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교차 평가 모델의 고정 가중치 유형의 확장 연구

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Extensions on The Fixed Weighting Nature of Cross-Evaluation Model

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DEA 모델중 널리 사용되는 교차평가모델(cross efficiency model)은 가중치에 제한을 두지 않고 어떤 특정분야에 탁월한 성과를 내는 DMU(Decision Making Unit)보다는 보다 전반적인 분야에서 두각을 나타내는 DMU를 선발함으로써 많은 연구자들이 DEA문헌에서 적용하여 왔다. 본 연구에서는 이러한 교차평가모델이 실제에 있어서는 암묵적으로 고정 가중치를 사용한다는 것과 동일한 결과를 나타낸다는 것을 분석적으로 밝혔다(one input, multi output case). 또한 multi-input, multi-output case의 경우에도 overall performer의 cluster에 근접한 대다수 DMU의 경우에는 고정 가중치를 사용한 경우와 거의 차이가 없음을 보였다. 교차평가 모델에 적용된 변수의 가중치를 보다 명확히 함으로써 연구자들이 모델의 평가결과를 이해하는데 도움이 될 수 있을 것이다. 또한 교차 평가의 가중치 도식을 더 명확히 보여주기 위해 biplot을 제안한다.

Keywords : Data Envelopment Analysis, Cross-efficiency, Multiple-Input, Multiple-Output

1. Introduction

In the earlier paper, Anderson et al. [1] demonstrated that in a single-input, multiple-output constant returns to scale model with input orientation, cross-efficiency evaluation in effect applies implicitly fixed weights to each and every DMU, which is a weighted average of the weights used by all of the DMUs in the sample. They also stated that 1) The common set of weights also exists in the multiple-input, single-output constant returns to scale model with output orientation 2) The multiple-input, multiple-output models do not exhibit this fixed weighting phenomena because of the inability to normalize the weights. Based on their ideas, in this paper we made an extension to the multiple-inputs, multiple-outputs constant returns to scale with input orientation to show that 1) The cross-evaluation does not use the column average of cross-efficiency multipliers as fixed weights in the multiple-input, multiple-output situation 2) Even though cross-evaluation doesn't use the fixed weights, the column averages of cross-efficiency multipliers can be considered as the fixed set of weights to each variable without much difference in many cases 3) Moreover the above difference is

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much smaller in the DMUs that are located among the cluster of majority DMUs in the sample. Therefore, if a 1st ranker in cross-evaluation shows very small difference, it can be said that it is among the majority of DMUs, i.e. winner with many competitors and almost fixed set of weights (i.e. average values of each multiplier column) are used for evaluation.

2. Performing a Cross-Evaluation

Cross-evaluation was first proposed by Sexton et al. [12] and Doyle and Green [3, 4] suggested their cross-evaluation formulations with signification of the essence of the ranking criteria. After that, many researchers have applied cross-evaluation in their research [6, 8, 10, 11, 13]. Doyle and Green [3] indicated that decision makers do not always have a reasonable mechanism to choose assurance regions and therefore they recommended cross-evaluation for ranking units. Also Doyle and Green [4] justified the ranking criteria of cross-evaluation such that "To get a first rank in cross-efficiency evaluation is equivalent to win in a big race with many competitors. It could be said that coming second in a race where there were a thousand competitors must surely be better than coming first in a walkover." Cross-evaluation requires the following three-step calculation.

Step 1 : Find the CCR efficiency score θ_{j_o} for DMU j_0.

Step 2 : Find the virtual multipliers (μ_j, ν_j) by (1) with restriction that the efficiency score for DMU j_0 is set to the CCR efficiency score θ_{j_o}, Here input $x_{i,j}$ and output $y_{r,j}$ represent input i and output r of DMU j.

$$Min \quad \sum_{r=1}^{s}\left(\mu_{r,j}\sum_{j\neq j_0} y_{r,j}\right)$$

$$\text{subject to} \quad \sum_{i=1}^{m}\left(v_{r,j}\sum_{j\neq j_0} x_{r,j}\right)=1,$$

$$\sum_{r=1}^{s}\mu_{r,j}\,y_{r,j}-\sum_{i=1}^{m}v_{i,j}\,x_{i,j}\le 0, \qquad \forall j\neq j_0 \tag{1}$$

$$\sum_{r=1}^{s}\mu_{r,j}\,y_{j_0}-\theta_{j_0}\sum_{i=1}^{m}v_{i,j}\,x_{i,j}=0,$$

$$\mu_{r,j},\ v_{i,j}\ge 0, \qquad \forall r \text{ and } i$$

Step 3 : After making cross-efficiency matrix (where each element is $E_{j,k}$), calculate the cross-efficiency score by (2)

$$E_{j,k}=\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}},\ CE_k=\frac{1}{n}\sum_{j=1}^{n}E_{j,}\,k \tag{2}$$

3. Single-Input, Multiple Outputs Case

3.1 The Case that Each DMU's Input Value is Unified to 1

For illustration purpose, we will begin with brief explanation of earlier work and example from Anderson et al. [1].

