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FUZZY SOFT MODULES

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Abstract. In this paper, the concept of fuzzy soft module is introduced, some of their properties are discussed. Furthermore, the concept of fuzzy soft exactness is also given.

1. Introduction

Most of the problems in economics, engineering, medical science, environments etc. have various uncertainties. To solve these uncertainties, we cannot successfully use classical methods because of various uncertainties typical for the those problems. So some kinds of theories were given like theory of fuzzy sets [20], rough sets [15], i.e., which we can use as mathematical tools for dealing with uncertainties. However, all of these theories have their own difficulties which are pointed out in [14]. In 1999, Molodtsov [14] introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainties. There are many authors study soft set theory, for example, Maji et al. [13] have demonstrated the application of soft sets in a decision making problem by considering rough sets. Chen et al. [3] presented a new definition and compared this definition to the related concept of attributes reduction in rough set theory. Maji et al. [11] defined and studied several operations on soft sets. Xiang et al.[19] have studied a review on soft set theory. In particular, first proposed the definition of soft groups by Akataş and Çağam [1]. Liu [10] have also considered normal soft groups. Some authors have considered some soft algebraic structures and extensively investigated, Jun [6] considered soft BCK/BCI-algebras and Jun-Park [7] investigated the applications of soft set in the ideal theory of BCK/BCI-algebras. We also noticed that Feng et al. [4] have recently started to investigate the structure of soft semirings, Liu et al. [9] have studied isomorphism theorems for soft rings ,Sun et al. [18] have studied soft sets and soft modules. Furthermore, some interesting results in classical modules are still being explored nowadays, for example, J.J. Rotman [16] have studied classical modules. The notion of fuzzy soft sets, as a generalization of the standard soft sets, was introduced in [12], and an application of fuzzy

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soft sets in a decision making problem is presented. Feng et al. [5] studied an adjustable approach to fuzzy soft set based decision making. Roy et al. [17] presented some results on an application of fuzzy soft sets in decision making problem. Aygünoğlu et al. [2] introduced the notion of fuzzy soft group and studied its properties. Jun et al [8] have studied fuzzy soft set theory applied to BCK/BCI-algebras.

The work of this paper is organized as follows. In the second section as preliminaries, we give basic concepts of soft sets and fuzzy soft sets. In Section 3, we introduce fuzzy soft modules and study their characteristic properties. In Section 4, we give the definition of fuzzy soft exactness and study some of their basic properties.

2. Preliminaries

In this section as a beginning, the concepts of soft sets introduced by Molodsov [14] and the notions of fuzzy soft set introduced by Maji et al. [12] will be presented.

Throughout this paper, let I be a closed unit interval, i.e., $I = [0, 1]$, and E be all of convenient parameter set for the universe X. Then we formulate the following definition.

Definition 1. ([14]) A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

A soft set over U can be regarded as a parameterized family of subsets of the universe U. For any $x \in A$, $F(x)$ may be regarded as the set of x-elements of the soft set (F, A) or as the set of x-approximate elements of the soft set.

Definition 2. ([11]) Let (F, A) and (G, B) be two soft sets over U. Then (F, A) is said to be a soft subset of (G, B) if

(1) $A \subset B$ and
(2) for all $x \in A$, $F(x)$

and $G(x)$ are identical approximations.

We now denote the above inclusion relationship by $(F, A)\widetilde{\subset}(G, B)$. Similarly, (F, A) is called a soft superset of (G, B) if (G, B) is a soft subset of (F, A). Denoted the above relationship by $(F, A)\widetilde{\supset}(G, B)$.

Equipped with the above notations, we give the following definitions.

Definition 3. ([11]) Two soft sets (F, A) and (G, B) over U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 4. ([11]) The intersection of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \bigcap B$ and for all $x \in C$, either $H(x) = F(x)$ or $H(x) = G(x)$. This intersection is denoted by $(F, A)\widetilde{\bigcap}(G, B) = (H, C)$.

