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The Mediation of Embodied Symbol on Combinatorial Thinking[1]

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This research investigated if the embodied symbol using a turtle metaphor in a microworld environment works as a cognitive tool to mediate the learning of combinatorics. It was found that students were able to not only count the number of cases systematically by using the embodied symbols in a situated problem regarding Permutation and Combination, but also find the rules and infer a concept of Combination through the activities manipulating the symbols. Therefore, we concluded that the embodied symbol, as a bridge that connects learners' concrete experiences with abstract mathematical concepts, can be applied to introduction of various mathematical concepts as well as a combinatorics concept.

Keywords: combinatorics, embodied, symbols, mediation, bridge, microworld
MESC Classification: K20, U70
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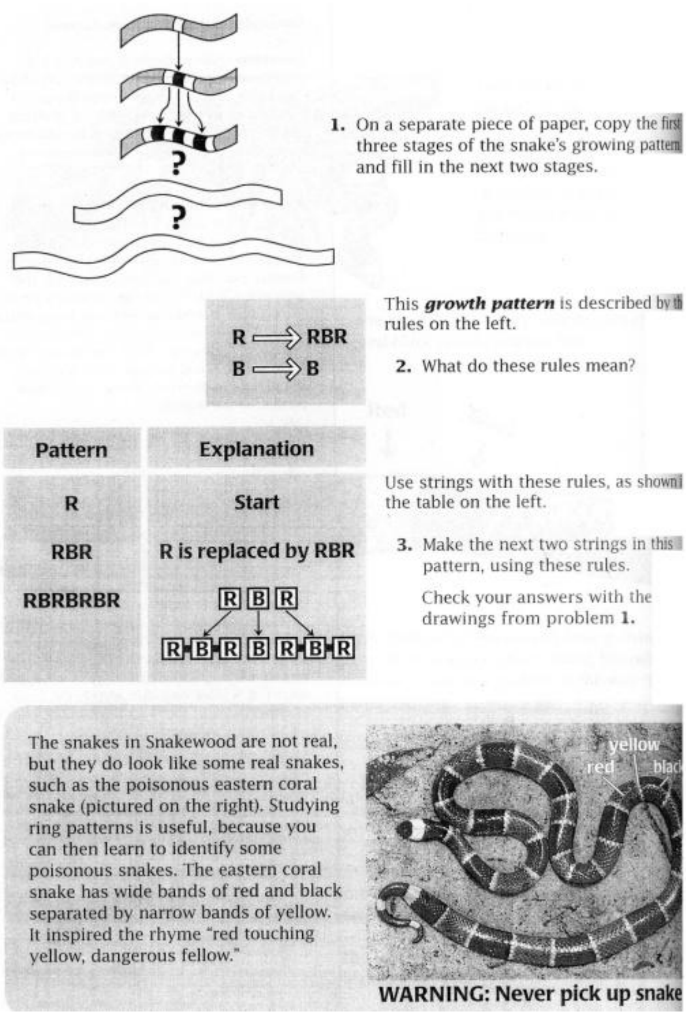
INTRODUCTION

Combinatorics is an essential component of discrete mathematics and it has an important role to play in school mathematics. Kaput (1970) suggested that Combinatorics can be used to train pupils in enumeration, making conjectures, generalization and systematic thinking. Combinatorial capacity is a fundamental and very important thing to be acquired before learning Probability. However, most students regard it abstract and difficult because Permutation or Combination is approached with a formula or calculation in school education (Batanero, Navarro-Pelayo & Godino, 1997). On this, while Fischbein & Gazit (1988) stated teaching a Permutation formula actually disturbs learner's intuitive experiential strategy on Combinatorics, they found out a 10-year-old child who is not capable of formal thinking may understand a combinatorial concept with a help of tree diagram.

Wilensky (1991) pointed out that "formal is often abstract because we haven't yet constructed the connections that will concretize it." In addition, he explained that "concreteness is not a property of an object but rather a property of a person's relationship to an object." That is, it may not be considered abstract if a new mathematical concept is introduced by being connected to learner's concrete experience. From an educational perspective, therefore, how to provide a bridge that connects an abstract concept with a concrete experience should be considered very critical. Then, how do we provide learners with the bridge that connects their personal experiences with very dry and abstract combinatorial concepts?