Cross-efficiency score of DMU k is calculated as (3).

$$CE_k=\frac{1}{n}\left[\sum_{j=1}^{n}\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}}\right] \tag{3}$$

In the single-input, multiple-output case (when $x_{1,k}=1$), it becomes (4)

$$CE_k=\frac{1}{n}\left[\sum_{j=1}^{n}\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}}\right]=\frac{1}{n}\left[\sum_{j=1}^{n}\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}}\right]$$

$$=\frac{1}{nx_{1,k}}\left[\sum_{j=1}^{n}\sum_{r=1}^{s}\frac{\mu_{r,j}\,y_{r,k}}{v_{1,j}}\right]=\frac{1}{nx_{1,k}}\left[\sum_{r=1}^{s}y_{r,k}\sum_{j=1}^{n}\left(\frac{\mu_{r,j}}{v_{1,j}}\right)\right]$$

$$=\sum_{r=1}^{s}y_{r,k}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\mu_{r,j}}{v_{1,j}}\right)\right] \tag{4}$$

As a result, the multipliers $\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\mu_{r,j}}{v_{1,j}}\right)\right]$ for each output $y_{r,k}$ are independent of DMU k. That is, in the following example, the weights for each output are 0.0323175, 0.05257 respectively. And they also said that these weight results match those obtained using the standard column average method to four decimal places of accuracy.

However the weights actually applied for each output in cross-evaluation (single-input, multiple-outputs case with all

<Table 1> Example 1 Artificial Data and Cross Efficiency Weights

Data				CE Multipliers-Input Oriented				
DMU	x	y_1	y_2	DMU	CCR	ν_1	μ_1	μ_2
1	1	10.7	12.0	1	1	0.2	0.0015	0.0153
2	1	11.6	2.5	2	1	0.2	0.0172	0
3	1	2.8	12.8	3	1	0.2	0	0.0156
4	1	10.5	11.6	4	0.9799	0.2	0.0169	0.0016
5	1	10.1	11.8	5	0.9801	0.2	0.0015	0.0153
6	1	10.2	11.5	6	0.9579	0.2	0.0015	0.0153
				mean		0.2	0.0065	0.0105

<Table 2> Cross-Evaluation Results for Example

DMU	1	2	3	4	5	6
1	1.000	0.281	1.000	0.968	0.980	0.958
2	0.922	1.000	0.241	0.905	0.871	0.879
3	0.938	0.195	1.000	0.906	0.922	0.898
4	1.000	1.000	0.339	0.980	0.948	0.954
5	1.000	0.281	1.000	0.968	0.980	0.958
6	1.000	0.281	1.000	0.968	0.980	0.958
CE	0.97664	0.50631	0.76339	0.94915	0.94673	0.93419

input values are unified to 1) is exactly same with the value $\frac{\bar{\mu}_{r,j}}{\bar{v}}$. That is column average of each output multipliers/ column average of input multipliers. This simple error is due to the fact that they didn't show the following in deriving (4). That is, by the first constraint of cross-evaluation, input-multipliers for all DMUs should be the same and therefore the column mean of input-multipliers is also the same with input multiplier for each DMU.

Therefore, in the single-input, multiple-output case (when $x_{1,j}=1$), the above (4) can be developed to (5) as follows.

- By the first constraint of cross-evaluation

$$\sum_{i=1}^{m}\left(v_{i,j}\sum_{\substack{j=1\\ j\neq j_0}}^{n}x_{i,j}\right)=(n-1)v_{1,j}=1,$$

thus $$v_{1,j}=\frac{1}{n-1},\ \forall j$$

(For example, in the following example $\nu_{1,j}=\frac{1}{6-1}=0.2$, for all j).

- Also $\bar{\nu}=\frac{1}{n}\sum_{j=1}^{n}\nu_{1,j}=\frac{1}{n}\times\frac{n}{n-1}=\frac{1}{n-1}=\nu_{1,j}$

- Therefore

$$CE_k=\frac{1}{n}\left[\sum_{j=1}^{n}\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}}\right]=\frac{1}{n}\left[\sum_{j=1}^{n}\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}}\right]$$

$$=\frac{1}{nx_{1,k}\,v_{1,j}}\left[\sum_{j=1}^{n}\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}\right]=\frac{1}{n\bar{v}}\left[\sum_{r=1}^{s}\sum_{j=1}^{n}\mu_{r,j}\,y_{r,k}\right]$$

$$=\sum_{r=1}^{s}y_{r,k}\left[\sum_{j=1}^{n}\left(\frac{\mu_{r,j}/n}{\bar{v}}\right)\right]=\sum_{r=1}^{s}y_{r,k}\left[\frac{\bar{\mu}_r}{\bar{v}}\right] \qquad (5)$$

where, $\bar{\mu}_r$ is the average value of each output weights, $\nu_{1,j}=\bar{\nu}$ (for all j) is the average value of input weights.