Definition 5. ([11]) Let (F, A) and (G, B) be two soft sets. Then we denote (F, A) and (G, B) by $(F, A)\widetilde{\bigwedge}(G, B)$. The soft set $(F, A)\widetilde{\bigwedge}(G, B)$ is defined by $(H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \bigcap G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 6. ([11]) The union of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \bigcup B$ and for all $x \in C$, we define

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B, \\ G(x) & \text{if } x \in B - A, \\ F(x) \bigcup G(x) & \text{otherwise.} \end{cases}$$

The above relationship is denoted by $(F, A) \widetilde{\bigcup} (G, B) = (H, C)$.

Definition 7. ([12]) Let I^X denote the set of all fuzzy sets on X and $A \subset E$. A pair (f, A) is called a fuzzy soft set over X, where f is a mapping given by $f : A \to I^X$. That is, for all $a \in A$, $f(a) = f_a$, is a fuzzy set on X.

Definition 8. ([12]) Let (f, A) and (g, B) be two fuzzy soft sets over X. Then (f, A) is said to be a fuzzy soft subset of (g, B) if

(1) $A \subset B$ and
(2) for all $a \in A$, f_a is a fuzzy subset of g_b.

We now denote the above inclusion relationship by $(f, A) \widetilde{\subset} (g, B)$. Similarly, (f, A) is called a fuzzy soft superset of (g, B) if (g, B) is a fuzzy soft subset of (f, A). Denoted the above relationship by $(f, A) \widetilde{\supset} (g, B)$.

Equipped with the above notations, we give the following definitions.

Definition 9. ([12]) Two fuzzy soft sets (f, A) and (g, B) over X are said to be fuzzy soft equal if (f, A) is a fuzzy soft subset of (g, B) and (g, B) is a fuzzy soft subset of (f, A).

Definition 10. ([12]) The intersection of two fuzzy soft sets (f, A) and (g, B) over X is the soft set (h, C), where $C = A \bigcap B$ and for all $c \in C$, either $h_c = f_c$ or $h_c = g_c$. This intersection is denoted by $(f, A) \widetilde{\bigcap} (g, B) = (h, C)$.

Definition 11. ([12]) Let (f, A) and (g, B) be two fuzzy soft sets. Then we denote (f, A) and (g, B) by $(f, A) \widetilde{\bigwedge} (g, B)$. The fuzzy soft set $(f, A) \widetilde{\bigwedge} (g, B)$ is defined by $(h, A \times B)$, where $h(a, b) = h_{a,b} = f_a \bigcap g_b$, for all $(a, b) \in A \times B$.

Definition 12. ([12]) The union of two fuzzy soft sets (f, A) and (g, B) over X is the fuzzy soft set (h, C), where $C = A \bigcup B$ and for all $c \in C$, we define

$$h_c = \begin{cases} f_c & \text{if } c \in A - B, \\ g_c & \text{if } c \in B - A, \\ f_c \bigcup g_c & \text{if } c \in A \cap B. \end{cases}$$

The above relationship is denoted by $(F, A) \widetilde{\bigcup} (G, B) = (H, C)$.

3. Fuzzy soft modules

Throughout this section, M is a module with zero element and A a nonempty set. Now, we use ρ to refer an arbitrary binary relation between an element of A and an element of M. Thus, a set-valued function $F : A \to P(M)$ can be defined by $F(x) = \{y \in M \mid (x, y) \in \rho, x \in A \text{ and } y \in M\}$.

Definition 13. ([18]) Let (F, A) be a soft set over M. Then (F, A) is said to be a soft module over M if and only if $F(x)$ is a submodule of M for all $x \in A$. For our convenience, the empty set $\emptyset$ is regarded as a submodule of M.