Math in Context (Mathematics in Context Development Team, 2003), the textbook written by Freudenthal Research Institute, introduces the mathematical concept of recursion to the 5th grade students by representing a pattern composed of R(Red) and B(Black) which means rattlesnake's colour pattern as a string of symbols shown in Figure 1. By writing a string of symbols, learners may go through what Davis (1984) calls a 'visually moderated sequence', which means considering the symbols, manipulating, considering the new symbols, manipulating, ..., and so on, until a solution is found. Unlike a ready-made mathematical formula, a string of symbols is semi-formal. Hence, manipulating a string of symbols helps learners conjecture, observe, reflect, and explore mathematical ideas. This leads to a cognitive tool which helps learners represent and construct an object easily and effectively.

This research applies a string of symbols introduced in *Math in Context* to an introduction of Permutation and Combination concepts by merging with Logo's turtle metaphor in a technology-based microworld. By doing so, we analyse the power and educational implications of a string of symbols as a bridge that connects concreteness with abstraction.



1. On a separate piece of paper, copy the first three stages of the snake's growing pattern and fill in the next two stages.

This ***growth pattern*** is described by the rules on the left.

R ⟹ RBR
B ⟹ B

2. What do these rules mean?

Pattern	Explanation
R	Start
RBR	R is replaced by RBR
RBRBRBR	R B R → R B R B R B R

Use strings with these rules, as shown in the table on the left.

3. Make the next two strings in this pattern, using these rules.

Check your answers with the drawings from problem **1.**

The snakes in Snakewood are not real, but they do look like some real snakes, such as the poisonous eastern coral snake (pictured on the right). Studying ring patterns is useful, because you can then learn to identify some poisonous snakes. The eastern coral snake has wide bands of red and black separated by narrow bands of yellow. It inspired the rhyme "red touching yellow, dangerous fellow."

Figure 1. Recursive pattern and symbols

THEORETICAL BACKGROUND

The semiotic approach: The semiotic mediation and microworld

Vygotsky (1978) mentioned a concept of 'semiotic mediation' which sees knowledge construction as a consequence of instrumented activity where signs emerge and evolve within social interaction. Through this, he focused on how tools impact on the human activity and support it. Vygotsky distinguished tools from signs: tools are externally directed, toward changes in the non-social, material world, while signs are used internally, within a social group. That is, semiotic mediation does not refer to the concrete act of using a tool to accomplish a task, but rather to the fact that new meanings related to the use of a tool may be generated and evolve in learner's mind (Mariotti, 2002). Shaffer & Clinton (2006) introduced a concept of 'Toolforthought' regarding the close relation between tools and thoughts. In this ontology, there are no tools without thinking, and there is no thinking without tools. There are only 'Toolforthoughts', which represent the reciprocal relation between tools and thoughts that exists in both. Moving a step forward from continuous interaction between tools and thoughts, in other words, Shaffer and Clinton regarded human cognition works together with 'Toolforthoughts', filling the gap between the two.

An instrumental approach based on Vygotsky's perspective has recently appeared in mathematics education. The instrumental approach distinguishes an artifact from an instrument. The term artifact is used to describe a given human-made object, whilst the instrument is used to represent a material and psychological hybrid, in which the artifact has become associated with a set of schemes for use. The instrumental approach introduces the notion of instrumental genesis, a process by which artifacts become transformed into instruments. The transformation of an artifact into an instrument proceeds in two directions: towards the self and towards the context in which it is employed. The first direction, instrumentation, involves the construction of schemes of instrumented actions to be integrated into the individual's own cognitive structure. The second, instrumentalization, refers to how the functionalities and affordances of the artifact in question are adjusted and transformed for specific uses (Shaffer & Clinton, 2006). By relating the instrumental approach with the perspective of Microworld research, Healy & Kynigos (2010) stated activity being 'shaped by tools' is analogous to 'instrumentation' and activity 'shaping of tools' is analogous to 'instrumentalization'. Along with this, they raised the following question: How can a given microworld become transformed into an instrument? Pratt (2000) argued the microworld works as a semiotic mediator that connects learners' intuition with mathematical concepts in the interaction that occurs between learners' informal intuition and Chance-Maker microworld in understanding number of cases occurred in playing a dice game. With the object of answering the question of Healy & Kynigos (2010) regarding Probability Combinatorics, this research will design the JavaMAL microworld environment that guides and fosters learners and study its semiotic mediation.