After all, the multipliers $\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\mu_{r,j}}{\nu_{1,j}}\right)\right]$ for each output $y_{r,k}$ can be expressed as which $\left[\frac{\bar{\mu}_r}{\bar{v}}\right]$ are independent of DMU k. We will show this by the following example 1.

<Table 1> shows an example 1 data (left table), CCR efficiency score (2nd column in right table) and cross-efficiency multipliers (3rd~5th column in right table). <Table 2> shows

<Table 3> Example 2 Artificial Data and Cross Efficiency Weights

Data				CE ultipliers-Input Oriented				
DMU	x	y_1	y_2	DMU	CCR	ν_1	μ_1	μ_2
1	1	7	7	1	1	0.0345	0.0031	0.0018
2	5	50	10	2	1	0.0400	0.0040	0.0000
3	5	15	45	3	1	0.0400	0.0000	0.0044
4	8	48	48	4	0.8571	0.0455	0.0041	0.0024
5	10	50	70	5	0.9048	0.0500	0.0024	0.0048
6	1	5	8	6	1	0.0500	0.0024	0.0048
				mean		0.04332	0.00265	0.00304

<Table 4> Cross-Evaluation Results for Example 2

DMU	1	2	3	4	5	6
1	1.000	1.000	0.750	0.857	0.821	0.875
2	0.700	1.000	0.300	0.600	0.500	0.500
3	0.778	0.222	1.000	0.667	0.778	0.889
4	1.000	1.000	0.750	0.857	0.821	0.875
5	1.000	0.667	1.000	0.857	0.905	1.000
6	1.000	0.667	1.000	0.857	0.905	1.000
CE	0.91294	0.75925	0.79997	0.78252	0.78834	0.85645
CE^*	0.91962	0.75206	0.81538	0.78824	0.79729	0.86750

across-efficiency matrix and cross-efficiency scores represented as CE in the last row. From the above (3), cross-efficiency scores of DMU 1 and 2 are calculated as follows.

$$CE_1 = \frac{1}{v}[y_{11}\bar{\mu}_1 + y_{21}\bar{\mu}_2]$$

$$= \frac{1}{0.2}(y_{11}(0.0064635) + y_{21}(0.010514))$$

$$= 0.97664$$

$$CE_2 = \frac{1}{v}[y_{12}\bar{\mu}_1 + y_{22}\bar{\mu}_2]$$

$$= \frac{1}{0.2}(y_{12}(0.0064635) + y_{22}(0.010514))$$

$$= 0.50631$$

As a conclusion, in the case that single-input, multiple-outputs with all input values are equal to 1, cross-evaluation uses the fixed weights, which are exactly same as the column mean of the multipliers.

3.2 The Case that Each DMU's Input Value is not Unified to 1

However unlike the above example, if the input variables are not unified to 1 and have different values, then above (4) should be modified as (6). Because input variables are not unified to 1, the input multipliers and input values $x_{1,k}$ have different values and cannot be extracted out of the bracket. Therefore we cannot simply get the fixed weights as above example.

$$CE_k = \frac{1}{n}\left[\sum_{j=1}^{n}\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}}\right] = \frac{1}{n}\left[\sum_{j=1}^{n}\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}}\right]$$

$$= \frac{1}{n}\left[\sum_{r=1}^{s}y_{r,k}\sum_{j=1}^{n}\left(\frac{\mu_{r,j}}{v_{1,j}\,x_{1,k}}\right)\right] = \sum_{r=1}^{s}y_{r,k}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\mu_{r,j}}{v_{1,j}\,x_{1,k}}\right)\right] \quad (6)$$

<Table 3> shows an example 2 data (left table), CCR efficiency score (2^{nd} column in right table) and cross-efficiency multipliers (3^{rd}~5^{th} column in right table). <Table 4> shows across-efficiency matrix and efficiency scores as represented

as mean in the second last row. The last row of <Table 4> represented as CE^* is calculated by (7) using the column mean as the fixed common set of weights to each variable.

$$CE^* = \frac{\sum_{r=1}^{s} y_{r,j}\overline{\mu_r}}{x_{1,j}\overline{v}} \tag{7}$$

where, $\overline{\nu}, \overline{\mu_r}$ is the mean value of input and multipliers. When we compare the results of example 2 with those in which all input values are changed to 1 (i.e. if input value of DMU j is multiplied by k, each output value is also multiplied by k respectively), 1) CCR efficiency scores and the final cross-efficiency scores (mean) are the same. 2) However, the results of CE (real cross-efficiency score) and CE^*(scores using fixed set of weights) are not the same. Even though we cannot find the common fixed weights in this case, column means of each multiplier values can be served as a good indicator in this example too. Since the values from those two methods are very similar to each other. Until this we have examined the single-input, multiple-outputs case, and from next section we will extend this to the multiple-input, multiple-output case to see the characteristics of fixed weighting nature in cross-evaluation.