Example 3.1. Let M and A be Z-module and $F(x) = \{y \mid x\rho y \Longleftrightarrow x+y \in Z\}$. Then for all $y_1, y_2 \in F(x)$, $x \in A$, $r \in R = Z$, we have $(x + y_1) + (x + y_2) = x + (y_1 + y_2) \in Z$ so $y_1 + y_2 \in F(x)$, $r(x + y) = rx + ry \in Z$, we imply that for all $yr \in F(x)$ and for all $r \in R$. Thus, for each $x \in A$, we can verify that $F(x)$ is a submodule of M and hence, (F, A) is a soft module over M.

Definition 14. Let M be a module and (f, A) be a fuzzy soft set over M. Then (f, A) is said to be a fuzzy soft module over M if and only if for all $a \in A$, $x, y \in M$ and $r \in R$,

(1) $f_a(x + y) \geq min\{f_a(x), f_a(y)\}$,
(2) $f_a(-x) = f_a(x)$,
(3) $f_a(0) = 1$,
(4) $f_a(rx) \geq f_a(x)$.

That is, for each $a \in A$, f_a is a fuzzy submodule.

We now illustrate the above definition by the following example.

Example 3.2. Since each soft set can be considered as fuzzy soft set and each characteristic function of a submodule of a module is a fuzzy module, we can consider a soft module as a fuzzy soft module.

Example 3.3. Let N be the set of all natural numbers and define $f : N \longrightarrow I^M$ by $f(n) = f_n : M \longrightarrow I$, for all $n \in N$ where

$$f_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x = kz^n, \exists k \in z \ and \ k \neq 0, \\ 1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

where I is the set of all integers. Then the pair (f, N) forms a fuzzy soft set over M and the fuzzy soft set (f, N) is a fuzzy soft module over M.

Example 3.4. Let $M = Z_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ is a Z-module, consider a set of parameters $A = \{\overline{0}, \overline{1}, \overline{2}\}$.

(1) Let (f, A) be a fuzzy soft set over M. Then $f_{\overline{0}}$, $f_{\overline{1}}$ and $f_{\overline{2}}$ are fuzzy sets in M. We define them as follows:

f	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	1	0.7	0.7	0.7
$\overline{1}$	1	0.6	0.6	0.6
$\overline{2}$	1	0.4	0.4	0.4

The (f, A) is a fuzzy soft module over M.

(2) Let (g, A) be a fuzzy soft set over M. Then $g_{\overline{0}}$, $g_{\overline{1}}$ and $g_{\overline{2}}$ are fuzzy sets in M. We define them as follows:

g	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	1	0.7	0.7	0.7
$\overline{1}$	1	0.6	0.6	0.6
$\overline{2}$	1	0.4	0.5	0.2

Then (g, A) is not a fuzzy soft module over module M, because $g_{\overline{2}}(\overline{1} + \overline{2}) = g_{\overline{2}}(\overline{3}) = 0.2 \ngeq 0.4 = min\{g_{\overline{2}}(\overline{1}), g_{\overline{2}}(\overline{2})\}$.

Proposition 3.5. *If (f, A) is a fuzzy soft module over M. Then $f_a(0) \geq f_a(x)$ $(\forall x \in M)$ where a is any parameter in A.*

Proof. It is straightforward. □

Theorem 3.6. *Let (f, A) and (g, B) be two fuzzy soft modules over M. Then $(f, A)\widetilde{\cap}(g, B)$ is also a fuzzy soft module over M.*