The embodied approach: embodied cognition and embodied symbol

As more attentions have been paid to the relation between tools and thoughts, a belief on cognitive root has been changed. Researchers realized that they cannot understand human mind if focusing only on brain, separating a human body from an environment, and such realizations have been developed to an 'embodied cognition' perspective. Lakoff & Núñez (2000) explained embodied cognition from a mathematical perspective. Mathematical concepts are formed by human brain and sense of body, and made through concepts in human mind. That is, represented math to human is embodied in human body and mind. This cognitive perspective provides a viewpoint that math learning is abstraction as a result of certain manipulation on mathematical objects.

Embodied approach brings a new perspective on the role of body based on the understanding that even the most abstract of symbols have physical grounding (Healy & Kynigos, 2010). In other words, body movement may contribute on the process of mathematical conceptualization and the importance of perception and action is emphasized. Such an embodied approach can be understood in the same context as 'body syntonicity' and 'ego syntonicity' introduced in microworld's Logo by Papert (1980). In relation to 'body syntonicity', Papert′s vision was that learners would be able to relate the behavior of the microworld objects with their own sense and knowledge about their own bodies, whilst the term 'ego syntonicity' stresses the intention that learners would also identify with these objects in ways coherent with their own senses of themselves as people with intentions, goals, desires, likes and dislikes. These two constructs then provided a glimpse of the theoretical perspective that first fuelled the constructionist.

External representation is an 'idea' which has a certain concept in the perspective that a mathematical concept should be expressed in any ways to be represented in mind (Davis, 1984). The ways to represent a mathematical object vary. It could be symbolic representation which was defined as an agreement in a society or could be visual representation such as a grape or a diagram. Learners may or may not find mathematical concepts easy, depending on the ways of external representations.

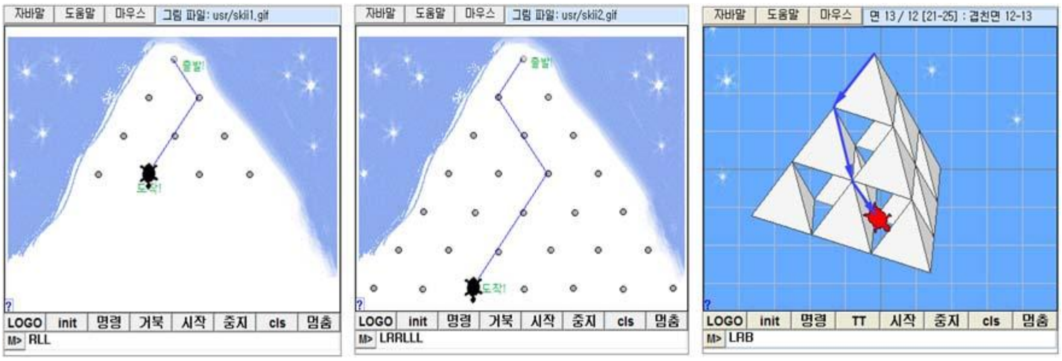
Godino, Batanero & Roa (2005) observed symbolic notation such as $C_{4,2}$ and visual representation such as tree diagram by analyzing students' semiotics used in combinatorial problem solving. A symbol, $C_{4,2}$, is symbolic which is hard to expect learners' semiotic operation before they learn the agreement system. However, it is manipulatable, so enables learners to reach out to a higher level of concepts once they learn the agreement system. On the other hand, tree diagram is iconic, which is roughly understandable intuitively without prior knowledge. However, it is not manipulatable, so unlikely to further to the next level. In the embodied approach, we would like to consider semi-visual and semi-symbolic embodied symbols which learners can perceive intuitively and manipulate at the same time.