4. Multiple-Input, Multiple-Output Case

Like the single-input, multiple-output case (if $x_{i,k}$ is not all changed to 1), cross-evaluation doesn't use each column mean of multiplier values as the fixed weights in the multiple-input, multiple-output case. However, when we assume that the cross-evaluation uses each column mean as the fixed set of weights and calculate the efficiency score by (8), the result is very similar with the real cross-efficiency score (CE) in many cases.

$$CE^* = \frac{\sum_{r=1}^{s} y_{r,j}\,\overline{\mu_r}}{\sum_{i=1}^{m} x_{i,j}\,\overline{v}_i} \tag{8}$$

Moreover the difference is much smaller in the DMUs that are located among the cluster of majority DMUs in the sample. To confirm that we will compare two equations, one is an equation to find the real cross-efficiency score (CE) and the other is (6) using the column mean of multipliers as the fixed set of weights (CE^*).

The real cross-efficiency score of DMU k with multiple-input, multiple-output case is calculated as (9)

$$CE_k = \frac{1}{n}\left[\sum_{j=1}^{n}\frac{\sum_{r=1}^{s}\mu_{r,j}\,y_{r,k}}{\sum_{i=1}^{m}v_{i,j}\,x_{i,k}}\right]$$

$$= \frac{1}{n}\left(\left[\frac{y_{1,k}\,\mu_{1,1}+y_{2,k}\,\mu_{2,1}+y_{3,k}\,\mu_{3,1}+\cdots y_{s,k}\,\mu_{s,1}}{x_{1,k}\,v_{1,1}+x_{2,k}\,v_{2,1}+x_{3,k}\,v_{3,1}+\cdots+x_{m,k}\,v_{m,1}}\right]\rightarrow\frac{b(1)}{a(1)}\right.$$

$$+\left[\frac{y_{1,k}\,\mu_{1,2}+y_{2,k}\,\mu_{2,2}+y_{3,k}\,\mu_{3,2}+\cdots y_{s,k}\,\mu_{s,2}}{x_{1,k}\,v_{1,2}+x_{2,k}\,v_{2,2}+x_{3,k}\,v_{3,2}+\cdots+x_{m,k}\,v_{m,2}}\right]\rightarrow\frac{b(2)}{a(2)}$$

$$+\cdots\vdots$$

$$\left.+\left[\frac{y_{1,k}\,\mu_{1,j}+y_{2,k}\mu_{2,j}+y_{3,k}\,\mu_{3,j}+\cdots y_{s,k}\mu_{s,j}}{x_{1,k}\,v_{1,j}+x_{2,k}\,v_{2,j}+x_{3,k}\,v_{3,j}+\cdots+x_{m,k}\,v_{m,j}}\right]\right)\rightarrow\frac{b(j)}{a(j)}\Bigg\}$$

$$= \frac{1}{n}\left[\frac{b(1)}{a(1)}+\frac{b(2)}{a(2)}+\frac{b(3)}{a(3)}+\cdots+\frac{b(j)}{a(j)}\right]_k \tag{9}$$

where, $a(j)$ = virtual input when we use DMU j's input multipliers

$b(j)$ = virtual output when we use DMU j's output multipliers

And when we assume that cross-evaluation would use each multiplier column mean values as the fixed set of weights, cross-efficiency score of DMU k(8) can be restated by (10).

$$CE_k^* = \frac{y_{1,k}\times\frac{1}{n}(\mu_{1,1}+\cdots+\mu_{1,j})+y_{2,k}\times\frac{1}{n}(\mu_{2,1}+\cdots+\mu_{2,j}) + \cdots + y_{s,k}\times\frac{1}{n}(\mu_{s,1}+\cdots+\mu_{s,j})}{x_{1,k}\times\frac{1}{n}(v_{1,1}+\cdots+v_{1,j})+x_{2,k}\times\frac{1}{n}(v_{2,1}+\cdots+v_{2,j}) + \cdots + x_{m,k}\times\frac{1}{n}(v_{m,1}+\cdots+v_{m,j})}$$

$$= \frac{(y_{1,k}\,\mu_{1,1}+\cdots+y_{s,k}\,\mu_{s,1})+(y_{1,k}\,\mu_{1,2}+\cdots+y_{s,k}\,\mu_{s,2}) + \cdots+(y_{1,k}\,\mu_{1,j}+\cdots+y_{s,k}\,\mu_{s,j})}{(x_{1,k}\,v_{1,1}+\cdots x_{m,k}\,v_{m,1})+(x_{1,k}\,v_{1,2}+\cdots+x_{m,k}\,v_{m,2}) + \cdots+(x_{1,k}\,v_{1,j}+\cdots x_{m,k}\,v_{m,j})}$$

$$= \left[\frac{b(1)+b(2)+b(3)+\cdots+b(j)}{a(1)+a(2)+a(3)+\cdots+a(j)}\right]_k \tag{10}$$

After all, it is clear from above two equations (9) and (10) that the results are not the same. The purpose of deriving

(10) is not to show that the results from two equations (9) and (10) is the same but to show the characteristics of differences according to DMUs.