Proof. Let $(f, A)\widetilde{\cap}(g, B) = (h, C)$, where $C = A \cap B$ and $h_c = f_c \cap g_c$, that is $h_c(x) = f_c(x) \cap g_c(x), \forall c \in C$ and $\forall x \in M$. For arbitrary $c \in C$, we have
(1) $f_c(x + y) \geq min\{f_c(x), f_c(y)\}$,
(2) $f_c(-x) = f_c(x)$,
(3) $f_c(0) = 1$,
(4) $f_c(rx) \geq f_c(x)$, and
(1) $g_c(x + y) \geq min\{g_c(x), g_c(y)\}$,
(2) $g_c(-x) = g_c(x)$,
(3) $g_c(0) = 1$,
(4) $g_c(rx) \geq g_c(x)$, for all $a \in A$, $x, y \in M$ and $r \in R$. Then we obtain,
(1)
$$\begin{aligned}(f_c \cap g_c)(x + y) &= f_c(x + y) \cap g_c(x + y)\\ &\geq \min\{f_c(x), f_c(y)\} \cap \{g_c(x), g_c(y)\}\\ &= \min\{f_c(x) \cap g_c(x), f_c(y) \cap g_c(y)\}\\ &= \min\{(f_c \cap g_c)(x), (f_c \cap g_c)(y)\},\end{aligned}$$
(2) $(f_c \cap g_c)(-x) = (f_c \cap g_c)(x)$,
(3) $(f_c \cap g_c)(0) = 1$,
(4) $(f_c \cap g_c)(rx) \geq (f_c \cap g_c)(x)$. □

Theorem 3.7. *Let (f, A) and (g, B) be two fuzzy soft modules over M. If $A \bigcap B = \emptyset$, then $(f, A)\widetilde{\bigcup}(g, B)$ is a fuzzy soft module over M.*

Proof. Let $(f, A)\widetilde{\bigcup}(g, B) = (h, C)$. Since $A \bigcap B = \emptyset$, it follows that either $c \in A - B$ or $c \in B - A$ for all $c \in C$. If $c \in A - B$, then $h_c = f_c$ is a fuzzy module of M and if $c \in B - A$, then $h_c = g_c$ is a fuzzy module of M. Thus $(f, A)\widetilde{\bigcup}(g, B)$ is a fuzzy soft module over M. □

The following example shows that Theorem 3.7 is not valid if A and B are not disjoint.

Example 3.8. Let $M = Z_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ is a Z-module, consider the sets of parameters $A = \{\overline{0}, \overline{1}, \overline{2}\}$ and $B = \{\overline{0}, \overline{1}\}$, let (f, A) be a fuzzy soft set over M. Then $f_{\overline{0}}$, $f_{\overline{1}}$ and $f_{\overline{2}}$ are fuzzy sets in M. We define them as follows:

f	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	1	0.7	0.7	0.7
$\overline{1}$	1	0.6	0.6	0.6
$\overline{2}$	1	0.4	0.4	0.4

Then (f, A) is a fuzzy soft module over M. Let (g, B) be a fuzzy soft set over M. Then $g_{\overline{0}}$, and $g_{\overline{1}}$ are fuzzy sets in M. We define them as follows:

g	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	1	0.6	0.6	0.6
$\overline{1}$	1	0.5	0.5	0.5

Then (g, B) is a fuzzy soft module over M. But the union $(f, A)\widetilde{\bigcup}(g, B)$ is not a fuzzy soft module over M, since $(f_{\overline{0}} \cup g_{\overline{0}})(\overline{2} + \overline{2}) = (f_{\overline{0}} \cup g_{\overline{0}})(\overline{0}) = 1$ and

$$\begin{aligned}(f_{\overline{0}} \cup g_{\overline{0}})(\overline{2} + \overline{2}) &\geq \min\{(f_{\overline{0}} \cup g_{\overline{0}})(\overline{2}), (f_{\overline{0}} \cup g_{\overline{0}})(\overline{2})\}\\ &= \min\{max\{f_{\overline{0}}(\overline{2}), g_{\overline{0}}(\overline{2})\}, \max\{f_{\overline{0}}(\overline{2}), g_{\overline{0}}(\overline{2})\}\}\\ &= \min\{\max\{0.7, 0.6\}, max\{0.7, 0.6\}\}\\ &= 0.7.\end{aligned}$$

We have $(f_{\overline{0}} \cup g_{\overline{0}})(\overline{2} + \overline{2}) \neq (f_{\overline{0}} \cup g_{\overline{0}})(\overline{2} + \overline{2})$.