B and W themselves in *Math in Context* are just symbols which do not mean any. However, once the rule is made that B means black and W means white, B and W mean a certain thing to learners and they become a way of semiotic mediation, generating a higher level of thinking through manipulation. A string of symbols has been used in the previous Combinatorics education. By using symbols like H and T, for example, we express it HHHT for the case that three heads and one tail come out when flipping four coins. That is, these symbols like H and T are the tools to represent the situation easily as meaning head and tail of the coin respectively. Taking a step forward from here, a string of symbols introduced in *Math in Context* is a tool that can represent, visualize, and explore dynamically changing recursion in sequence. We implemented a string comprised of direction indicating symbols, L(Left), R(Right), and B(Back), in the JavaMAL microworld environment by applying the idea of a string of symbols to Logo's turtle metaphor. This is something embodied to learners physically and mentally by projecting learners themselves to the turtle on the screen and we will call it an "embodied symbol". The embodied symbol is expected to be a symbolic mediation tool to concrete and elaborate learners' thoughts and help learners communicate with others.

METHOD

To observe the process of combinatorial problem solving which does not rely on any algorithms and the role of embodied symbols, this research was conducted to 73 students aged 12 to 13, who have learned basic concept of Probability, but have not learned formulas on Permutation and Combination. The problem situations were fully explained to the students through online video lectures, texts and experimental environment and we let students manipulate and experiment combinatorial situations on their own in the microworld environment.

The experiment was preceded with a test conducting before and after learning the embodied symbols in the online environment. The problems in both pre- and post-test were designed in an identical pattern, but we encouraged the students to use the embodied symbols that they learned when they answered the problems in the post-test. The problems asking a concept of Permutation and Combination were presented by difficulty level as shown in Figure 2 and the students were supposed to answer number of cases in the context where a turtle slides down on skis from the top of the hill to the right or left and finally arrives at a certain point and explain how they get to their answer. While Level 1 and Level 2 problems are binominal, using L(Left) and R(Right) on two dimensions, Level 3 problem is trinomial, using L(Left), R(Right), and B(Back) on three dimensions. In addition, Level 1 and Level 2 problems include a concept of 'Combination' by choosing a particular number of R or L out of all possible paths or a concept of 'Combination with Repetition' by choosing L or R by the number of paths as allowing repetition. Adding one more dimension, on the other hand, Level 3 problem includes a concept of 'Permutation' which enumerates a total of three paths with three symbols. We asked the students to get number of cases with their own strategy in the pre-test since the students who participated in the experiment have not yet learned a concept or a formula of Permutation and Combination while we asked them to use the embodied symbols to get the answer in the post-test after learning such symbols. In the test results, we analyzed the correct-answer rate quantitatively and analyzed students' reasoning process on their answer qualitatively.



a: Level 1 problem b: Level 2 problem c: Level 3 problem

Figure 2. Task problems

RESULT

We analyzed the pre- and post-test result to see if the embodied symbols work as a mediating tool to solve combinatorial problems.

Table 1. Result

	Level 1	Level 2	Level 3
Pre-test	46.3%	24.1%	20.35%
Post-test	62.9%	26.6%	33.33%

In Level 1 problem, the correct-answer rates in the pre- and post-test were 46.3% and 62.9% respectively; thus the correct-answer rate in the post-test was increased compared to the one in the pre-test. We may presume the correct-answer rate was increased because students, by using a symbol of L and R, felt more concrete towards the vague problem asking number of cases.

However, Level 2 problem did not show a big difference as the correct-answer rates in the pre- and post-test showed 24.1% and 26.6% respectively. On this, expanding from the problem simply asking number of cases, we asked students in the post-test if they were able to discover the rules related to a concept of Combination. Approximately 70% of the students discovered the rules, 4 Ls and 2 Rs came out in common, in their enumeration of symbols in the post-test and justified their reasoning. That is, the students were able to reach out to a concept of Combination which number of cases is the same when choosing 4 Ls or 2 Rs out of a string of 6 symbols through the concrete experience enumerating the paths which a turtle skis down the hill with the symbols.