When we evaluate the cross-efficiency score of DMU k, if most values of $b(j)/a(j)$ $(j=1,\cdots,n)$ are close to the DMU k's CCR efficiency score $c_k = b(k)/a(k)$, then the difference in results of above two equations should be very small. For example when we assume that DMU 1's CCR efficiency score is 1.0, then $b(1)/a(1)=1$. If all the other values of $b(2)/a(2)$, $\cdots$, $b(j)/a(j)$ are very close to 1, above two equation values should be very similar

5. The Characteristics of Differences between Two Equations

When we let DMU k's CCR efficiency score as c_k, then we can say

$$\frac{b(1)}{a(1)} = c_k - \alpha_1, \cdots \frac{b(k)}{a(k)} = c_k, \cdots \frac{b(j)}{a(j)} = c_k - \alpha_j \tag{11}$$

(For example, DMU 3's CCR efficiency score in the following FMS data is $c_3 = 0.982$, and $\frac{b(1)}{a(1)} = c_k - \alpha_1 = 0.982 - \alpha_1 = 0.630$. Therefore $\alpha_1 = 0.352$).

Therefore (10) becomes

$$CE_k^* = \left[\frac{b(1)+b(2)+b(3)+\cdots+b(j)}{a(1)+a(2)+a(3)+\cdots+a(j)}\right]_k$$

$$= \left[\frac{\{a(1)c_k - a(1)\alpha_1\}+\cdots+\{a(j)c_k - a(j)\alpha_j\}}{a(1)+a(2)+a(3)+\cdots+a(j)}\right]_k$$

$$= \left[c_k - \frac{a(1)\alpha_1+\cdots+a(j)\alpha_j}{a(1)+a(2)+a(3)+\cdots+a(j)}\right]_k \tag{12}$$

On the other hand, (9) becomes

$$CE_k = \frac{1}{n}\left[\frac{b(1)}{a(1)}+\frac{b(2)}{a(2)}+\frac{b(3)}{a(3)}+\cdots+\frac{b(j)}{a(j)}\right]_k$$

$$= \frac{1}{n}\left[\frac{a(1)c_k - a(1)\alpha_1}{a(1)}+\cdots+\frac{a(j)c_k - a(j)\alpha_j}{a(j)}\right]_k$$

$$= \left[c_k - \frac{\alpha_1+\cdots+\alpha_j}{n}\right]_k \tag{13}$$

Finally, when we subtract equation (12) from equation (13)

$$CE_k - CE_k^* = \left[c_k - \frac{\alpha_1+\cdots+\alpha_j}{n}\right]_k - \left[\frac{a(1)\alpha_1+\cdots a(j)\alpha_j}{a(1)+\cdots+a(j)}\right]_k$$
$$= \alpha_1\left[\frac{a(1)}{a(1)+\cdots+a(j)} - \frac{1}{n}\right] + \cdots + \alpha_j\left[\frac{a(j)}{a(1)+\cdots+a(j)} - \frac{1}{n}\right] \tag{14}$$

In the single input, multi outputs case ($x_{1k}=1$, $\nu_{1j}=1/n-1=constant \; \forall j$)

$$\frac{a(k)}{a(1)+\cdots+a(j)} = \frac{x_{1k}v_{1k}}{x_{1k}v_{1,1}+\cdots+x_{1k}v_{1j}} = \frac{1}{n},$$

$$\text{therefore} \qquad CE_k - CE_k^* = 0 \tag{15}$$

Therefore, (15) verifies the result of (5).

We cannot directly see from (14) that the difference between two results is small. The fact that the final cross-efficiency scores are not much different with those obtained from assuming fixed set of weights in many cases will be shown in next section through some application examples. However we can see from (14) that for a certain DMU k, when many values of α_j are zero or very small, then the difference between the results from two equations (9) and (10) should be very small. And for a certain DMU k, "many values of α_j are zero or very small" means that DMU $k$locates very close to the majority of DMUs, not located far apart from other DMUs. Therefore, we can say that using fixed weights (column mean of each multipliers), we can get closer results from two equations when DMU is located in a cluster of majority of DMUs, than located far apart. If a 1stranker in cross-evaluation shows very small difference between the results, it can be said that it is among them ajority of DMUs, i.e. winner with many competitors and almost fixed weights (average values of each multiplier column) are used for evaluation.

Also, we can define average difference, maximum difference between the results as follows (16), (17).