Theorem 3.9. *Let (f, A) and (g, B) be two fuzzy soft modules over M. Then $(f, A)\widetilde{\bigwedge}(g, B)$ is also a fuzzy soft module over M.*

Proof. Let $(f, A)\widetilde{\bigwedge}(g, B) = (h, A \times B)$. We know that $f_a, \forall a \in A$, and $g_b, \forall b \in B$, are fuzzy module of M and so is $h(a, b) = h_{a,b} = f_a \cap g_b, \forall (a, b) \in A \times B$, because the intersection of two fuzzy module is also a fuzzy module. Hence, $(h, A \times B) = (f, A)\widetilde{\bigwedge}(g, B)$ is fuzzy soft module over M. □

Theorem 3.10. *Let (f, A) be a fuzzy soft module over M. If B is a subset of A, then $(f\mid_B, B)$ is a fuzzy soft module over M.*

Proof. It is straightforward. □

The following example shows that there exists a fuzzy soft set (f, A) over a module M such that

(1) (f, A) is not a fuzzy soft module over a module M.

(2) There exists a subset B of A such that $(f\mid_B, B)$ is a fuzzy soft module over module M.

Example 3.11. Let $M = Z_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ is a Z-module, consider the sets of parameters $A = \{\overline{0}, \overline{1}, \overline{2}\}$ and $B = \{\overline{0}, \overline{1}\}$, let (f, A) be a fuzzy soft set over M. Then $f_{\overline{0}}$, $f_{\overline{1}}$ and $f_{\overline{2}}$ are fuzzy sets in M. We define them as follows:

f	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	1	0.7	0.7	0.7
$\overline{1}$	1	0.6	0.6	0.6
$\overline{2}$	0.6	0.4	0.5	0.4

Then (f, A) is not a fuzzy soft module over M. But (f, B) is a fuzzy soft module over M, then $(f\mid_B, B)$ is a fuzzy soft module over module M.

We now give the following definition.

Definition 15. ([2]) Let $\varphi : X \to Y$ and $\psi : A \to B$ be two functions, where A and B are parameter sets for the crisp sets X and Y, respectively. Then the pair (φ, ψ) is called a fuzzy soft function from X to Y.

Definition 16. ([2]) Let (f, A) and (g, B) be two fuzzy soft sets over X and Y, respectively and let (φ, ψ) be a fuzzy soft function from X to Y.

(1) The image of (f, A) under the soft function (φ, ψ), denoted by $(\varphi, \psi)(f, A)$ is the fuzzy soft set over Y defined by where

$$\varphi(f)_k(y) = \begin{cases} \bigvee\limits_{\varphi(x)=y} \bigvee\limits_{\psi(a)=k} f_a(x) & \text{if } x \in \varphi^{-1}(y), \\ 0 & \textit{otherwise}. \end{cases} \quad \text{for all } k \in \psi(A), y \in Y.$$

(2) The pre-image of (g, B) under the fuzzy soft function (φ, ψ) denoted by $(\varphi, \psi)^{-1}(g, B)$, is the fuzzy soft set over X,
defined by $(\varphi, \psi)^{-1}(g, B) = (\varphi^{-1}(g), \psi^{-1}(B))$, where $\varphi^{-1}(g)_a(x) = g_{\psi(a)}(\varphi(x))$, for all $a \in \psi^{-1}(B), x \in X$. If φ and ψ is injective (surjective), then (φ, ψ) is said to be injective (surjective).

Definition 17. ([2]) Let (φ, ψ) be a fuzzy soft function from X to Y. If φ is a homomorphism from X to Y, then (φ, ψ) is said to be a fuzzy soft homomorphism. If φ is a isomorphism from X to Y and ψ is a one-to-one mapping from A onto B, then (φ, ψ) is said to be a fuzzy soft isomorphism.

