x(t), y(t))$ be a line. Then there are some $a, b, c \in \mathbb{R}$ such that $ax(t) + by(t) + c = 0$ with $a^2 + b^2 \neq 0$. The equation $a^2 + b^2 \neq 0$ results in three cases: (A) $a \neq 0$ and $b \neq 0$; (B) $a = 0$ and $b \neq 0$; (C) $a \neq 0$ and $b = 0$. (A) In case of $a \neq 0$ and $b \neq 0$, we differentiate both sides of $ax(t) + by(t) + c = 0$ so that we obtain $ax'(t) + by'(t) = 0$. Since $x'(t) = -\frac{b}{a}y'(t)$ and since $u(t)^2 - v(t)^2$ and $2u(t)v(t)$ are relatively prime, we conclude that $u(t)^2 - v(t)^2$ and $2u(t)v(t)$ are constants so that $u(t)$ and $v(t)$ are constant, i.e., $\max\{\deg(u(t)), \deg(v(t))\} = 0$. (B) In case of $a = 0$ and $b \neq 0$, we have $w(t)[2u(t)v(t)] = y'(t) = 0$. Since $w(t)$ is a monic, either $u(t)$ or $v(t)$ must be the zero constant function. But from $\gcd(u(t), v(t)) = 1$, we conclude that the other function must be a nonzero constant function, which implies that $\max\{\deg(u(t)), \deg(v(t))\} = 0$. (C) In case of $a \neq 0$ and $b = 0$, by some similar steps, we conclude that $\max\{\deg(u(t)), \deg(v(t))\} = 0$. $\square$

Now we solve the geometric Hermite interpolation problem for PH cubics in the standard condition:

Theorem 3.2. *For given geometric Hermite data* $\mathbf{p}_i = (0, 0)$, $\mathbf{p}_f = (a, 0)$ $(a > 0)$, $\mathbf{d}_i = (\cos\theta_i, \sin\theta_i)$, *and* $\mathbf{d}_f = (\cos\theta_f, \sin\theta_f)$ *with* $-\pi < \theta_i < 0$ *and* $0 < \cos\theta_f < \pi$.

(a) *If* $\frac{\pi}{3} \leq \theta_f - \theta_i - \pi$, *then there is no solution;*
(b) *If* $-\frac{\pi}{3} \leq \theta_f - \theta_i - \pi < \frac{\pi}{3}$, *then there is a unique simple solution;*
(c) *If* $\theta_f - \theta_i - \pi < -\frac{\pi}{3}$, *then there are two solutions: a simple one and a non-simple one.*

Proof. Let

$$\mathbf{r}(t) = \mathbf{p}_0(1-t)^3 + \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3$$

be a planar polynomial cubic in the Bernstein-Bézier form such that

$$\mathbf{r}(0) = \mathbf{p}_i, \quad \mathbf{r}(1) = \mathbf{p}_f, \quad \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \mathbf{d}_i, \quad \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \mathbf{d}_f.$$

Since

$$\mathbf{r}'(t) = 3(\mathbf{p}_1 - \mathbf{p}_0)(1-t)^2 + 6(\mathbf{p}_2 - \mathbf{p}_1)(1-t)t + 3(\mathbf{p}_3 - \mathbf{p}_2)t^2,$$

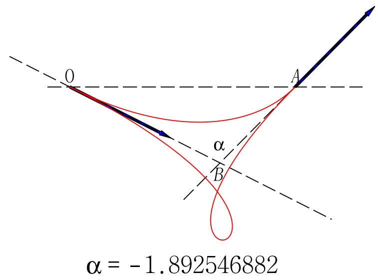

FIGURE 1. $-\pi < \alpha < -\frac{\pi}{3}$

we know that $\mathbf{p}_1$ and $\mathbf{p}_2$ are below the x. Let $\mathbf{L}_k = \mathbf{p}_k - \mathbf{p}_{k-1}$ for $k = 1, 2, 3$ be the direction legs of the Bézier control polygon and $L_k = \|\mathbf{L}_k\|$ for $k = 1, 2, 3$. Let θ_1 and θ_2 are control-polygon angles at the interior vertices $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. Here we know that the conditions $L_2^2 = L_1 \cdot L_3$ and $\theta_1 = \theta_2$ are necessary and sufficient for $\mathbf{r}(t)$ to be a Pythagorean-hodograph curve. We therefore suppose that $\theta_1 = \theta_2$, and set $\theta = \theta_1 = \theta_2$.