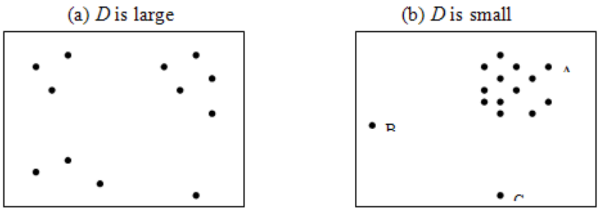
$$D = \sum_{k=1}^{n} d_k = \frac{1}{n}\sum_{k=1}^{n}(CE_k - CE_k^*) \tag{16}$$

$$\max(d_k) = \max(CE_k - CE_k^*) \tag{17}$$

<Figure 1> shows the characteristics of differences between two equations.

(a) shows the sparse data and seems to have a larger D value than (b).
(b) has a majority (cluster) DMUs and seems to have a smaller D value than (a).

Among the DMUs in (b), If DMUs A, B and C are CCR efficient, DMU A will show the smaller difference in results from two equations than those of DMU B or C.



<Figure 1> The Characteristics of Difference between Two Equations

6. Application Examples

To confirm above derivations, we compared the cross-evaluation results between (9) and (10) of FMS data from [13]. Also to show the fact that the final cross-efficiency scores are not much different with those obtained from assuming fixed set of weights in many cases, we compared the results of 5 more application examples, which can be found in previous DEA literature. However for simplicity, each data and more detailed results of these 5 application examples are not shown in this paper except the results of (16) and (17).

<Table 5> FMS Data[13]

(FMS)	x_1	x_2	y_1	y_2	y_3	y_4
1	17.02	5.0	42	45.3	14.2	30.1
2	16.46	4.5	39	40.1	13.0	29.8
3	11.76	6.0	26	39.6	13.8	24.5
4	10.52	4.0	22	36.0	11.3	25.0
5	9.50	3.8	21	34.2	12.0	20.4
6	4.79	5.4	10	20.1	5.0	16.5
7	6.21	6.2	14	26.5	7.0	19.7
8	11.12	6.0	25	35.9	9.0	24.7
9	3.67	8.0	4	17.4	0.1	18.1
10	8.93	7.0	16	34.3	6.5	20.6
11	17.74	7.1	43	45.6	14.0	31.1
12	14.85	6.2	27	38.7	13.8	25.4

<Table 5> shows the FMS data, which have two inputs and 4 outputs. <Table 6> shows the CCR efficiency scores, input/output cross efficiency multipliers of each DMU. The final cross-efficiency scores for each DMU are represented as CE in the <Table 7> (second from last row), which is calculated by (9) in previous section. Each DMU's self-efficiency score (CCR) is the corresponding diagonal elements of cross-efficiency matrix. The final row (CE^*) in <Table 6> represents the efficiency score by (10) in previous section.

Among 12 DMUs, 7 DMUs (1, 2, 4, 5, 6, 7, and 9) are evaluated as CCR efficient (58.3%). DMU 5 is the 1st ranker and DMU 9 is ranked the last. Also it is interesting to see that the results by (10), i.e. CE^* shows very similar to CE through all DMUs while some of the DMUs have very close scores and the others show relatively big differences.

DMU 5 (1st ranker) shows the smallest difference (0.0029) between CE and CE^* among the CCR efficient DMUs, and DMU 9 (12th ranker) shows the largest difference (0.2089) between those scores. This represents that DMU 5 is located among the cluster of majority DMUs and DMU 9 located apart from the majority of DMUs. When we compare the DMUs 5 and 9 in detail, 6 DMUs (1, 2, 6, 7, 9 and 11) are the cause of difference in results of DMU 5 and 9 DMUs (1, 2, 3, 4, 5, 8, 10, 11 and 12) are the cause of difference in results of DMU 9. Especially we can see that DMUs 1, 2, 3, 4, 5 would cause large difference in results of DMU 9 while only DMU 9 cause large difference in results to DMU 5.

The average difference by (16) among all DMUs is $D = 0.0386$. Only three DMUs of 6, 7 and 9 have more than average values of d_k ($d_9 = 0.2089$, $d_6 = 0.0947$, $d_7 = 0.0731$) and all the other DMUs have the value of d_k less than average.

Above (14) can be confirmed by the followings.

(DMU 5)

$$CE_k - CE_k^* = \left[\frac{a(1)\alpha_1 + \cdots + a(j)\alpha_j}{a(1) + \cdots + a(j)} - \frac{\alpha(1) + \cdots + \alpha(j)}{n}\right]_k$$
$$= \left[\frac{0.111247}{0.854493} - \frac{1.59757}{12}\right] = -0.0029$$

This results is exactly equal to $CE_k - CE_k^* = 0.86684 - 0.86978 = -0.0029$

(DMU 9)

$$CE_k - CE_k^* = \left[\frac{0.4796066}{0.743540} - \frac{5.233927}{12}\right] = 0.20887$$