Definition 18. Let (f, A) and (g, B) be two fuzzy soft modules over M and N, respectively and let (φ, ψ) be a fuzzy soft function from M to N. We define

$Im(\varphi, \psi) = \{g_b(y) \mid g_b(y) = \varphi(f)_k(y)\}$ and $Ker(\varphi, \psi) = \{f_a(x) \mid \varphi(f)_k(y) = 1\}$ for all $k \in \psi(A), y \in N$.

Theorem 3.12. *Let (f, A) be a fuzzy soft module over M, (φ, ψ) be a monic fuzzy soft homomorphism from M to N. Then $(\varphi, \psi)(f, A)$ is a fuzzy soft module over N.*

Proof. Let $k \in A$ and $y_1, y_2 \in N$. If $\varphi^{-1}(y_1) = \emptyset$ or $\varphi^{-1}(y_2) = \emptyset$; the proof is straightforward. Let assume that there exist $x_1, x_2 \in M$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y_2$.

$$\begin{aligned} \varphi(f)_k(y_1 + y_2) &= \bigvee_{\varphi(t)=y_1+y_2} \bigvee_{\psi(a)=k} f_a(t) \\ &\geq \bigvee_{\psi(a)=k} f_a(x_1 + x_2) \\ &\geq \bigvee_{\psi(a)=k} min\{f_a(x_1), f_a(x_2)\} \\ &\geq \min\{\bigvee_{\psi(a)=k} f_a(x_1), \bigvee_{\psi(a)=k} f_a(x_2)\}. \end{aligned}$$

This inequality is satisfied for each $x_1, x_2 \in M$, which satisfy $\varphi(x_1) = y_1, \varphi(x_2) = y_2$. Since we have

$$\begin{aligned} \varphi(f)_k(y_1 + y_2) &\geq \min\{\bigvee_{\varphi(t_1)=y_1} \bigvee_{\psi(a)=k} f_a(t_1), \bigvee_{\varphi(t_2)=y_2} \bigvee_{\psi(a)=k} f_a(t_2)\} \\ &= \min\{\varphi(f)_k(y_1), \varphi(f)_k(y_2)\}. \end{aligned}$$

□

Theorem 3.13. *Let (g, B) be a fuzzy soft module over N and (φ, ψ) be a monic fuzzy soft homomorphism from M to N. Then $(\varphi, \psi)^{-1}(g, B)$ is a fuzzy soft module over M.*

Proof. Let $a \in \psi^{-1}(B)$ and $x_1, x_2 \in M$.

$$\begin{aligned} \varphi^{-1}(g)_a(x_1 + x_2) &= g_{\psi(a)}(\varphi(x_1 + x_2)) \\ &= g_{\psi(a)}(\varphi(x_1) + \varphi(x_2)) \end{aligned}$$

$$\begin{aligned} &\geq \min\{g_{\psi(a)}\varphi(x_1), g_{\psi(a)}\varphi(x_2)\} \\ &= \min\{\varphi^{-1}(g)_a(x_1), \varphi^{-1}(g)_a(x_2)\}. \end{aligned}$$

□

4. Fuzzy soft exactness

In this section, we consider all nonempty sets are modules.

Definition 19. Let (F, A), (G, B) and (H, C) be three fuzzy soft modules over modules M, N and K, respectively. Then we say soft exact at (G, B), if the following conditions are satisfied:

$$Im(\varphi_1, \psi_1) = Ker(\varphi_2, \psi_2),$$

which is denoted by $(f, A) \xrightarrow{(\varphi_1,\psi_1)} (g, B) \xrightarrow{(\varphi_2,\psi_2)} (h, C)$. In this definition, if every (f_i, A_i), $i \in I$ is fuzzy soft exactness, then we say that $(f_i, A_i)_{i\in I}$ is fuzzy soft exactness.

Definition 20. If $M = 0$, then $(f, A) = 1$. We call (f, A) is fuzzy soft singular module.