Suppose that $-\theta_i > \theta_f$. We set $B = L_2$ and $C = L_3$ when $L_1 = 0$. We now let L_1 be any positive number. Then we obtain

$$L_2 = B + 2L_1 \cos\theta, \qquad L_3 = C + L_1.$$

Consider the function of L_1:

$$f(L_1) = L_2^2 - L_1 L_3 = (4\cos^2\theta - 1)L_1^2 + (4B\cos\theta - C)L_1 + B^2.$$

Let $\alpha = \theta_f - \theta_i - \pi$.

Case of $-\pi < \alpha < -\frac{\pi}{3}$. In this case, we have $\frac{2\pi}{3} < \theta < \pi$ so that the leading coefficient $4\cos^2\theta - 1$ of the quadratic $f(L_1)$ is positive. Since

$$f(0) = B^2 > 0 \quad \text{and} \quad f(L) < 0,$$

where L is the positive number L_1 when $\mathbf{p}_1 = \mathbf{p_2}$, f have two positive zeros ℓ_1 and ℓ_2 with $0 < \ell_1 < L < \ell_2$. When $L_1 = \ell_1$, $\mathbf{r}(t)$ is a simple PH cubic and when $L_1 = \ell_2$, $\mathbf{r}(t)$ is a non-simple cubic. We clearly see that

$$\ell_1 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha B^2}}{2\alpha} \quad \text{and} \quad \ell_2 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha B^2}}{2\alpha},$$

where $\alpha = 4\cos^2\theta - 1$ and $\beta = 4B\cos\theta - C$.

Case of $\alpha = -\frac{\pi}{3}$. In this case, we have

$$f(L_1) = -\left(\frac{B}{2} + C\right) L_1 + B^2$$

Therefore, f has the unique positive zero $B^2/(2B + C)$. Since $f(0) = B^2 > 0$ and $f(L) < 0$, we know that $B^2/(2B + C) < L$.

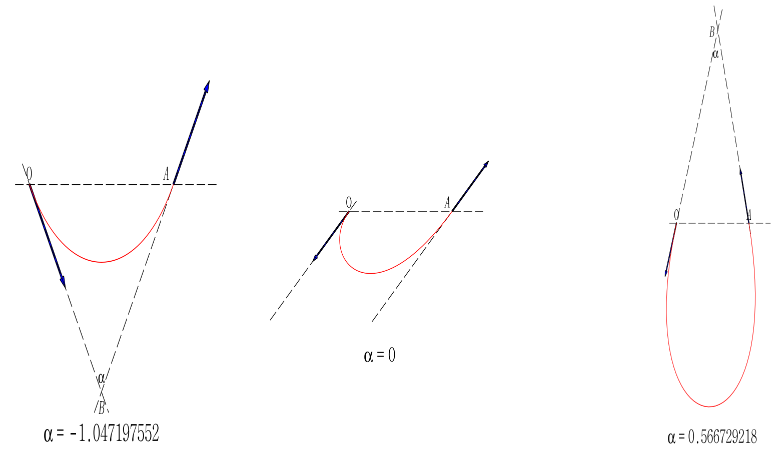

FIGURE 2. $-\frac{\pi}{3} \le \alpha < \frac{\pi}{3}$

Case of $-\frac{\pi}{3} < \alpha < \frac{\pi}{3}$. In this case, we have $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$ so that the leading coefficient $4\cos^2\theta - 1$ of the quadratic $f(L_1)$ is negative. Since $f(0) = B^2 > 0$, f has one positive zero ℓ_2. In case of $0 \le \alpha <\le \frac{\pi}{3}$, then $\mathbf{r}(t)$ is clearly a simple PH cubic. In case of $-\frac{\pi}{3} < \alpha < 0$, since $f(0) = B^2 > 0$ and $f(L) < 0$, we know that $\ell_2 < L$ so that $\mathbf{r}(t)$ is a simple PH cubic. Here we see that

$$\ell_2 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha B^2}}{2\alpha}.$$

Case of $\alpha = \frac{\pi}{3}$. In this case, we have $\theta = \frac{\pi}{3}$. Since

$$4B\cos\theta - C = 2B - C = B + (B - C) > 0,$$

$f(t)$ has no positive zero.