This results is exactly equal to $CE_k - CE_k^* = 0.56384 - 0.35479 = 0.20887$

<Table 6> Cross Efficiency Multipliers of FMS Data

	Efficiency	Input weights		Output weights			
(FMS)	(CCR)	ν_1	ν_2	μ_1	μ_2	μ_3	μ_4
1	1	0.0024	0.0112	0.0023	0	0	0
2	1	0	0.0155	0.0018	0	0	0
3	0.9824	0.0083	0	0.0028	0	0.0016	0.0001
4	1	0.0006	0.0142	0	0	0.0029	0.0012
5	1	0	0.0153	0	0	0.0048	0
6	1	0.0078	0	0	0	0.0023	0.0016
7	1	0.0079	0	0	0.0009	0.0015	0.0007
8	0.9614	0.0070	0.0024	0.0019	0.0004	0	0.0011
9	1	0.0078	0	0	0	0	0.0016
10	0.9536	0.0069	0.0024	0	0.0022	0	0
11	0.9831	0.0087	0	0.0032	0	0	0.0005
12	0.8012	0.0072	0.0023	0.0020	0	0.0006	0.0014
mean		0.00539	0.00527	0.00117	0.00029	0.00115	0.00067

<Table 7> Cross Efficiency Results of FMS Data

DMU	1	2	3	4	5	6	7	8	9	10	11	12
1	**1.000**	1.000	0.630	0.725	0.742	0.322	0.384	0.615	0.094	0.371	0.813	0.593
2	0.969	**1.000**	0.500	0.634	0.638	0.214	0.260	0.481	0.058	0.264	0.699	0.502
3	1.000	0.959	**0.982**	0.927	1.000	0.932	1.000	0.927	0.422	0.760	0.977	0.801
4	0.957	1.000	0.758	**1.000**	1.000	0.434	0.482	0.610	0.190	0.418	0.703	0.732
5	0.899	0.915	0.728	0.895	**1.000**	0.293	0.358	0.475	0.004	0.294	0.624	0.705
6	0.601	0.597	0.764	0.794	0.804	**1.000**	0.970	0.685	1.000	0.678	0.585	0.617
7	0.630	0.599	0.807	0.820	0.857	1.000	**1.000**	0.732	1.000	0.795	0.610	0.637
8	1.000	0.977	0.949	1.000	1.000	0.962	1.000	**0.961**	0.753	0.833	0.951	0.794
9	0.359	0.367	0.422	0.482	0.435	0.698	0.643	0.450	**1.000**	0.468	0.355	0.347
10	0.764	0.704	0.904	0.956	1.000	0.951	1.000	0.860	0.849	**0.953**	0.714	0.720
11	1.000	0.966	0.924	0.896	0.927	0.954	1.000	0.945	0.672	0.783	**0.983**	0.759
12	1.000	0.985	0.953	1.000	1.000	0.968	1.000	0.950	0.724	0.796	0.953	**0.801**
CE	0.848	0.839	0.777	0.844	0.867	0.727	0.758	0.724	0.564	0.618	0.747	0.667
CE^*	0.837	0.820	0.781	0.848	0.870	0.633	0.685	0.727	0.355	0.587	0.755	0.671

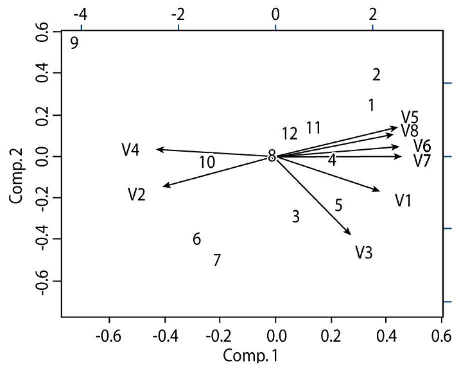
Also from the results (16), (17) of the following application examples which can be found in previous DEA literature, we can confirm the fact that the final cross-efficiency scores are not much different with those obtained from assuming fixed set of weights in many cases.

- Evaluating regions in Serbia (30 DMUs, 4 input 4 output variables) $D=0.030$, $d_k=0.090$ [6]
- Location of solid waste system (22 DMUs, 5 input 3 output variables) $D=0.010$, $d_k=0.022$ [8]
- Car selection problem (28 DMUs, 4 input 2 output variables) $D=0.043$, $d_k=0.104$ [5]
- Economic performance of Chinese cities (18 DMUs, 2 input 3 output) $D=-0.010$, $d_k=-0.065$ [9]
- Location of hydro-electrical power station (6 DMUs, 4 input 2 output variables) $D=0.183$, $d_k=0.415$ [2, 5, 7]

Among the 6 cases including FMS data, the average differences D (16) in 5 cases are very small and only the result of case 5 (Location of hydro electrical power station) shows are latively large difference. It can be said that case 5 has very small number of DMUs compared to that of variables and we can see from the biplot that each DMU locates very sparse. Therefore we can say that even though cross- evaluation doesn't use the fixed weights, the column averages of cross-efficiency multipliers can be considered as the fixed set of weights to each variable without much difference in many cases. Moreover the above difference is much smaller in the DMUs that are located among the cluster of majority DMUs in the sample. If a 1st ranker in cross- evaluation shows very small difference, it can be said that it is among the majority of DMUs, i.e. winner with many competitors and almost fixed set of weights (i.e. average values of each multiplier column) are used for evaluation.