Proposition 4.1. *Let (f, A) and (g, B) be two fuzzy soft modules over modules M and N, respectively. If (φ_1, ψ_1) and (φ_2, ψ_2) are two soft fuzzy homomorphisms from 0 to M and M to N respectively. Then (φ_2, ψ_2) is monic if and only if $1 \xrightarrow{(\varphi_1,\psi_1)} (f, A) \xrightarrow{(\varphi_2,\psi_2)} (g, B)$ is fuzzy soft exactness.*

Proof. Since $0 \xrightarrow{\varphi_1} M \xrightarrow{\varphi_2} N$, we have $Im(\varphi_1, \psi_1) = 1$, and so $Ker(\varphi_2, \psi_2) = \{f_a(x) \mid \varphi_2(f)_k(y) = 1\}$ and $y = 0$. Since (φ_2, ψ_2) is monic, we have $x = 0$ and $Ker(\varphi_2, \psi_2) = 1$. It is clear that $Im(\varphi_1, \psi_1) = Ker(\varphi_2, \psi_2)$. Whence $1 \xrightarrow{(\varphi_1,\psi_1)} (f, A) \xrightarrow{(\varphi_2,\psi_2)} (g, B)$ is fuzzy soft exactness.

Conversely, if $1 \xrightarrow{(\varphi_1,\psi_1)} (f, A) \xrightarrow{(\varphi_2,\psi_2)} (g, B)$ is fuzzy soft exactness, then we have $Im(\varphi_1, \psi_1) = Ker(\varphi_2, \psi_2) = 1$. It is clear that (φ_2, ψ_2) is monic. □

Proposition 4.2. *Let (f, A) and (g, B) be two fuzzy soft modules over modules M and N, respectively.*

(1) *If (φ, ψ) is epic, then $Im(\varphi, \psi) = (g, B)$.*
(2) *If $(f, A) \xrightarrow{(\varphi,\psi)} 1$, then $Ker(\varphi, \psi) = (f, A)$ and (φ, ψ) is epic.*
(3) *If $1 \xrightarrow{(\varphi,\psi)} (f, A)$, then $Im(\varphi, \psi) = (f, A)$ and (φ, ψ) is monic.*
(4) *If $(f, A) \xrightarrow{(\varphi,\psi)} (g, B)$ and (φ, ψ) is monic, then $Ker(\varphi, \psi) = 1$.*
(5) *If $1 \to (f, A) \xrightarrow{(\varphi,\psi)} (g, A) \to 1$ is fuzzy soft exact, then (φ, ψ) is isomorphism.*

Proof. It is straightforward. □

Proposition 4.3. *Let (f, A) and (g, B) be two fuzzy soft modules over modules M and N, respectively. If (φ_1, ψ_1) and (φ_2, ψ_2) are two soft fuzzy homomorphisms from M to N and N to 0 respectively. Then (φ_1, ψ_1) is epic if and only if $(f, A) \xrightarrow{(\varphi_1,\psi_1)} (g, B) \xrightarrow{(\varphi_2,\psi_2)} 1$ is fuzzy soft exactness.*

Proof. Assume $(f, A) \xrightarrow{(\varphi_1,\psi_1)} (g, B) \xrightarrow{(\varphi_2,\psi_2)} 1$ is fuzzy soft exactness, so $Im(\varphi_1, \psi_1) = ker(\varphi_2, \psi_2)$. By Proposition4.2 $ker(\varphi_2, \psi_2) = (g, B)$. We have $Im(\varphi_1, \psi_1) =$

(g, B), so (φ_1, ψ_1) is epic. Conversely, let (φ_1, ψ_1) be epic. By Proposition4.2, we have $ker(\varphi_2, \psi_2) = (g, B)$ and $Im(\varphi_1, \psi_1) = (g, B)$, whence $Im(\varphi_1, \psi_1) = ker(\varphi_2, \psi_2)$. It is clear that $(f, A)\xrightarrow{(\varphi_1,\psi_1)}(g, B)\xrightarrow{(\varphi_2,\psi_2)}1$ is fuzzy soft exactness. □