Case of $\frac{\pi}{3} < \alpha < \pi$. In this case, we have $0 < \theta < \frac{\pi}{3}$ so that the leading coefficient $4\cos^2\theta - 1$ of the quadratic $f(L_1)$ is positive. Since

$$4B\cos\theta - C > 2B - C = B + (B - C) > 0,$$

f has no positive zero. $\square$

4. Examples

In this section, we will see several examples for geometric Hermite interpolations.

Let $\mathbf{p}_i = (0, 0)$ and $\mathbf{p}_f = (1, 0)$ be the initial and terminal points, respectively. Let $\mathbf{d}_i$ and $\mathbf{d}_f$ be direction vectors at $\mathbf{p}_i$ and $\mathbf{p}_f$, respectively. Let θ_i and θ_f be the angles of $\mathbf{d}_i$ and $\mathbf{d}_f$ with respect to the positive x-axis, respectively. Here we assume that $-\pi < \theta_i < 0$ and $0 < \theta_f < \pi$. Let $\alpha = \theta_f - \theta_i - \pi$.

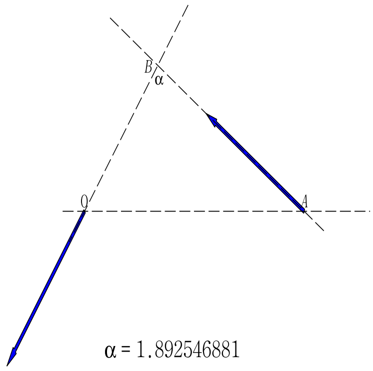

FIGURE 3. $\frac{\pi}{3} \leq \alpha < \pi$

In case of $-\pi < \alpha < -\frac{\pi}{3}$, we can see two the geometric Hermite interpolants satisfying the data $\{\mathbf{p}_i, \mathbf{p}_f, \mathbf{d}_i = (\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}), \mathbf{d}_f = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$ in Figure 1. One interpolant is simple, but the other is not.

In case of $-\frac{\pi}{3} \leq \alpha < \frac{\pi}{3}$, there exists the unique geometric Hermite interpolant, which is simple. Figure 2 shows the geometric Hermite interpolants satisfying the data

$$\{\mathbf{p}_i, \mathbf{p}_f, \mathbf{d}_i = (\frac{1}{2}, -\frac{\sqrt{3}}{2}), \mathbf{d}_f = (\frac{1}{2}, \frac{\sqrt{3}}{2})\},$$

$$\{\mathbf{p}_i, \mathbf{p}_f, \mathbf{d}_i = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \mathbf{d}_f = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\},$$

and

$$\{\mathbf{p}_i, \mathbf{p}_f, \mathbf{d}_i = (-\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}), \mathbf{d}_f = (-\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}})\}.$$

In case of $\frac{\pi}{3} \leq \alpha < \pi$, there exists no geometric Hermite interpolant satisfying the data $\{\mathbf{p}_i, \mathbf{p}_f, \mathbf{d}_i = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \mathbf{d}_f = (-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})\}$. See Figure 3.

In Figure 4, the curve 1 and the curve 2 are a piecewise PH cubic and a quadric Bézier curve, respectively. The bending energy of the piecewise PH cubic and the quadric Bézier curve are 1.363535924 and 2.332705060, respectively. We can see that the bending energy of PH cubic is lower than that of the quadric Bézier curve.

Figure 5 shows an offset curve at the distance 0.5 from the curve 1. To obtain an offset of a polynomial curve, we must go through some approximation step. But in case of a PH curve, we do not need this step.

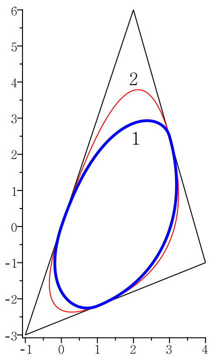

FIGURE 4. 1: Piecewise PH cubics and 2: quadratic Bézier curves

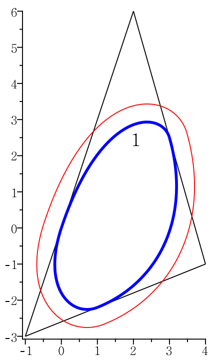

FIGURE 5. The offset curve

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