To understand the characteristics of these differences more clearly, biplot of FMS data is presented in <Figure 2>. This biplot is made using correlation of the following relation (18) of variables. All variables are made to have maximizing criteria.

$$(V_1, V_2, \cdots, V_n) = \left(\frac{y_1}{x_1}, \frac{y_2}{x_1}, \cdots, \frac{y_s}{x_1}, \frac{y_1}{x_2}, \frac{y_2}{x_2}, \cdots, \frac{y_s}{x_2}, \cdots\cdots, \frac{y_1}{x_m}, \frac{y_2}{x_m}, \cdots, \frac{y_s}{x_m}\right) \quad (18)$$



<Figure 2> Biplot of the FMS data

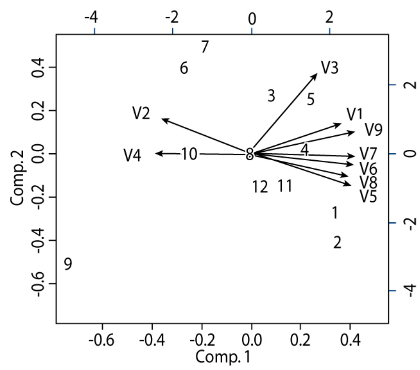
From above variable relations (18), we have 8 variables in this case. And the proportion of variance explained from this biplot is 88.1% of total variance due to two principal components. Biplot permits the visual inspection of one DMU relative to another and the relative importance of each of the two variables to the position of any DMU.

Above biplot shows well the relative position of each DMU to changed variables. Even though the variables are changed according to relation (18), this biplot can satisfy our major interest of hoping to see the relative position among DMUs. It is also interesting to see that all of the CCR efficient DMUs (1, 2, 4, 5, 6, 7, 9) are positioned relatively at the end of each variable and all of CCR inefficient DMUs (3, 8, 10, 11, 12) are positioned near the origin of each variable. We explained in the previous section that 6 DMUs (1, 2, 6, 7, 9 and 11) are the cause of difference in results of DMU 5 and 9 DMUs (1, 2, 3, 4, 5, 8, 10, 11 and 12) are the cause of difference in results of DMU 9 in which DMUs 1, 2, 3, 4, 5 would cause large difference in results of DMU 9 while only DMU 9 cause large difference in results to DMU 5. As expectedly, DMUs 1, 2, 6, 7, and 9 (the major cause to difference to DMU 5) are positioned most apart from DMU 5 in the above biplot too. For DMU 9, it is positioned far apart from most of the DMUs, and especially DMUs 1, 2, 3, 4, 5 in the above biplot. After all the relative positions among DMUs in the biplot coincide the results from the calculation of cross-evaluation. From above biplot, we can see that among CCR efficient DMUs (1, 2, 4, 5, 6, 7, 9), DMUs 6, 7, 9, are positioned relatively apart from the majority of DMUs and all of them have relatively low cross-efficiency score than other DMUs (1, 2, 4, 5). Biplot in Figure 3 includes another variable, which represents the direction of cross-evaluation with fixed weights. That is $(V_1, V_2, \cdots, V_n, V_{n+1})$.

$$V_{n+1} = CE^{*} = \frac{\sum_{r=1}^{s} y_{r,j}\bar{\mu}_r}{\sum_{i=1}^{m} x_{i,j}\bar{v}_i} \quad (19)$$

where, $V_1, V_2, \cdots, V_n$ are the same as (18)

In <Figure 3>, $V_{n+1} = V_9$ represents the direction of cross-efficiency evaluation with fixed weights. From the biplot with additional variable, we can see more clearly that DMUs 5, 1, 4, 2, 3 can be positioned higher rankers, DMUs 7, 6 are middle rankers and DMU 9 should be the lowest ranker among CCR efficient DMUs when we use cross-evaluation with fixed weights. And even though this biplot cannot display total variance (proportion of variance is 89.1% in this case) these look to pretty well coincide with the calculation results.



<Figure 3> Biplot of the FMS Data with Showing the Direction of Fixed Weights

7. Conclusion

In this paper, we have derived and demonstrated the weighting characteristics of cross-evaluation in multiple- input, multiple-output case with input orientation under constant returns to scale. It is true that the same process also can be applied to output orientation under constant returns to scale. Anderson et al. [1] noted that the reasonability and acceptability of this model's fixed weights in single-input, multiple-output case depends on the judgment of the modeler.

However, when we recall the basic assumption of many DEA ranking models, i.e. there is no given preferences (criteria) from decision makers and which is often the case in real life applications, cross-evaluation can be a useful tool to select the winner with many competitors or at least can suggest useful information to decision makers.

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