The followings are two episodes about two students who received a help of the embodied symbols when solving a combinatorial problem. First, a student, Ye-won answered 5,040 in the pre-test in Level 2 problem, but she described in the post-test "a string of 4 Ls and 2 Rs should be used. On arriving point's basis, 2 dots are on the left side and 4 dots are on the right side in the bottom line. Turning it the other way, it is correct that 4 Ls and 2 Rs are used." From this, we were able to verify that Ye-won reasoned out the number of Ls and Rs with the location of the dot in the bottom line. Another student, Han-woul, answered 57 by counting all possible paths one by one in the pre-test, but after learning, he described in the post-test, "The turtle can arrive at the bottom point with 4 Ls and 2 Rs. For example, in case of arriving at the second left point, 5 Ls and 1 R are required. Out of the 6 symbols, 1 L is removed as L moves to the left one time and 1 R is added. Therefore, the total number of paths is 15 with LLLLRR, LLLRLR, LLLRRL, LLRLLR, LLRLRL, LLRRLL, LRLLLR, LRLLRL, LRLRLL, LRRLLL, RLLLLR, RLLLRL, RLLRLL, RLRLLL, and RRLLLL." Since it was too difficult to come up with 15 cases at once, Han-woul was able to count number of cases systematically by switching L with R from the previous stage which he was able to think easily. Although many students felt difficult in counting all 15 cases accurately, we may interpret that students' concrete experience enumerating the embodied symbols actually worked as a bridge that connects to the abstract concept of combination.

Finally, Level 3 problem asked number of cases that a turtle arrives at the center point of the bottom (1st floor) from the peak of the regular tetrahedron shape of mountain (3rd floor). The correct-answer rate in the pre-test was as low as 20.35% and the rate went up to 33.3% in the post-test after learning the embodied symbols of L, R and B (Left, Right and Back). It is assumed that the extended problem situation from binomial to trinomial and limitation of visualization led to the low correct-answer rate in both pre- and post-test. Given that the correct answer is 6, however, approximately 40% of students incorrectly answered a nonsensically large number above 20 in the pre-test, but only 5.56% of students answered such a big number above 20 in the post-test. This result confirms us that those students who did not have any clue on how to approach the problem before learning the usage of embodied symbols and therefore ended up with a nonsensically large numbers got to be more capable of presuming not exactly correct, but approximate values by enumerating number of cases with the symbols.

In the following, we looked into some of students' explanation to see what kinds of misconception appeared by analysing student's reasoning in the pre-test of Level 3 problem.

Student A: There are 3 ways to go down from the 3rd floor and 9 ways to go down from the 2nd floor. To get to the arrival point, the turtle needs to go down to the 3 triangles which are connected to the arrival point and this narrows down to 6 ways. There are 3 ways to go down from the 2nd floor to the 1st floor. Therefore, the total number of paths to reach the arrival point is 54, as 3*6*3 equals to 54.

Student B: A regular tetrahedron has 4 vertices and there are 6 regular tetrahedrons; therefore 24, 6*4.

Student C: The answer is 486 because the number of all cases possible is 18 on the 1st floor, 9 on the 2nd floor and 3 on the 3rd floor. Therefore, 18*9*3 equals to 486.

Student D: 1st floor: 3, 2nd floor: 6, 3rd floor: 3. 3*6*3=54. Likewise, I solved it the number of all cases possible on each floor.

Student E: 3 ways on the 3rd floor, 6 ways adding the number of ways on the 2nd floor, and 24 adding the number of ways on 1st floor. The sum is 24.

Student F: The answer is 21. Three edges on the first regular tetrahedron + 9 edges on the second 3 regular tetrahedrons + 9 edges on the last 3 regular tetrahedrons.

Student A and Student D counted number of cases repetitively as counting 3 ways to go down from the 3rd floor and 6 ways to go down from the 2nd floor while the correct number of cases to go down from 3rd to 2nd floor is 6. In addition, they unnecessarily multiplied number of cases on each floor. Student C also multiplied all number of cases occurred on each floor. Student E counted correctly number of cases on the 3rd and 2nd floor, but did not count number of cases on the 1st floor systematically. Student F counted the number of cases on each floor systematically, but he made a mistake by adding all numbers on each floor. Student B made an error by multiplying the number of tetrahedron by the number of corner irrelevantly.