Definition 21. $1\xrightarrow{(\varphi_1,\psi_1)}(f, A)\xrightarrow{(\varphi_2,\psi_2)}(g, B)\xrightarrow{(\varphi_3,\psi_3)}(h, C)\xrightarrow{(\varphi_4,\psi_4)}1$ is called a fuzzy soft exact sequence, if the exactness holds at (f, A), (g, B) and (h, C), where the two 1's express appropriate fuzzy soft singular modules and (φ_1, ψ_1),(φ_4, ψ_4) are the monic and epic respectively.

Proposition 4.4. *Let (f, A), (g, B) and (h, C) be three fuzzy soft modules over modules M, N and K, respectively. If $(f, A)\xrightarrow{(\varphi_1,\psi_1)}(g, B)\xrightarrow{(\varphi_2,\psi_2)}(h, C)$ is a fuzzy soft exactness with (φ_1, ψ_1) epic and (φ_2, ψ_2) monic, then $(g, B) = 1$.*

Proof. It is straightforward. □

Proposition 4.5. *Let (f_i, A_i) $i = 1, 2, 3, 4$ be fuzzy soft modules over modules M_i $i = 1, 2, 3, 4$, respectively. If $(f_1, A_1)\xrightarrow{(\varphi_1,\psi_1)}(f_2, A_2)\xrightarrow{(\varphi_2,\psi_2)}(f_3, A_3)\xrightarrow{(\varphi_3,\psi_3)}(f_4, A_4)$ is fuzzy soft exactness. Then (φ_1, ψ_1) is epic if and only if (φ_3, ψ_3) is monic.*

Proof. Let (φ_1, ψ_1) be epic. Because $(f_1, A_1)\xrightarrow{(\varphi_1,\psi_1)}(f_2, A_2)\xrightarrow{(\varphi_2,\psi_2)}(f_3, A_3)\xrightarrow{(\varphi_3,\psi_3)}(f_4, A_4)$ is fuzzy soft exactness, so $Im(\varphi_1, \psi_1) = Ker(\varphi_2, \psi_2)$. By Proposition 4.2 $Im(\varphi_1, \psi_1) = (f_2, A_2)$. Hence, we have $Ker(\varphi_2, \psi_2) = (f_2, A_2)$. By Proposition4.2. So $(f_3, A_3) = 1$, it is clear that (φ_3, ψ_3) is monic. Conversely, if (φ_3, ψ_3) is monic. By Proposition4.2, we have $Ker(\varphi_3, \psi_3) = 1$. So $Im(\varphi_2, \psi_2) = 1$. By Proposition4.2, so $(f_2, A_2) = 1$, by Proposition4.2, it is clear that (φ_1, ψ_1) is epic. □

Proposition 4.6. *Let (f_i, A_i) $i = 1, 2, 3, 4, 5$ be fuzzy soft modules over modules M_i $i = 1, 2, 3, 4, 5$, respectively. If $(f_1, A_1)\xrightarrow{(\varphi_1,\psi_1)}(f_2, A_2)\xrightarrow{(\varphi_2,\psi_2)}(f_3, A_3)\xrightarrow{(\varphi_3,\psi_3)}(f_4, A_4)\xrightarrow{(\varphi_4,\psi_4)}(f_5, A_5)$ is fuzzy soft exactness, with (φ_1, ψ_1) is epic and (φ_4, ψ_4) is monic. Then $(f_3, A_3) = 1$.*

Proof. Let (φ_1, ψ_1) be epic. By Proposition4.5, we have (φ_3, ψ_3) is monic. Let (φ_4, ψ_4) be monic. By Proposition4.5, we have (φ_2, ψ_2) is epic. By Proposition4.4. It is clear that $(f_3, A_3) = 1$. □

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