Based on students' explanations, we may classify these students into a certain type of error as follows:

Error 1: Misusing of a multiplication rule -------------- Student A, B, C, D
Error 2: Applying incorrect addition rules --------------------- Student E, F
Error 3: Repetitively counting ------------------------------- Student A, D, F
Error 4: Unsystematic counting in a certain part ----------------- Student E
Error 5: Irrelevantly applying the number of solids or edges ---Student B

We observed that high percentage of students who provided a wrong answer actually applied multiplication or addition rules when they were not supposed to do so or counted some numbers repetitively. This tells us that students tend to solve the problem according to algorithm, rather than enumerating sample space systematically. We were able to conclude that enumerating number of cases with symbols can be a proper pedagogic prescription to those students.

Finally, the students responded that they used the embodied symbols easily and fount it useful to the question asking if they feel comfortable and useful in using these symbols affectively. “I had to count all number of cases one by one in my mind in the pre-test, but it felt easier when I used the symbols in the post-test. Also this looks good to me and the other people since it can be written simply. It is much better if I use symbols because I do not forget it well and also can find the rules with ease.”, “I did not get confused by using symbols as solving the problem. Also this helped me find the rules in the number of cases.” Such answers confirm that enumerating number of cases by using the embodied symbols works as not only a cognitive tool to solve combinatorial problems and but also a bridge that connects abstraction with concretion.

CONCLUSION AND DISCUSSION

This research investigated if the embodied symbol using a turtle metaphor in a JavaMAL microworld environment may work as a cognitive tool to mediate the learning of Combinatorics. First, as Mariotti (2009) proposed guiding and fostering as a component of mediator in a semiotic approach, we were able to examine a guiding and fostering role of JavaMAL microworld. Starting from a very familiar situation like a skiing turtle, JavaMAL microworld guided learners to approach mathematical concepts gradually through video clips, images and texts. In addition, JavaMAL microworld fostered the evolution of learners’ personal meaning to mathematical meaning on Combinatorics through the activities to manipulate embodied symbols, discover regularity and make an inference.

Also in a perspective of embodied approach, we were able to confirm that learners approached combinatorial thinking with ease by using embodied symbols which make learners feel that they move their own body by their avatar. Increased correct-answer rate proves usefulness of such embodied symbols’ usage, but more importantly, we find it useful in that combinatorial sense and feeling, ‘something will be likely’, is created. Students were able to conjecture approximate values close to the correct answer in a trinomial situation of Level 3 problem once embodied symbols were given as a tool for thoughts while they provided wrong answers, far away from the correct one, by using meaningless algorithms when no tool was given. This can be an alternative answer to the question

Healy & Kynigos (2010) previously raised. Thus, we suppose embodied symbols become an instrument as a semiotic mediator which helps learners generate meaning in the JavaMAL microworld environment.

This research confirmed that the embodied symbols, as a tool that can be easily used by learners, are useful for solving a basic combinatorial problem. We expect, furthermore, the embodied symbols can be applied to the studies of various mathematical concepts. For example, Figure 3 shows a recursion problem situation presented in the Figure 1 — growth occurs by three times as R is replaced by RWR and W is replaced by W — which is transferred to the microworld environment.

In here, students accept a mathematical concept of recursion concrete through the visual images represented as a result of manipulating symbols and receiving feedback. Thus, microworld based on the embodied symbols can be a “Mathematics laboratory” where learners observe, experiment, conjecture various mathematical situations from learners’ intuition and generate a new mathematical concept through ‘what if questions’.

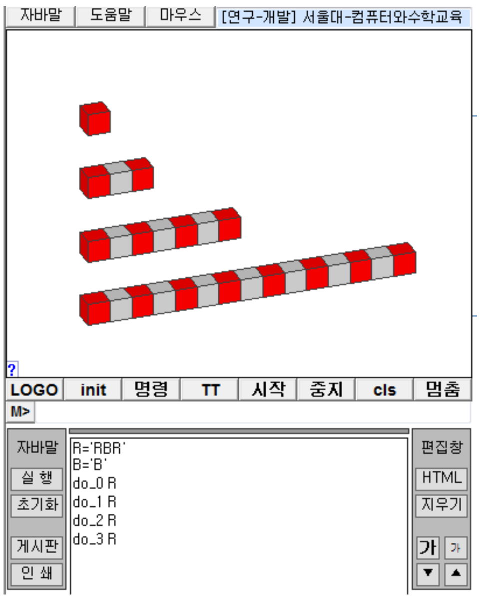


Figure 3. Embodied symbols for recursion

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